Anomalous dimension of 2d non-linear Sigma Models

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March 1, 2016

GATIS training event 2016

Based on 1512.07472
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Plan of the talk

1. Introduction
2. symmetric and semi-symmetric backgrounds
3. Z_2 coset sigma model (Candu, Mitev, Schomerus '13)
4. Z_4 coset sigma model
5. Conclusions

Why sigma models?

Sigma models appear many times in low and high energy physics. We focus on 2d non-linear sigma models, that are renormalizable. In particular we focus on the case in which we have target-space supersymmetry.
Examples:

- String theory
- Disordered fermions
- ...

What is a non-linear sigma model?

$$(\mathcal{M}, g) \qquad \phi : \Sigma \to \mathcal{M} \tag{1}$$

$$S_{SM}[\Phi(\sigma)] = \frac{1}{2}\int_\Sigma d^d\sigma \, \eta^{\mu\nu} g_{ab}\partial_\mu\phi^a\partial_\nu\phi^b \tag{2}$$

We focus on $d = 2$. This is not the most general action one can write, in principle one could other terms, like a B-field. The form of the possible actions is very much determined by the background.

Coset

Today we are going to focus on sigma model on cosets: $\mathcal{M} = \frac{G}{H}$

$$g \sim gh \qquad \forall h \in H \subset G \tag{3}$$

Lie algebra:

$$\mathfrak{g} = \mathfrak{h} \oplus_{\perp} \mathfrak{m} \qquad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m} \qquad (\cdot\,,\cdot)\text{bilinear form} \tag{4}$$

P is the projector on $\mathfrak{m}$, and using the currents $j = g^{-1}dg$, we can rewrite the action:

$$S = \frac{1}{2}\int_{\Sigma} d^d\sigma\, \eta^{\mu\nu}(P(j_\mu), P(j_\nu)) \tag{5}$$

Symmetric spaces $\frac{G}{H}$

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 \tag{6}$$

$\sigma : G \to G$ leaves invariant $\mathfrak{h}$ elements. So $\mathfrak{h} = \mathfrak{g}_0$

$$[T_0, T_0] \subset \mathfrak{g}_0 \tag{7}$$

$$\sigma^2 = 1 \qquad \sigma(T_A) = (-)^{|A|} T_A \tag{8}$$

$$[T_1, T_1] \subset \mathfrak{g}_0, \qquad [T_1, T_0] \subset \mathfrak{g}_1 \tag{9}$$

Semi-Symmetric spaces $\frac{G}{H}$

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3 \tag{6}$$

$\sigma : G \to G$ leaves invariant $\mathfrak{h}$ elements. So $\mathfrak{h} = \mathfrak{g}_0$

$$[T_0, T_0] \subset \mathfrak{g}_0 \tag{7}$$

$$\sigma^4 = 1 \qquad \sigma(T_A) = (i)^{|A|} T_A \tag{8}$$

$$[T_A, T_B] \subset \mathfrak{g}_{A+B \text{ mod } 4} \tag{9}$$

Sigma models on coset spaces are very important integrable models.

Symmetric	Semisymmetric
$\mathbb{Z}_2$	$\mathbb{Z}_4$

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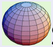 Supersphere	 AdS spaces
$S^{2S+1\|2S}$	$AdS_n \times S^n$

The study of non linear sigma models becomes difficult if we go in the strong curvature regime. For the study of this regime are widely used dualities (strong/weak coupling).

AdS/CFT

Sigma models action on Z_{2N} cosets

$$S = \frac{1}{2} \int_{\Sigma} \frac{d^2 z}{\pi} \sum_{A=1}^{2N-1} (p_A + q_A) \, Str \left(P_{A\mathcal{J}}, P_{A'\bar{\mathcal{J}}} \right) \tag{10}$$

Symmetric $N = 1$

$$p_1 = 1, q_1 = 0 \tag{11}$$

Semi-symmetric $N = 2$

$$p_A = 1 \; , \quad q_A = 1 - \frac{A}{2} \; , \quad \text{for } A = 1, 2, 3 \text{ hybrid model} \tag{12}$$

One more thing before the examples... Fields

At level $(h, \bar{h})$ in the spectrum we find the field:

$$\jmath_{\mathbf{m}\mu}((h-n)\partial, n\jmath) \tag{13}$$

$\mu \rightarrow$ representation of H

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At level $(h, \bar{h})$ in the spectrum we find the field:

$$\jmath_{\mathbf{m}\mu}((h-n)\partial, n\jmath) \otimes \bar{\jmath}_{\bar{\mathbf{m}}\bar{\mu}}((\bar{h}-m)\bar{\partial}, m\bar{\jmath})^{\lambda} \tag{13}$$

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$\bar{\mu}$, $\mu \rightarrow$ representation of H
$\lambda \rightarrow$ representation of H, diagonal representation
$\mathrm{d}_{\lambda\mu\bar{\mu}} \rightarrow$tensor product

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At level $(h, \bar{h})$ in the spectrum we find the field:

$$\Phi_\Lambda(z,\bar{z}) = \mathrm{d}_{\lambda\mu\bar{\mu}} V_{\Lambda\lambda}\, \jmath_{\mathbf{m}\mu}((h-n)\partial, n\jmath) \otimes \bar{\jmath}_{\bar{\mathbf{m}}\bar{\mu}}((\bar{h}-m)\bar{\partial}, m\bar{\jmath})^{\lambda}\,(z,\bar{z}) \quad (13)$$

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The dilation operator will therefore act on a space

$$L^2(G) \otimes \mathfrak{m}^r \otimes \mathfrak{m}^{\bar{r}} \quad (14)$$

One-loop anomalous dimension

$$\langle \Phi_\Lambda(u,\bar{u}) \otimes \Phi_\Xi(v,\bar{v}) \rangle_1 = \langle 2\delta\mathbf{h} \cdot \Phi_\Lambda(u,\bar{u}) \otimes \Phi_\Xi(v,\bar{v}) \rangle_0 \log \left| \frac{\epsilon}{u-v} \right|^2 + \dots \tag{15}$$

$$\langle \Phi_\Lambda(u,\bar{u}) \otimes \Phi_\Xi(v,\bar{v}) \rangle = \int_{G/H} d\mu \left\langle \Phi_\Lambda(u,\bar{u} \mid g_0) \otimes \Phi_\Xi(v,\bar{v} \mid g_0) e^{-S_{\text{int}}} \right\rangle \ , \tag{16}$$

Z_2 sigma model

$$S = \frac{1}{2}\int_\Sigma \frac{d^2 z}{\pi}(\jmath_z, \bar{\jmath}_{\bar{z}}) \tag{17}$$

where $\jmath$ has grading one.

$$\jmath = e^{-\phi}\partial e^{\phi} = \left[\partial\phi - \frac{1}{2}\left[\phi\,,\,\partial\phi\right] + \frac{1}{6}\left[\phi\,,\,\left[\phi\,,\,\partial\phi\right]\right]\right] + \cdots \ . \tag{18}$$

$$S^{(1)}_{\text{int}} = \int d^2 z\ \Omega_4 = \int d^2 z\ \frac{1}{3}\left[\phi\,,\,\partial\phi\right]\left[\phi\,,\,\bar{\partial}\phi\right] \quad \phi : \Sigma \to \frac{G}{H} \tag{19}$$

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One loop anomalous dimension

$$\langle \phi(z,\bar z) \otimes \phi(w,\bar w) \rangle_0 = -\log\left|\frac{z-w}{\epsilon}\right|^2 t_i \otimes t^i \qquad \Phi_\Lambda = \mathrm{d}V \jmath_{\mathbf{m}} \otimes \bar\jmath_{\bar{\mathbf{m}}} \quad (20)$$

Two correlators contribute:

$$\left\langle V^{(1)} \otimes \jmath^{(0)}_{\mathbf{m}}(u) \otimes \bar\jmath^{(0)}_{\bar{\mathbf{m}}}(\bar u) \otimes V^{(1)} \otimes \jmath^{(0)}_{\mathbf{m}}(v) \otimes \bar\jmath^{(0)}_{\bar{\mathbf{m}}}(\bar v) \right\rangle \quad (21)$$

$$\int_{\mathbb{C}_\epsilon} d^2z \left\langle V^{(0)} \otimes \jmath^{(0)}_{\mathbf{m}}(u) \otimes \bar\jmath^{(0)}_{\bar{\mathbf{m}}}(\bar u) \otimes V^{(0)} \otimes \jmath^{(0)}_{\mathbf{m}}(v) \otimes \bar\jmath^{(0)}_{\bar{\mathbf{m}}}(\bar v) \Omega_4(z,\bar z) \right\rangle \quad (22)$$

$$\delta h = \frac{1}{2R^2}\big(\underbrace{C_\Lambda}_{G} + \underbrace{C_\mu + C_{\bar\mu}}_{H}\big)$$

Candu, Mitev, Schomerus '13

$$S = \frac{1}{2}\int_\Sigma \frac{d^2z}{\pi}\sum_{A=1}^{3}(p_A + q_A)\,(P_A\jmath, P_{A'}\bar{\jmath}) \ , \quad p_0 = 0,\ q_0 = 0 \tag{23}$$

Conformal, hybrid model:

$$S = \frac{1}{2}\int_\Sigma \frac{d^2z}{\pi}(\jmath_2, \bar{\jmath}_2) + \frac{3}{2}(\jmath_1, \bar{\jmath}_3) + \frac{1}{2}(\jmath_3, \bar{\jmath}_1) \tag{24}$$

$$S_{\text{int}} = \int \frac{d^2z}{\pi}\Omega_3 + \Omega_4 \tag{25}$$

$$j^(1) = \frac{1}{2}[\phi, \partial\phi] \tag{26}$$

Dilation operator

acts on $L^2(G) \otimes \mathfrak{m}^r \otimes \mathfrak{m}^{\bar{r}}$

$$D^{\text{1-loop}}_{r,\bar{r}} = H^{\text{h}} + H^{\text{ht}} + H^{\text{tt}} \ . \tag{27}$$

$$H^{\text{h}} = C_{\mathfrak{g}} - C_{\mathfrak{h}} \ .$$

$$H^{\text{ht}} = -\sum_{\rho=1}^{r} ((-1)^c + Q_{\alpha\gamma}) R(t_c)\,(t^c)^{\rho}_{\alpha} - \sum_{\bar{\rho}=1}^{\bar{r}} ((-1)^c + Q_{\alpha\gamma}) R(t_c)\,(t^c)^{\bar{\rho}}_{\alpha}$$

$$Q_{\alpha\beta} \equiv q_\alpha + q_\beta - q_{\alpha+\beta} \ . \tag{28}$$

H^{tt}

$$H^{\mathrm{XXZ}}_{\mathrm{pair}} = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta\sigma^z \otimes \sigma^z = (1+\Delta)(I+P) + (1-\Delta)K \ .$$

as for the XXZ model we have three operators

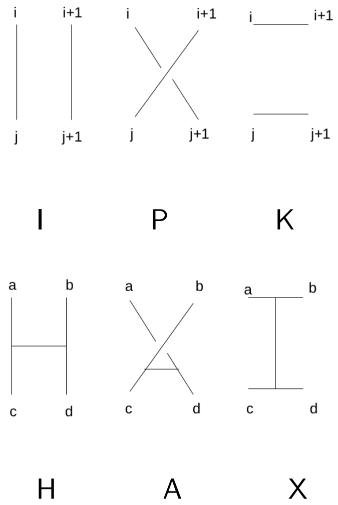
$$X^{abcd}_{\alpha\beta\gamma\delta} = (-1)^{|c||d|} f^{bai} f^{cd}{}_i \ , \tag{29}$$

$$A^{abcd}_{\alpha\beta\gamma\delta} = (-1)^{|c|} f^{adi} f^{cb}{}_i \ , \tag{30}$$

$$H^{abcd}_{\alpha\beta\gamma\delta} = (-1)^{|b|+|b||d|} f^{aci} f^{bd}{}_i \ . \tag{31}$$

$$H^{tt} = f(p,q)X + g(p,q)A + h(p,q)H \tag{32}$$

H^{tt}



Beta-function

Let's take the case of $AdS_5 \times S^5$. We can show that the operator

$$(\jmath_{2^+}, \bar{\jmath}_{2^+}) + (\jmath_{2^-}, \bar{\jmath}_{2^-}) + \frac{3}{2}(\jmath_1, \bar{\jmath}_3) + \frac{1}{2}(\jmath_3, \bar{\jmath}_1) \tag{33}$$

has zero anomaluos dimension. This is analogous to say that the beta function is vanishing. "+" and "-" refer to the sphere and AdS respectively

$$D_{1,1}^{\text{1-loop}} = C \cdot \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \qquad C = f_{132^+} f^{132^+}$$

$$\Rightarrow \begin{pmatrix} \frac{3}{2} \\ 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

Conclusions and outlook

- We found the one-loop anomalous dimension for Z_4 coset sigma model with a particular choice of parameters that guarantees to have a conformal and integrable system.
- Our result is applicable in the study of string theories described by the hybrid model formalism.
- The result is obtain without using integrability, thus it can give complementary and independent information.
- We have seen that our result reproduce the vanishing of the beta function in the $AdS_5 \times S^5$ case. The same thing happen for the vanishing of the one-loop anomalous dimension of the Noether currents (not presented in this talk).

Conclusions and outlook

- To study the spectrum of the sigma model we now need a bit of harmonic analysis on cosets with a non-compact stability group.
- It would be interesting to generalize the study to non conformal cases.
- The case of Green-Schwarz formalism still need to be studied.
- If in this case one would observe instability like in the symmetric case, we could use the many integrability results to address this problem. The problem of instabilities in sigma model is a long standing problem, that still need to be clarified.

Thank you