Recent new methods and applications of the differential equation approach to master integrals

Chris Wever (N.C.S.R. Demokritos)

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C. Papadopoulos [arXiv: 1401.6057 [hep-ph]]
C. Papadopoulos, D. Tommasini, C. Wever [to appear]

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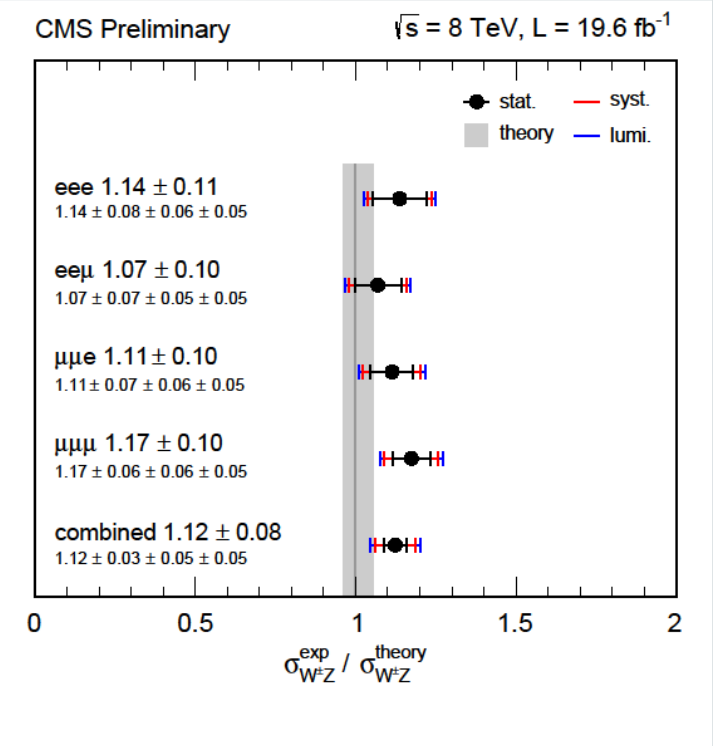
Outline

- Introduction and traditional differential equations method to integration
- Simplified differential equations method
- Application
- Summary and outlook

Motivation

- Mismatch between theory and experimental result
- Theory prediction up to NLO, full NNLO calculation might resolve the discrepancy
- NLO calculations fully automated thanks to NLO reduction methods to Master integrals (MI): (pentagons), boxes, triangles, bubbles and tadpoles



[CMS 2013]

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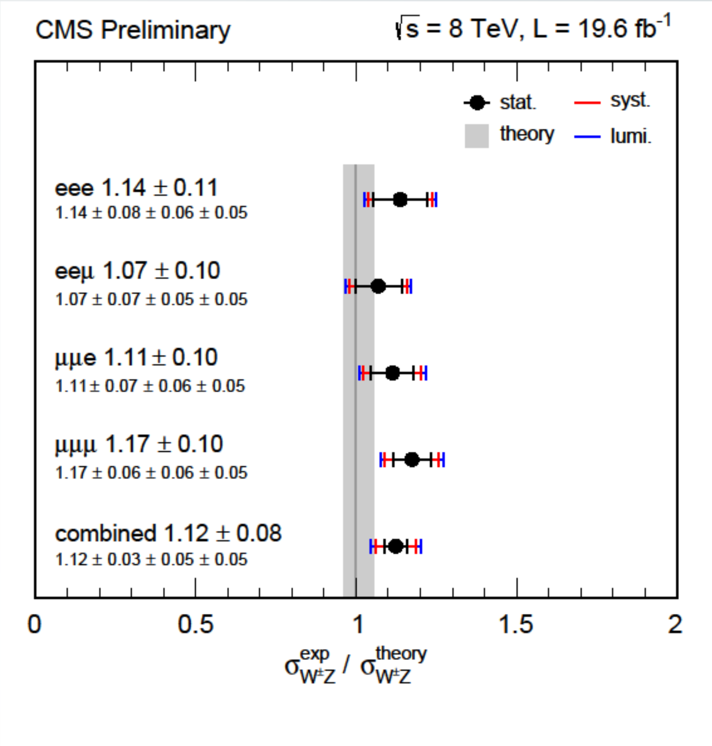


[CMS 2013]

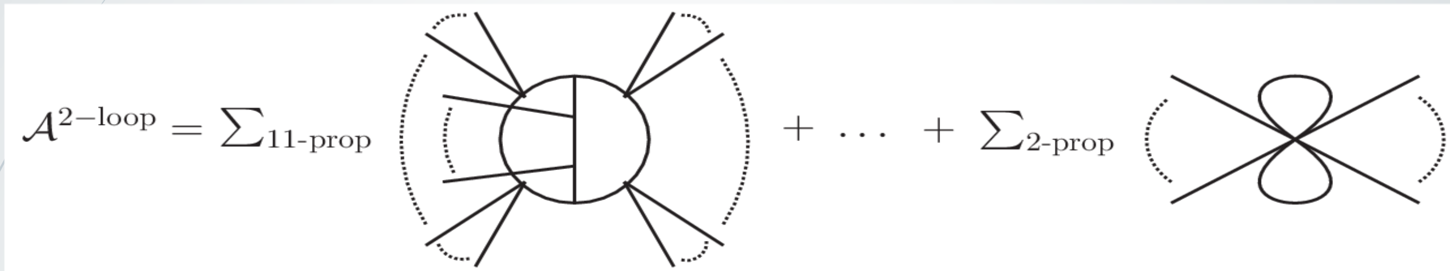
Many numerical NLO tools: Formcalc [Hahn '99], Golem (PV) [Binoth, Cullen et al '08], Rocket [Ellis, Giele et al '09], NJet [Badger, Biederman, Uwer & Yundin '12], Blackhat (***see D. Kosower and D. Maitre talks***) [Berger, Bern, Dixon et al '12], Helac-NLO [Bevilacqua, Czakon et al '12], MCFM (***see K. Ellis's talk***), GoSam (***see G. Heinrich's talk***), OpenLoops (***see P. Maierhofer's talk***), Recola (***see S. Uccirati's talk***), MadGolem, MadLoop, MadFKS, aMC@NLO, …

→ Next step in automation: **NNLO**

→ Bottleneck: virtual-virtual two-loop corrections

Two-loop overview

- A finite basis of Master Integrals exists as well at two-loops:



- Master integrals may contain loop-dependent numerators as well (*tensor integrals*)

Coherent framework for reductions for two- and higher-loop amplitudes:

- In N=4 SYM [Bern, Carrasco, Johansson et al. '09-'12]
- Maximal unitarity cuts in general QFT's [Johansson, Kosower, Larsen et al. '12-'13]
- Integrand reduction with polynomial division in general QFT's (***see P. Mastrolia and S. Badger talks***) [Ossola & Mastrolia '11, Zhang '12, Badger, Frellesvig & Zhang '12-'13, Mastrolia, Mirabella, Ossola & Peraro '12-'13, Kleis, Malamos, Papadopoulos & Verheynen '12]

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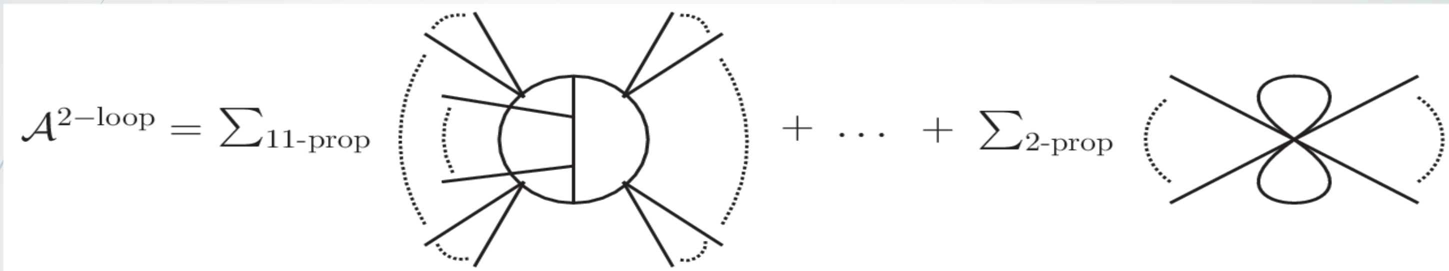

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- By now reduction substantially understood for two- and (multi)-loop integrals
- **Missing ingredient: library of Master integrals (MI)**
- Reduction to MI used for specific processes: *Integration by parts* (IBP) [Tkachov '81, Chetyrkin & Tkachov '81]

Methods for calculating MI

Rewriting of integrals in different representations:

- Parametric: Feynman/alpha parameters → Sector decomposition
- Mellin-Barnes and nested sums (***see C. Raab and J. Gluza talks***) [Bergere & Lam '74, Ussyukina '75, V. Smirnov '99, Tausk '99, Vermaseren '99, Blumlein et al '99,…]

Using relations and/or (cut) identities:

- Dimensional shifting relations [Tarasov '96, Lee '10, Lee, V. Smirnov & A. Smirnov '10]
- Loop-tree duality (***see G. Rodrigo's talk***) [Catani, Gleisberg, Krauss, Rodrigo and Winter '08, Bierenbaum, Catani, Draggiotis, Rodrigo et al '10-'14]
- Integral reconstruction with cuts and coproducts [Abreu, Britto, Duhr & Gardi '14]

As solutions of differential equations (DE): (method of current talk)

- Differentiation w.r.t. invariants (***see V. Smirnov's talk***) [Kotikov '91, Remiddi '97, Caffo, Cryz & Remiddi '98, Gehrmann & Remiddi '00, Henn '13, Henn, Smirnov et al '13-'14]
- Differentiation w.r.t. externally introduced parameter [Papadopoulos '14]

Many more: Dispersion relations, dualities, …

DE method for MI

[Kotikov '91, Remiddi '97, Caffo, Cryz & Remiddi '98, Gehrmann & Remiddi '00 , Henn '13, Henn, Smirnov et al '13-'14]

- Assume one is interested in a multi-loop Feynman integral:

$$G_{a_1\cdots a_n}(\tilde{s}) := \int \Big(\prod_i \frac{d^d k_i}{i\pi^{d/2}}\Big)\frac{1}{D_1^{a_1} D_2^{a_2}\cdots D_n^{a_n}}$$

$$D_i = c_{ijl} k_j.k_l + c_{ij} k_j.p_j + m_i^2$$

$$\tilde{s} = \{\tilde{s}_1, \tilde{s}_2 \cdots\} = \{f_1(p_i.p_j), f_2(p_i, p_j), \cdots\}$$

IBP identities →

$$\int \Big(\prod_i d^d k_i\Big)\frac{\partial}{\partial k_j^\mu}\Big(\frac{v^\mu}{D_1^{a_1} D_2^{a_2}\cdots D_n^{a_n}}\Big) \overset{DR}{=} 0$$

solve →

$$G_{a_1\cdots a_n}(\tilde{s}) = \sum_a f_a(\tilde{s}, d) G_a^{MI}(\tilde{s}, d)$$

- Differentiate w.r.t. external momenta and reduce by IBP to get DE:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) \overset{IBP}{=} \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

DE method for MI

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- Differentiate w.r.t. external momenta and reduce by IBP to get DE:

$$\frac{\partial}{\partial \tilde{s}_k} \vec{G}^{MI}(\tilde{s}, \epsilon) \overset{IBP}{=} \overline{\overline{M}}_k(\tilde{s}, \epsilon) . \vec{G}^{MI}(\tilde{s}, \epsilon)$$

- **Conjecture:** by rotation $\vec{G}^{MI} \to \overline{\overline{A}} . \vec{G}^{MI}$

$$\frac{\partial}{\partial \tilde{s}_k} \vec{G}^{MI}(\tilde{s}, \epsilon) = \epsilon \overline{\overline{M}}_k(\tilde{s}) . \vec{G}^{MI}(\tilde{s}, \epsilon)$$

[Henn '13]

Comments: [Argeri et al '14, Gehrmann et al '14, Hehn et al '14]

- **If** set of invariants $\tilde{s} = \{f(p_i . p_j)\}$ correct: $\overline{\overline{M}}_k(\tilde{s}) = \sum_{\text{poles } \tilde{s}_k^{(0)}} \frac{\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}}{(\tilde{s}_k - \tilde{s}_k^{(0)})}$ → ***Uniform Goncharov Polylogarithm (GP) solution***

- Boundary condition $\vec{G}^{MI}(\tilde{s}_k = \tilde{s}_{k,0})$ found (among other ways) by solving DE's in other invariants

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x-Parametrization

[Papadopoulos '14]

- Introduce extra parameter *x* in the denominators of loop integral
- *x*-parameter describes off-shellness of (some) external legs:

Massive external legs: p_1 → x-parametrize → $p_1 + (1-x)q$

Massless external legs: p_2 → x-parametrize → p_2 or xp_2

(back with x=1)

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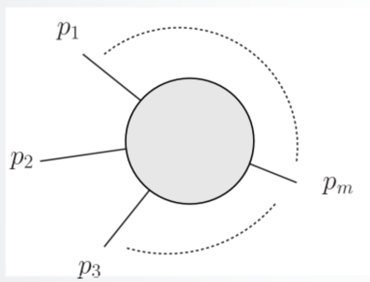
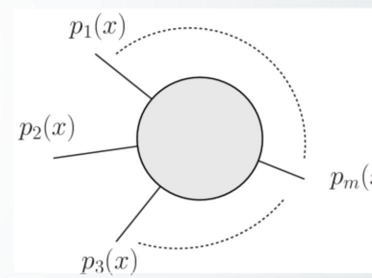
General:



x-parametrize

x=1



$$p_i(x) = p_i + (1-x)q_i$$
$$\sum_i q_i = 0$$

$$G_{a_1\cdots a_n}(s,\epsilon) = \int \left(\prod_i d^d k_i\right) \frac{1}{D_1^{2a_1}(k,p)\cdots D_n^{2a_n}(k,p)} \longrightarrow G_{a_1\cdots a_n}(x,s,\epsilon) = \int \left(\prod_i d^d k_i\right) \frac{1}{D_1^{2a_1}(k,p(x))\cdots D_n^{2a_n}(k,p(x))}$$

$$D_i(k,p) = c_{ij}k_j + d_{ij}p_j, \quad s = \{p_i.p_j\}|_{i,j}$$

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(x=1 returns to the original)

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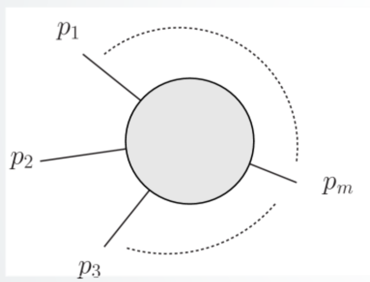
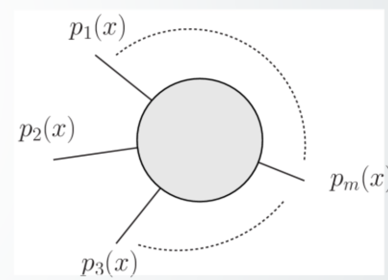


x-parametrize →
← x=1



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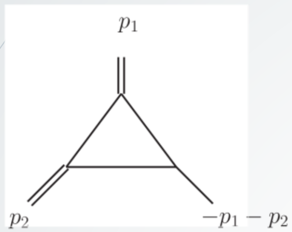
- Take derivative of integral *G* w.r.t. *x*-parameter instead of w.r.t. invariants and reduce r.h.s. by IBP identities:

$$\frac{\partial}{\partial x}\vec{G}^{MI}(x,s,\epsilon) \overset{IBP}{=} \overline{\overline{M}}(x,s,\epsilon).\vec{G}^{MI}(x,s,\epsilon), \qquad s = \{p_i.p_j\}|_{i,j}$$

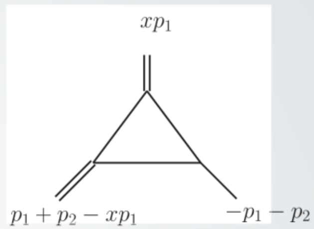
Example: one-loop triangle



$$G_{111}(m_1, m_2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2}$$

$$p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = 0$$

Parametrize p_2 off-shellness with x



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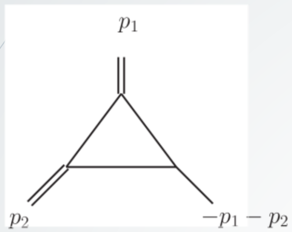

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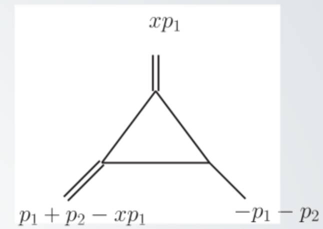

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- Differentiate to x and use IBP to reduce:

$$\frac{\partial}{\partial x} G_{111}(x) = \frac{-x^{-2-\epsilon}}{\epsilon^2 m_1}((-m_1 - i.0)^{-\epsilon}(1+2\epsilon)x^{-\epsilon} - (m_1 - i.0)^{-\epsilon}(1-x)^{-1-\epsilon}(1+\epsilon - x(1+2\epsilon)))$$

- Subtracting the singularities and expanding the finite part leads to:

$$\begin{aligned} G_{111}(x) &= G_{111}(0) + \int_0^x dx' \frac{-x'^{-2-\epsilon}}{\epsilon^2 m_1}((-m_1 - i.0)^{-\epsilon}(1+2\epsilon)x'^{-\epsilon} - (m_1 - i.0)^{-\epsilon}(1-x')^{-1-\epsilon}(1+\epsilon - x'(1+2\epsilon))) \\ &= \underbrace{G_{111}(0)}_{=0} + \frac{-(m_1 - i.0)^{-\epsilon}x^{-\epsilon} + (-m_1 - i.0)^{-\epsilon}x^{-2\epsilon}}{m_1 x \epsilon^2} + \frac{(m_1 - i.0)^{-\epsilon}(-x^{-\epsilon} + (x + GP(1;x)))}{m_1 x \epsilon} + \mathcal{O}(\epsilon^0) \end{aligned}$$

- Agrees with expansion of exact solution: $G_{111}(m_1 * x^2, m_2 = (-m_1)x(1-x)) = \frac{c_\Gamma(\epsilon)}{\epsilon^2} \frac{(-m_1 x^2)^{-\epsilon} - (-(-m_1)x(1-x))^{-\epsilon}}{m_1 x^2 - (-m_1)x(1-x)}$

Bottom-up approach

- Notation: upper index "(m)" in integrals $G^{(m)}_{\{a_1 \ldots a_n\}}$ denotes amount of positive indices, i.e. amount of denominators/propagators

$$G^{(m)}_{a_1 \cdots a_n} = \int \left(\prod_i d^d k_i \right) \underbrace{\frac{1}{D_1^{2a_1}(k,p) \cdots D_n^{2a_n}(k,p)}}_{m \text{ propagators, (positive indices) } a_i}$$

- In practice individual DE's of MI are of the form:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1 \cdots a_n}(x, s, \epsilon) = \sum_{m'=m_0}^{m} \sum_{b_1, \cdots b_n} \text{Rational}^{b_1, \cdots b_n}_{a_1 \cdots a_n}(x, s, \epsilon) G^{(m')}_{b_1 \cdots b_n}(x, s, \epsilon)$$

Bottom-up:

- Solve first for all MI with least amount of denominators m_0 (these are often already known to all orders in ϵ or often calculable with other methods)
- After solving all MI with m denominators ($m \geq m_0$), solve all MI with $m + 1$ denominators
- Often:

$$G^{(m_0)}_{a_1 \cdots a_n}(x, s, \epsilon) = \sum_{n,l} x^{-n+l\epsilon} \left(\sum \text{Rational}(x) GP(\cdots ; x) \right)$$

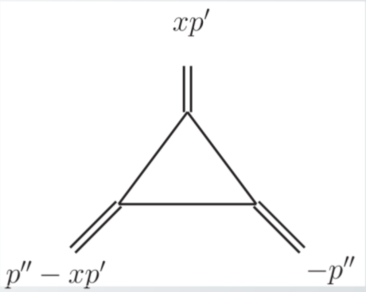
Choice of x-parametrization and boundary term

main criteria for choice of x-parametrization: *constant term* ($\epsilon = 0$) of *residues of homogeneous term* for *every* DE needs to be an *integer:*

$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) = \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon} \left(\sum \text{Rational}(x) GP(\cdots;x) \right) \longrightarrow$$

$$\frac{\partial}{\partial x}\left(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\right) = M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon} \left(\sum \text{Rational}(x) GP(\cdots;x) \right), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}$$

For all MI that we have calculated, the criteria could be easily met. Often it was enough to choose the external legs such that the corresponding massive MI triangles (found by pinching external legs) are as follows:



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$$\frac{\partial}{\partial x}(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)) = M(x,\epsilon)\sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}$$

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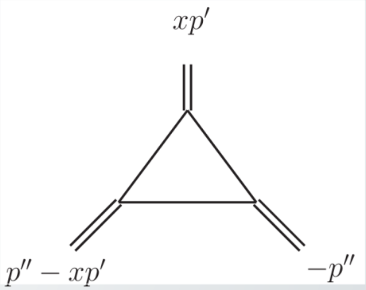

Boundary condition:

- Boundary condition almost always captured by singular subtraction in bottom-up approach
- Except in three cases, all loop integrals we have come across: $(M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} = 0$

→ Not well understood yet why this is so!

- If not zero, boundary condition $(M * G^{(m)}_{a_1\cdots a_n})_{x\to 0}$ may be found (in principle) by plugging in special values for x, via analytical/regularity constraints, asymptotic expansion in $x \to 0$ or some modular transformation like $x \to 1/x$

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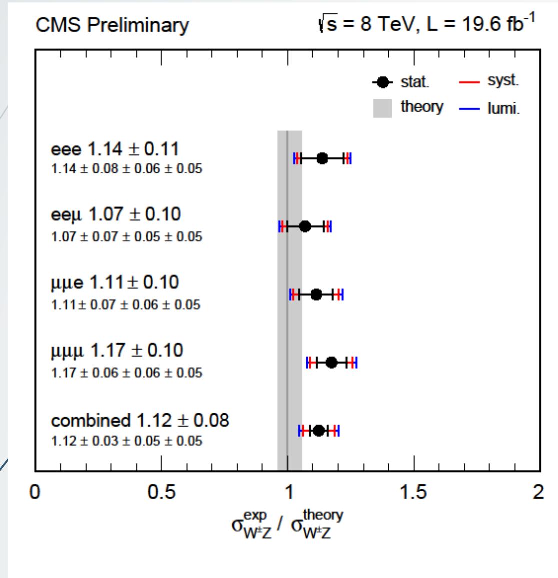
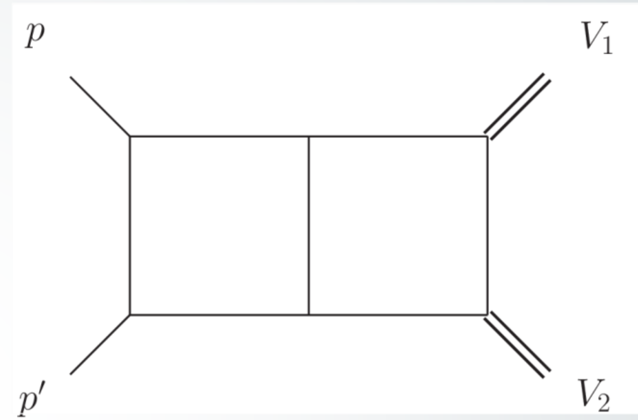
Two-loop planar double-box



Example of planar diagrams:

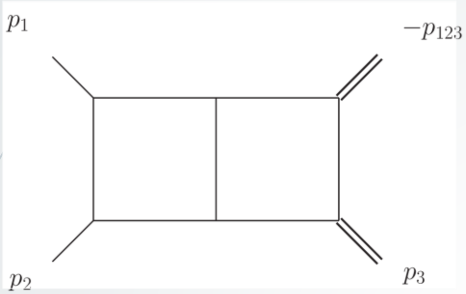


$$pp' \to V_1 V_2, \quad m_{V_1} \neq m_{V_2} \neq 0$$

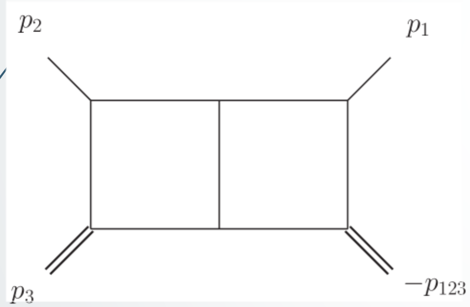
Require 4-point two-loop MI with 2 off-shell legs and massless internal legs (at LHC light-flavor quarks are massless to good degree): diboson production

- On-shell legs: $q_1^2 = \cdots = q_4^2 = 0$ [planar: V. Smirnov '99, V. Smirnov & Veretin '99, non-planar: Tausk '99, Anastasiou, Gehrmann, Oleari, Remiddi & Tausk '00]
- One off-shell leg (pl.+non-pl.): $q_1^2 = q^2,\ q_2^2 = q_3^2 = q_4^2 = 0$ [Gehrmann & Remiddi '00-'01]
- Two off-shell legs with same masses: $q_1^2 = q_2^2 = q^2,\ q_3^2 = q_4^2 = 0$ (***see A. von Manteuffel's talk***) [planar: Gehrmann, Tancredi & Weihs '13, non-planar: Gehrman, Manteuffel, Tancredi & Weihs '14]
- Two off-shell legs with different masses: $q_1^2 \neq 0, q_2^2 \neq 0,\ q_3^2 = q_4^2 = 0$ (***see V. Smirnov's talk***) [planar: Henn, Melnikov & Smirnov '14, non-planar: Caola, Henn, Melnikov & Smirnov '14]

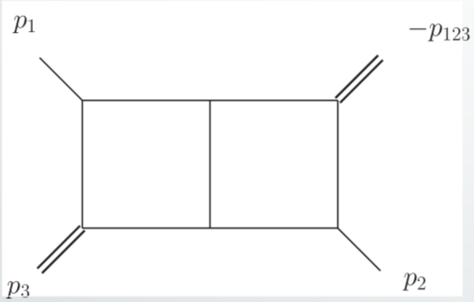
Double planar box: topologies



1st Adjacent mass topology → 31 MI



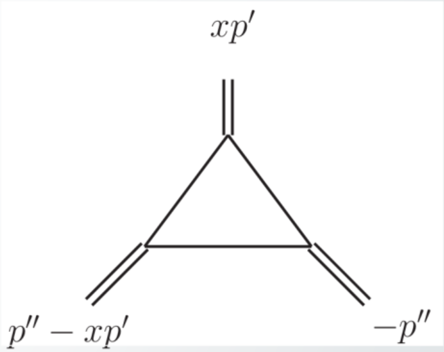
2nd Adjacent mass topology → 29 MI



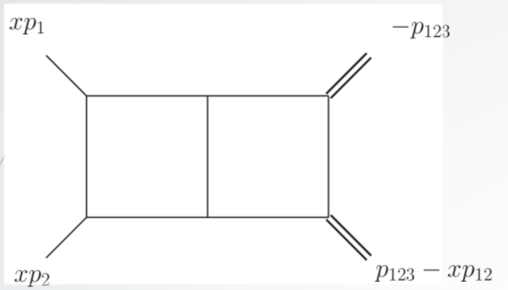
Opposite mass topology → 28 MI

condition for x-parametrization:

pinched massive triangle →

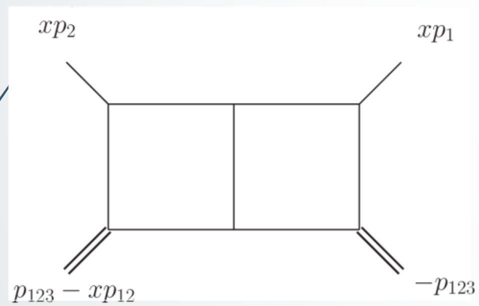


Double planar box: Parametrization



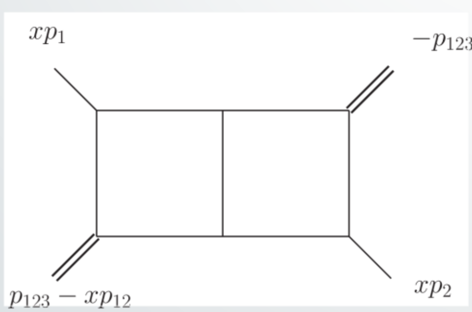
1[st] Adjacent mass topology → 31 MI

$$G^{(1)}_{a_1\cdots a_9}(x) := \int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-xp_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}$$



2[nd] Adjacent mass topology → 29 MI

$$G^{(2)}_{a_1\cdots a_9}(x) := \int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}$$

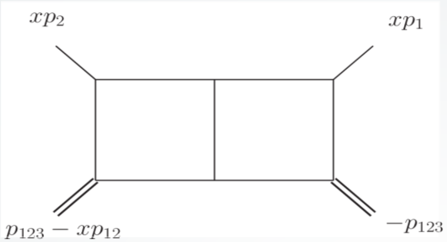


Opposite mass topology → 28 MI

$$G^{(3)}_{a_1\cdots a_9}(x) := \int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+p_{123}-xp_2)^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-p_1)^{2a_6}(k_2+xp_2-p_{123})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}$$

Solutions in GP

$$G^{(2)}_{011111011}(x) =$$



$$s_{12} = p_{12}^2,\;\; s_{23} = p_{23}^2,\;\; m_4 = p_{123}^2$$

solution of DE →

$$
\begin{aligned}
G^{(2)}_{011111011}(x) = {} & \frac{A_3(\epsilon)}{x^2 s_{12}(-m_4 + x(m_4 - s_{23}))^2}\Bigg(\frac{-1}{2\epsilon^4} + \frac{1}{\epsilon^3}\Big(-GP\Big(\frac{m_4}{s_{12}};x\Big) + 2GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big) + 2GP(0;x) - GP(1;x) + \log(-s_{12}) + \frac{9}{4}\Big) \\
& + \frac{1}{4\epsilon^2}\Big(18GP\Big(\frac{m_4}{s_{12}};x\Big) - 36GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big) - 8GP\Big(0,\frac{m_4}{s_{12}};x\Big) + 16GP\Big(0,\frac{m_4}{m_4 - s_{23}};x\Big) + 8GP\Big(\frac{s_{23}}{s_{12}} + 1,\frac{m_4}{m_4 - s_{23}};x\Big) \\
& + 8GP\Big(\frac{m_4}{s_{12}},\frac{s_{23}}{s_{12}} + 1;x\Big) - 8GP\Big(\frac{m_4}{s_{12}},\frac{m_4}{m_4 - s_{23}};x\Big) + 8GP\Big(\frac{m_4}{m_4 - s_{23}},1;x\Big) + 4\Big(-2GP\Big(\frac{s_{23}}{s_{12}} + 1;x\Big)GP\Big(\frac{m_4}{s_{12}};x\Big) \\
& + 2GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big)\Big(2GP\Big(\frac{m_4}{s_{12}};x\Big) - 2GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big) + GP(1;x)\Big) + GP(0;x)\Big(4GP\Big(\frac{m_4}{s_{12}};x\Big) - 8GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big) \\
& + 4GP(1;x) - 4\log(-s_{12}) - 9\Big) + 2\log(-s_{12})\Big(GP\Big(\frac{m_4}{s_{12}};x\Big) - 2GP\Big(\frac{m_4}{m_4 - s_{23}};x\Big) + GP(1;x)\Big) - 4GP(0;x)^2 - \log^2(-s_{12})\Big) \\
& - 8GP\Big(\frac{s_{23}}{s_{12}} + 1,1;x\Big) + 18GP(1;x) - 8GP(0,1;x) - 18\log(-s_{12}) - 9\Big) + \frac{1}{\epsilon}\Big(\cdots\Big) \\
& + \Big(-3GP\Big(0,\frac{m_4}{s_{12}};x\Big)^2 - 18GP\Big(0,\frac{m_4}{m_4 - s_{23}};x\Big)^2 - GP\Big(\frac{s_{23}}{s_{12}} + 1,\frac{m_4}{m_4 - s_{23}};x\Big)^2 - GP\Big(\frac{m_4}{s_{12}},\frac{s_{23}}{s_{12}} + 1;x\Big)^2 + GP\Big(\frac{m_4}{s_{12}},\frac{m_4}{m_4 - s_{23}};x\Big)^2 \\
& + GP\Big(\frac{m_4}{m_4 - s_{23}},1;x\Big)^2 - 2\Big(4GP\Big(0,0,0,\frac{m_4}{s_{12}};x\Big) - 8GP\Big(0,0,0,\frac{m_4}{m_4 - s_{23}};x\Big) - GP\Big(0,0,1,\frac{m_4}{s_{12}};x\Big) + 7GP\Big(0,0,1,\frac{m_4}{m_4 - s_{23}};x\Big) \\
& + 6\Big(GP\Big(0,0,1,\frac{m_4 s_{12} - \sqrt{m_4 s_{12} s_{23}(-m_4 + s_{12} + s_{23})}}{s_{12}(m_4 - s_{23})};x\Big) + GP\Big(0,0,1,\frac{m_4 s_{12} + \sqrt{m_4 s_{12} s_{23}(-m_4 + s_{12} + s_{23})}}{s_{12}(m_4 - s_{23})};x\Big)\Big) \\
& - 10GP\Big(0,0,1,\frac{s_{23}}{s_{12}} + 1;x\Big) + 4GP(0,0,0,1;x) - GP(0,0,1,1;x)\Big) - GP\Big(\frac{s_{23}}{s_{12}} + 1,1;x\Big)^2 - 3GP(0,1;x)^2 + \cdots\Big)\Bigg)
\end{aligned}
$$

ϵ^0 terms

- Numerical agreement in *Euclidean region* found with Secdec [Borowka, Carter & Heinrich]:

$$G^{(2)}_{011111011}(x = 1/3, s_{12} = -2, s_{23} = -5, m_4 = -9) = -\frac{0.0191399}{\epsilon^4} - \frac{0.0292887}{\epsilon^3} + \frac{0.0239971}{\epsilon^2} + \frac{0.340233}{\epsilon} + 0.870356 + \mathcal{O}(\epsilon)$$

Summary

- In LHC era multi-loop calculations are compulsory
- Two-loop automation is the next step: reduction substantially understood, library of MI mandatory but still missing
- Functional basis for large class of MI: *Goncharov polylogarithms*
- DE method is very fruitful for deriving MI in terms of GP
- Simplified DE method [Papadopoulos '14] (often) captures GP solution naturally, boundary constraints taken into account, very algorithmic
- Recent application: planar double box

Outlook

- Application to non-planar graphs
- Application/extension to (some) diagrams with massive propagators

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Thank you very much!

Backup slides

Functional basis for (class of) MI

- ϵ *expansion*:

$$\int dx_1 \cdots dx_n G[\vec{x}, s, \epsilon] = \int dx_1 \cdots dx_n G_{\text{sing}}[\vec{x}, s, \epsilon] + \int dx_1 \cdots dx_n (G[\vec{x}, s, \epsilon] - G_{\text{sing}}[\vec{x}, s, \epsilon])$$

$$= \sum_k \epsilon^k \left(\tilde{G}^{(k)}_{\text{sing}}[s] + \int dx_1 \cdots dx_n G^{(k)}_{\text{finite}}[\vec{x}, s] \right)$$

- The expansion in epsilon often leads to log's $\left(\cdots\right)^{a\epsilon} = 1 + a\epsilon \log\left(\cdots\right) + \frac{a^2}{2}\epsilon^2 \log^2\left(\cdots\right) + \cdots$
- (Some) integrals if parametrized correctly: $\sum \int (\text{Rational function}) * \log^n\left(\cdots\right)$
- The above integrals (often) naturally lead to *Goncharov Polylogarithms* (GP) [Goncharov '98, '01, Remiddi & Vermaseren '00]:

$$GP(\underbrace{a_1, \cdots, a_n}_{\text{weight n}}; x) := \int_0^x dx' \frac{GP(a_2, \cdots, a_n; x')}{x' - a_1}, \quad GP(;x) = 1, \quad GP(\underbrace{0, \cdots, 0}_{\text{n times}}; x) = \frac{1}{n!}\log^n(x)$$

$$GP(\vec{a}; x) GP(\vec{b}; x) = \sum_{\vec{c} = \text{shuffle}\{\vec{a}, \vec{b}\}} GP(\vec{c}; x), \qquad \int_0^x dx' \text{Rational}(x') GP(a_1, \cdots, a_n; x') \stackrel{*}{=} \sum_{i=0}^{n+1} \sum_{b_0 \cdots b_i} \text{Rational}^{b_0 \cdots b_i}(x) GP(b_1, \cdots, b_i; x)$$

GP's are fundamental building blocks for many MI

*Assuming convergence of integral, i.e. after subtracting singularities

→ DE method takes advantage of this fact

Comparison of DE methods

Traditional DE method:

- Choose $\tilde{s} = \{f(p_i.p_j)\}$ and use chain rule to relate differentials of (independent) momenta and invariants:

$$p_i.\frac{\partial}{\partial p_j}\mathrm{F}(\tilde{s}) = \sum_k p_i.\frac{\partial \tilde{s}_k}{\partial p_j}\frac{\partial}{\partial \tilde{s}_k}\mathrm{F}(\tilde{s})$$

- Solve above linear equations:

$$\frac{\partial}{\partial \tilde{s}_k} = g_k(\{p_i.\frac{\partial}{\partial p_j}\})$$

- Differentiate w.r.t. invariant(s) $\tilde{s}_k$:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = g_k(\{p_i.\frac{\partial}{\partial p_j}\})\vec{G}^{MI}(\tilde{s},\epsilon) \overset{IBP}{=} \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

- Make rotation $\vec{G}^{MI} \to \overline{\overline{A}}.\vec{G}^{MI}$ such that:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \epsilon\overline{\overline{M}}_k(\tilde{s}).\vec{G}^{MI}(\tilde{s},\epsilon)$$ [Henn '13]

- Solve perturbatively in ϵ to get GP's if $\tilde{s} = \{f(p_i.p_j)\}$ chosen properly

- Solve DE of different $\tilde{s}_k$, to capture boundary condition

Simplified DE method:

- Introduce external parameter x to capture off-shellness of external momenta:

$$G_{a_1\cdots a_n}(s,\epsilon) = \int \left(\prod_i d^d k_i\right) \frac{1}{D_1^{2a_1}(k,p(x))\cdots D_n^{2a_n}(k,p(x))}$$

$$p_i(x) = p_i + (1-x)q_i, \quad \sum_i q_i = 0, \quad s = \{p_i.p_j\}|_{i,j}$$

- Parametrization: pinched massive triangles should have legs (not fully constraining):

$$q_1(x) = xp', q_2(x) = p'' - xp', \quad p'^2 = m_1, p''^2 = m_3$$

- Differentiate w.r.t. parameter x:

$$\frac{\partial}{\partial x}\vec{G}^{MI}(x,s,\epsilon) \overset{IBP}{=} \overline{\overline{M}}(x,s,\epsilon).\vec{G}^{MI}(x,s,\epsilon)$$

- Check if *constant term* ($\epsilon = 0$) *of residues of homogeneous term* for *every* DE is an *integer*: 1) if yes, solve DE by "bottom-up" approach to express in GP's; 2) if no, change parametrization and check DE again

- Boundary term almost always captured, if not: try $x \to 1/x$ or asymptotic expnansion

Reduction by IBP

[Tkachov ’81, Chetyrkin & Tkachov ’81]

- Fundamental theorem of calculus: given integral, by IBP get linear system of equations

$$G = \int \Big(\prod_i d^d k_i\Big) I \quad \xrightarrow{\text{IBP identities:}} \quad \int \Big(\prod_i d^d k_i\Big) \frac{\partial}{\partial k_j^\mu} \Big(v^\mu I\Big) = \text{Boundary term} \overset{DR}{=} 0$$

$$I = \frac{\text{Num}(k,p)}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} \qquad D_i = c_{ijl} k_j.k_l + c_{ij} k_j.p_j + m_i^2, \qquad v \in \{k_1, \cdots, k_n, \text{external momenta}\}$$

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$$I = \frac{\mathrm{Num}(k,p)}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} \qquad D_i = c_{ijl} k_j . k_l + c_{ij} k_j . p_j + m_i^2, \qquad v \in \{k_1, \cdots, k_n, \text{external momenta}\}$$

In practice, *generate numerator with negative indices* such that w.l.o.g.:

$$G_{a_1 \cdots a_n}(s) := \int \left(\prod_i \frac{d^d k_i}{i\pi^{d/2}}\right) \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}, \quad s = \{p_i . p_j\}|_{i,j}$$

IBP identities: $\longrightarrow$

$$\sum_{a_1, \cdots a_n} \mathrm{Rational}^{a_1 \cdots a_n}(s,d) G_{a_1 \cdots a_n}(s) = 0$$

Solve: $\longrightarrow$

$$G_{a_1 \cdots a_n}(s) = \sum_{(b_1 \cdots b_n) \in \text{Master Integrals}} \mathrm{Rational}^{b_1 \cdots b_n}_{a_1 \cdots a_n}(s,d) G_{b_1 \cdots b_n}(s)$$

- Systematic algorithm: [Laporta '00]. Public implementations: AIR [Anastasiou & Lazopoulos '04], FIRE [A. Smirnov '08] Reduze [Studerus '09, A. von Manteuffel & Studerus '12-13], LiteRed [Lee '12], …
- Revealing independent IBP's: ICE [P. Kant '13]

Uniform weight solution of DE

- In general matrix in DE is dependent on ϵ:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

- **Conjecture:** possible to make a rotation $\vec{G}^{MI} \to \overline{\overline{A}}.\vec{G}^{MI}$ such that:

$$\boxed{\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \epsilon\overline{\overline{M}}_k(\tilde{s}).\vec{G}^{MI}(\tilde{s},\epsilon)}$$

[Henn '13]

- Explicitly shown to be true for many examples [Henn '13, Henn, Smirnov et al '13-'14]
- **If** set of invariants $\tilde{s} = \{f(p_i.p_j)\}$ chosen correctly: $\overline{\overline{M}}_k(\tilde{s}) = \sum_{\text{poles } \tilde{s}_k^{(0)}} \frac{\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}}{(\tilde{s}_k - \tilde{s}_k^{(0)})}$
- Solution is *uniform* in *weight* of GP's:

$$\vec{G}^{MI}(\tilde{s},\epsilon) = Pe^{\epsilon\int_{C[0,\tilde{s}]}\overline{\overline{M}}_k(\tilde{s}'_k)}\vec{G}^{MI}(0,\epsilon) = (\mathbf{1} + \epsilon\int_0^{\tilde{s}_k}\overline{\overline{M}}_k(\tilde{s}'_k) + \cdots)\underbrace{\vec{G}^{MI}(0,\epsilon)}_{\vec{G}_0^{MI}+\epsilon\vec{G}_1^{MI}+\cdots}$$

$$= \underbrace{\vec{G}_0^{MI}}_{\text{weight i}} + \epsilon\underbrace{(\underbrace{\vec{G}_1^{MI}}_{\text{weight i+1}} + \sum_{\text{poles } \tilde{s}_k^{(0)}} \overbrace{\left(\int_0^{\tilde{s}_k}\frac{d\tilde{s}'_k}{(\tilde{s}'_k - \tilde{s}_k^{(0)})}\right)}^{GP(\tilde{s}_k^{(0)};\tilde{s}_k)}\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}.\underbrace{\vec{G}_0^{MI}}_{\text{weight i}})}_{\text{weight i+1}} + \cdots$$

Reduction by IBP: one-loop triangle

One-loop triangle example:

$$G_{a_1 a_2 a_3} = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}, \quad p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = 0$$

IBP identities:

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{\partial}{\partial k^\mu}\left(v^\mu \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}\right) = 0$$

Choose $v = k, p_1, p_2$ respectively

$$0 \overset{v=k}{=} -a_3 G_{-1+a_1,a_2,1+a_3} - a_2 G_{-1+a_1,1+a_2,a_3} + (-2a_1 + d - a_2 - a_3) G_{a_1,a_2,a_3} + m_1 a_2 G_{a_1,1+a_2,a_3}$$

$$0 \overset{v=p_1}{=} a_2 G_{-1+a_1,1+a_2,a_3} + (a_1 - a_2) G_{a_1,a_2,a_3} + a_3 (G_{-1+a_1,a_2,1+a_3} - G_{a_1,-1+a_2,1+a_3} + m_2 G_{a_1,a_2,1+a_3}) \\ -m_1 a_2 G_{a_1,1+a_2,a_3} - a_1 G_{1+a_1,-1+a_2,a_3} + a_1 m_1 G_{1+a_1,a_2,a_3}$$

$$0 \overset{v=p_2}{=} a_3 G_{a_1,-1+a_2,1+a_3} + (a_2 - a_3) G_{a_1,a_2,a_3} - m_2 a_3 G_{a_1,a_2,1+a_3} - a_2 G_{a_1,1+a_2,-1+a_3} + m_2 a_2 G_{a_1,1+a_2,a_3} \\ +a_1 (G_{1+a_1,-1+a_2,a_3} - G_{1+a_1,a_2,-1+a_3} - m_1 G_{1+a_1,a_2,a_3})$$

Solve:

Master integrals: $\{G_{110}, G_{011}\}$

Triangle reduction by IBP:

$$G_{111} = \frac{2(d-3)}{(d-4)(m_1 - m_2)}(G_{011} - G_{110})$$

GP-structure of solution

- Assume for $m' < m$ denominators: $G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \mathrm{Rational}(x) GP(\cdots;x)\Big), \quad m' < m$

- For simplicity we assume here a non-coupled DE for a MI with m denominators:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = H(x,s,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) + \sum_{m'=1}^{m-1} \sum_{b_1,\cdots b_n} \mathrm{Rational}^{(b_1,\cdots b_n)}(x,s,\epsilon) G^{(m')}_{b_1\cdots b_n}(x,s,\epsilon)$$

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dependence on invariants s suppressed

$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \longrightarrow \\
\frac{\partial}{\partial x}\big(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\big) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

GP-structure of solution

- Assume for $m' < m$ denominators: $G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right), \quad m'<m$

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$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right) \longrightarrow \\
\frac{\partial}{\partial x}\left(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\right) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

- Formal solution:

$$\begin{aligned}
M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) &= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{n,l} \prod_{\text{poles } x^{(0)}} \int_0^x dx' \left(x'^{-n+l\epsilon}(x' - x^{(0)})^{-\epsilon c_{x^{(0)}}}\right)\left(\sum (x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}(x') GP(\cdots;x')\right) \\
&= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{\tilde{n},l} \int_0^x dx' x'^{-\tilde{n}+l\epsilon} I_{\tilde{n},l}(\epsilon) + \sum_k \epsilon^k \prod_{\text{poles } x^{(0)}} \sum \int_0^x dx' \underbrace{(x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}_k(x')}_{\text{Rational}_k(x') \text{ if } r_{x^{(0)}} \in \mathbb{Z}} GP(\cdots;x')
\end{aligned}$$

GP-structure of solution

- Assume for $m' < m$ denominators:

$$G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad m' < m$$

- For simplicity we assume here a non-coupled DE for a MI with m denominators:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = H(x,s,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) + \sum_{m'=1}^{m-1} \sum_{b_1,\cdots b_n} \text{Rational}^{(b_1,\cdots b_n)}(x,s,\epsilon) G^{(m')}_{b_1\cdots b_n}(x,s,\epsilon)$$

dependence on invariants *s* suppressed

$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \longrightarrow \\
\frac{\partial}{\partial x}\big(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\big) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

- Formal solution:

$$\begin{aligned}
M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) &= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{n,l} \prod_{\text{poles } x^{(0)}} \int_0^x dx' \Big(x'^{-n+l\epsilon}(x' - x^{(0)})^{-\epsilon c_{x^{(0)}}}\Big)\Big(\sum (x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}(x') GP(\cdots;x')\Big) \\
&= \underbrace{(M * G^{(m)}_{a_1\cdots a_n})_{x\to 0}}_{\text{boundary condition}} + \sum_{\tilde{n},l} \underbrace{\int_0^x dx' x'^{-\tilde{n}+l\epsilon} I_{\tilde{n},l}(\epsilon)}_{x^{-\tilde{n}+l\epsilon+1}\tilde{I}_{\tilde{n},l}(\epsilon)} + \sum_k \epsilon^k \prod_{\text{poles } x^{(0)}} \underbrace{\sum \int_0^x dx' \underbrace{(x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}_k(x')}_{\text{Rational}_k(x') \text{ if } r_{x^{(0)}} \in \mathbb{Z}} GP(\cdots;x')}_{\sum \text{Rational}_k(x) GP(\cdots;x) \text{ if } r_{x^{(0)}} \in \mathbb{Z}}
\end{aligned}$$

→ MI expressible in GP's:

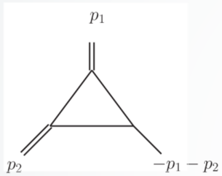
$$G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big)$$

Fine print for coupled DE's: if the non-diagonal piece of $\epsilon = 0$ term of *matrix H* is nilpotent (e.g. triangular) and if diagonal elements of *matrices* $r_{x^{(0)}}$ are integers, then above "GP-argument" is still valid

Example of tradition DE method: one-loop triangle (1/2)

- Consider again one-loop triangles with 2 massive legs and massless propagators:

$$G_{a_1a_2a_3}(\tilde{s}) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}, \quad p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = m_3 = 0$$

$G_{111} =$ 

- General function:

$$p_i \cdot \frac{\partial}{\partial p_j} \mathrm{F}(m_1, m_2, m_3) = \sum_{k=1}^{3} p_i \cdot \frac{\partial \tilde{s}_k}{\partial p_j} \frac{\partial}{\partial \tilde{s}_k} \mathrm{F}(m_1, m_2, m_3), \quad i, j \in \{1, 2\}$$

$$\tilde{s}_1 = p_1^2 = m_1, \tilde{s}_2 = p_2^2 = m_2, \tilde{s}_3 = (p_1+p_2)^2 = m_3$$

- Four linear equations, of which three independent because of invariance under Lorentz transformation [Remiddi & Gehrmann '00], in three unknowns: $\{\frac{\partial}{\partial m_1}, \frac{\partial}{\partial m_2}, \frac{\partial}{\partial m_3}\}$

- Solve linear equations: $\frac{\partial}{\partial m_k} = g_k(p_1 \cdot \frac{\partial}{\partial p_1}, p_2 \cdot \frac{\partial}{\partial p_2}, p_2 \cdot \frac{\partial}{\partial p_1}), \quad k = 1, 2, 3$

$$\frac{\partial}{\partial m_1} G_{111} = \frac{1-2\epsilon}{\epsilon(m_1 - m_2)^2} (G_{011} - (1 + \epsilon(1 - \frac{m_2}{m_1}))G_{110}), \quad \frac{\partial}{\partial m_2} G_{111} = \frac{\partial}{\partial m_1} G_{111} \; (m_1 \leftrightarrow m_2, G_{011} \leftrightarrow G_{110})$$

Example of tradition DE method: one-loop triangle (2/2)

$$\frac{\partial}{\partial m_1}G_{111} = \frac{1}{\epsilon^2(m_1-m_2)^2}((-m_2)^{-\epsilon}+(-m_1)^{-\epsilon}(1+\epsilon)-\epsilon m_2(-m_1)^{-1-\epsilon}) =: F[m_1,m_2], \quad \frac{\partial}{\partial m_2}G_{111} = F[m_2,m_1]$$

- Solve by usual subtraction procedure: $F_{\text{sing}}[m_1,m_2] = \frac{-1}{\epsilon m_2}(-m_1)^{-1-\epsilon}$

$$\begin{aligned} G_{111}(m_1,m_2) &= G_{111}(0,m_2) + \int_0^{m_1} F_{\text{sing}}[m_1',m_2] + \int_0^{m_1}(F[m_1',m_2]-F_{\text{sing}}[m_1',m_2]) \\ &= G_{111}(0,m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \int_0^{m_1}\Big(\frac{(1-(-m_2)^{-\epsilon})GP(;-m_1')}{\epsilon^2(m_2-m_1')^2} - \frac{(m_2-m_1')GP(;-m_1')+m_2GP(0;-m_1')}{\epsilon m_2(m_2-m_1')^2} + \mathcal{O}(\epsilon^0)\Big) \\ &= G_{111}(0,m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \Big(\frac{m_1(1-(-m_2)^{-\epsilon})}{\epsilon^2 m_2(m_1-m_2)} + \frac{m_1 GP(0;-m_1)}{\epsilon m_2(m_2-m_1)}\Big) + \mathcal{O}(\epsilon^0) \end{aligned}$$

- Boundary condition follows by plugging in above solution in $\frac{\partial}{\partial m_2}G_{111} = F[m_2,m_1]$

$$\frac{\partial}{\partial m_2}G_{111}(0,m_2) = \frac{(1+\epsilon)}{\epsilon^2}(-m_2)^{-2-\epsilon} \rightarrow G_{111}(0,m_2) = \frac{-(-m_2)^{-1-\epsilon}}{\epsilon^2} + \underbrace{G_{111}(0,0)}_{\text{scaleless}=0} = \frac{-(-m_2)^{-1-\epsilon}}{\epsilon^2}$$

- Agrees with exact solution: $G_{111} = \frac{c_\Gamma(\epsilon)}{\epsilon^2}\frac{(-m_1)^{-\epsilon}-(-m_2)^{-\epsilon}}{m_1-m_2} = \frac{c_\Gamma(\epsilon)}{m_1-m_2}\Big(-\frac{1}{\epsilon}\log(\frac{-m_1}{-m_2}) + \mathcal{O}(\epsilon^0)\Big)$

Open questions

- Is there a way to pre-empt the choice of x-parametrization without having to calculate the DE?
- Why are the boundary conditions (almost always) naturally taken into account?
- How do the DE in the x-parametrization method relate exactly to those in the traditional DE method?
- How to easily extend parameter x to whole real axis and extend the invariants to the *physical region*?