Perturbative and numerical aspects of string sigma models

based on 1407.4788, 1505.00783, 1508.07331, 1511.01091, 1601.04670
with M. Bianchi, V. Forini, B. Leder, E. Vescovi.

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Universität Hamburg

March 1^{st}, 2016

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AdS/CFT

$\mathcal{N} = 4$
Super Yang-Mills.
SCFT in 4d

Type IIB superstring
in $AdS_5 \times S^5$

$\lambda = \frac{g^2 N_c}{4\pi}$

AdS/CFT integrability

$\mathcal{N}=4$ Super Yang-Mills. SCFT in 4d	INTEGRABILITY ($N_c \to \infty$)	Type *IIB* superstring in $AdS_5 \times S^5$

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INTEGRABILITY ($N_c \to \infty$)

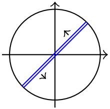
Type *IIB* superstring in $AdS_5 \times S^5$

$$\lambda = \frac{g^2 N_c}{4\pi}$$

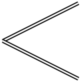
Folded spinning string

[Gubser, Klebanov, Polyakov, 2002; Frolov, Tseytlin, 2002; Belitsky, Gorsky, Korchemsky, 2006; Frolov, Tirziu, Tseytlin, 2007; Kruczenski, Roiban, Tirziu, Tseytlin, 2008]

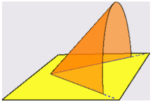
$$\mathrm{Tr}(\phi D_{(\mu_1} \dots D_{\mu_S)} \phi) \longleftrightarrow$$



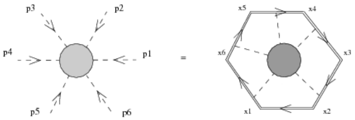
$\downarrow S \to \infty$ $\downarrow S \to \infty$



$\longleftrightarrow$



Scattering amplitudes



GKP string

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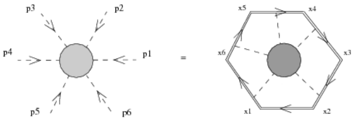


GKP string

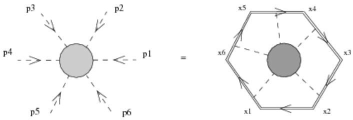
- Free energy: $\log Z = \Gamma_{\text{cusp}}(\lambda) V$

Square and pentagon [Drummond, Henn, Korchemsky, Sokatchev, 2007]

$$\log F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{ const}$$

$$\log F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(a)\sum_{i=1}^{5}\log\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right)\log\left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\right) + \text{ const}$$

Scattering amplitudes



GKP string

- Free energy: $\log Z = \Gamma_{\text{cusp}}(\lambda) V$
- Dispersion relation: $E_i = E_i(p_i) \Leftrightarrow \{E(\mathbf{u}), p(\mathbf{u})\}$
- S-matrix: $S(\mathbf{u}, \mathbf{v}) \Leftrightarrow P(\mathbf{u}|\mathbf{v})$

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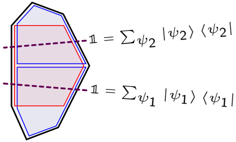
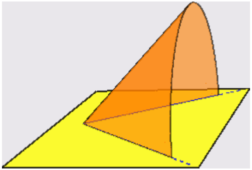
Hexagon and higher [Basso, Sever, Vieira, 2013]



$$\mathcal{W}_{\text{hep}} = \sum_{\psi_1,\psi_2} P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|0) \times e^{-E_1\tau_1 + ip_1\sigma_1 + im_1\phi_1 - E_2\tau_2 + ip_2\sigma_2 + im_2\phi_2}$$

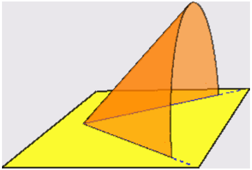
Perturbing the string theory side



Motivation

- Perturbative calculations are essential to give a solid foundation and inspiration to any integrability-based construction, and thus to guarantee its predictivity.
- Non-trivial checks of quantum integrability of the AdS/CFT systems.
- The choice of the regularization is crucial to find agreement with results from integrability (not clear for higher loops).

Perturbing the string theory side



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Strategy

Construct an action. (Green-Schwarz string in curved background). → Fix the light-cone gauge and consider the minimal surface ending on a null cusp. → Consider worldsheet quantum fluctuations around this vacuum (NO string interaction!)

The Lagrangian

Asymptotic spectrum

- Bosons: 1 mode ϕ $m^2 = 1$; 2 modes x, x^* $m^2 = 1/2$; 5 modes y^a $m^2 = 0$.
- Fermions: 8 modes θ^i, η^i $m^2 = \frac{1}{4}$.

$$S_{\rm cusp} = g \int dtds \, \Big\{ |\partial_t x + \tfrac{1}{2}x|^2 + \tfrac{1}{z^4}|\partial_s x - \tfrac{1}{2}x|^2 + \Big(\partial_t z^M + \tfrac{1}{2}z^M + \tfrac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j\Big)^2$$
$$+ \frac{1}{z^4}\Big(\partial_s z^M - \tfrac{1}{2}z^M\Big)^2 + i\Big(\theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i\Big) - \tfrac{1}{z^2}\left(\eta^i \eta_i\right)^2$$
$$+ 2i\Big[\tfrac{1}{z^3} z^M \eta^i (\rho^M)_{ij} \left(\partial_s \theta^j - \tfrac{1}{2}\theta^j - \tfrac{i}{z}\eta^j \left(\partial_s x - \tfrac{1}{2}x\right)\right)$$
$$+ \tfrac{1}{z^3} z^M \eta_i (\rho_M^\dagger)^{ij} \left(\partial_s \theta_j - \tfrac{1}{2}\theta_j + \tfrac{i}{z}\eta_j \left(\partial_s x - \tfrac{1}{2}x\right)^*\right)\Big]\Big\}$$

$$z = e^{\phi}\,, \qquad z^M = e^{\phi} u^M\,, \qquad M = 1, \ldots 6$$

$$u^a = \frac{y^a}{1 + \frac{1}{4}y^2}\,, \qquad u^6 = \frac{1 - \frac{1}{4}y^2}{1 + \frac{1}{4}y^2}\,, \qquad y^2 \equiv \sum_{a=1}^{5} (y^a)^2\,, \qquad a = 1, \ldots, 5$$

Summary of perturbative computations

Free energy (cusp anomaly)

- Computed at two loops in $AdS_5 \times S^5$ [Giombi, Ricci, Roiban, Tseytlin, Vergu, 2009]
- Computed at two loops in $AdS_4 \times \mathbb{CP}^3$ [LB, Bianchi, Bres, Forini, Vescovi, 2014]
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S-matrix

- Computed at tree-level for four and six legs in $AdS_5 \times S^5$ [LB, M. Bianchi 2015]
- Computed at one loop in $AdS_5 \times S^5$ for xx-scattering [LB, M. Bianchi 2015]
  - The calculation agrees with integrability as long as massless modes are not involved

Discretization and numerics

[LB, M. Bianchi, V.Forini, B.Leder, E. Vescovi, 2016]

Main idea

Discretize the two-dimensional string sigma model on a lattice and study the previous observables at finite coupling (still in the planar limit)

Various technical complications: fermion doubling, quartic fermionic interactions...

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Hubbard-Stratonovich

$$\exp\left\{-g\int dtds\left[-\tfrac{1}{z^2}\left(\eta^i\eta_i\right)^2+\left(\tfrac{i}{z^2}z_N\eta_i\rho^{MN\,i}{}_j\eta^j\right)^2\right]\right\}$$

$$\sim\int D\phi D\phi_M\,\exp\left\{-g\int dtds\left[\tfrac{1}{2}\phi^2+\tfrac{\sqrt{2}}{z}\phi\,\eta^2+\tfrac{1}{2}(\phi_M)^2-i\,\tfrac{\sqrt{2}}{z^2}\phi_M z_N\left(i\,\eta_i\rho^{MN\,i}{}_j\eta^j\right)\right]\right\}$$

$$\mathcal{L}=|\partial_t x+\tfrac{m}{2}x|^2+\tfrac{1}{z^4}|\partial_s x-\tfrac{m}{2}x|^2+(\partial_t z^M+\tfrac{1}{2}z^M)^2+\tfrac{1}{z^4}(\partial_s z^M-\tfrac{m}{2}z^M)^2+\tfrac{1}{2}\phi^2+\tfrac{1}{2}(\phi_M)^2+\psi^T O_F\psi$$

$$\int D\psi\; e^{-\int dtds\,\psi^T O_F\psi}=\mathrm{Pf}\,O_F\equiv(\det O_F\,O_F^\dagger)^{\frac{1}{4}}=\int D\xi D\bar{\xi}\,e^{-\int dtds\,\bar{\xi}(O_F O_F^\dagger)^{-\frac{1}{4}}\,\xi}$$

There is a sign problem.

Sign problem

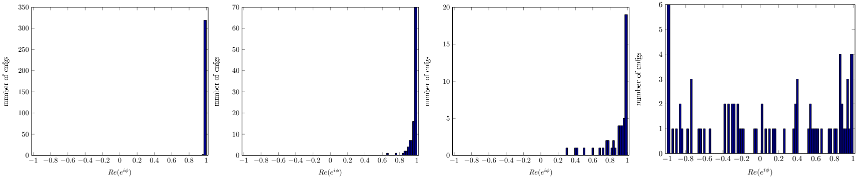

Figure: Histograms for the frequency of the real part of the phase factor $e^{i\theta}$ of the Pfaffian $\mathrm{Pf}\, O_F = |(\det O_F)^{\frac{1}{2}}|\, e^{i\theta}$, based on the ensembles generated at $g = 30, 10, 5, 1$ ($g = \frac{\sqrt{\lambda}}{4\pi}$).

For very large g our simulations are reliable.

The mass of the excitation x

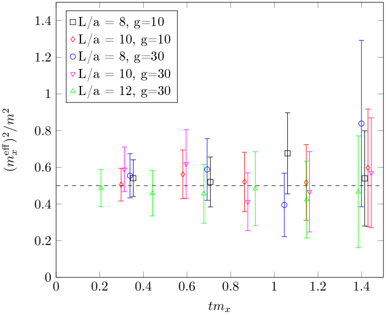

Figure: Effective mass plot $m_x^{\rm eff} = \frac{1}{a} \ln \frac{C_x(t)}{C_x(t+a)}$, as calculated from the correlator $C_x(t) = \sum_{s_1, s_2} \langle x(t, s_1) x^*(0, s_2) \rangle$ of bosonic fields x, x^* in presence of Wilson terms.

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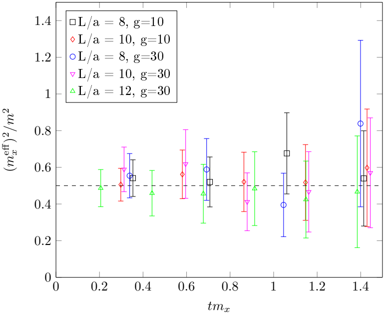

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THANK YOU

Cusp anomaly [Aharony, Bergman, Jafferis, Maldacena, 2008]

Prediction from the Bethe Ansatz [Gromov, Vieira, 2008]

$$f_{\mathrm{ABJM}}(\lambda) = \frac{1}{2}\, f_{\mathcal{N}=4}(\lambda_{\mathrm{YM}})\Big|_{\frac{\sqrt{\lambda_{\mathrm{YM}}}}{4\pi}\to h(\lambda)}$$

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Conjecture for $h(\lambda)$ [Gromov, Sizov, 2014]

$$\lambda = \frac{\sinh^2 2\pi h(\lambda)}{2\pi}\, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\sinh^2 2\pi h(\lambda)\right)$$

$$h(\lambda) \sim \sqrt{\frac{\tilde{\lambda}}{2}} - \frac{\log 2}{2\pi} + \mathcal{O}\left(e^{-2\pi\sqrt{2\lambda}}\right) \qquad \lambda \gg 1 \qquad \tilde{\lambda} = \lambda - \frac{1}{24} = \frac{T^2}{8}$$

[Bergman, Hirano, 2009]

No genuine two-loop contribution.

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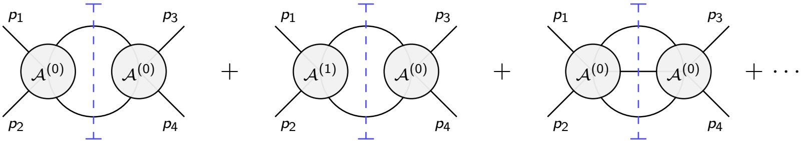
Cusp anomaly at strong coupling

$$f_{ABJM}\left(\tilde{\lambda}\right) = \sqrt{2\tilde{\lambda}} - \frac{5\log 2}{2\pi} - \frac{K}{4\pi^2\sqrt{2\tilde{\lambda}}} + \mathcal{O}\left(\tilde{\lambda}^{-1}\right)$$

Confirmed by two-loop computation [LB, M. Bianchi, A. Bres, V. Forini, E. Vescovi, 2014]

Standard unitarity in 4d [Bern, Dixon, Dunbar, Kosower, 1994]

$\mathcal{A}|_{\text{cut}} =$ 

Glue together the two amplitudes and uplift the integral with

$$i\pi\delta^{+}(p^2 - m^2) \to \frac{1}{p^2 - m^2 - i\epsilon}$$

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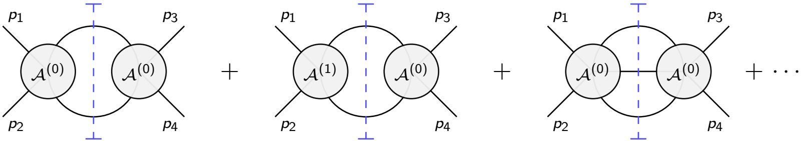

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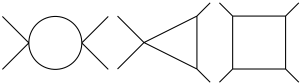
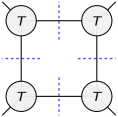
Generalized unitarity in 4d [Bern, Dixon, Kosower, 1998; Britto, Cachazo, Feng, 2004]

$$\mathcal{A}^L = \sum_i c_i \, \mathcal{I}_i^{(L)}$$ → Known basis of L-loop scalar integrals

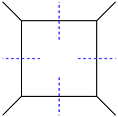
For L=1



$\Rightarrow$

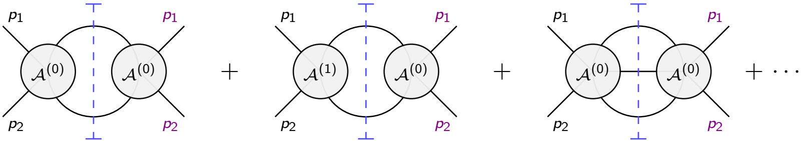


$= c_{\text{box}}$



Standard unitarity in 2d [LB, Forini, Hoare, 2013]

$\mathcal{A}|_{\text{cut}} =$



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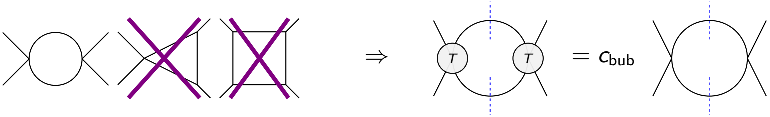
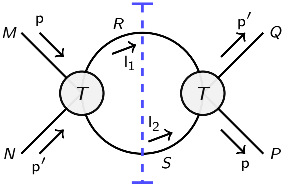


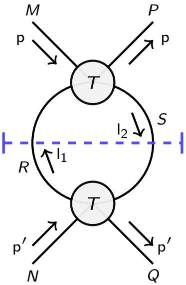
s-channel



$T^{RS}_{MN}(p,p')\,T^{PQ}_{RS}(p,p')$

$T \; \textcircled{s} \; T$

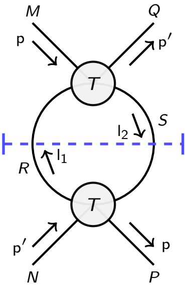
t-channel



$\frac{1}{2m^2}\, T^{SP}_{MR}(p,p)\, T^{RQ}_{SN}(p,p') + \frac{1}{2m'^2}\, T^{PS}_{MR}(p,p')\, T^{QR}_{SN}(p',p')$

$$\frac{1}{2m^2}\,\widetilde{T} \underset{\leftarrow}{\textcircled{t}} T + \frac{1}{2m'^2}\, T \underset{\rightarrow}{\textcircled{t}} \widetilde{T}$$

u-channel



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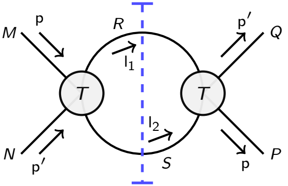
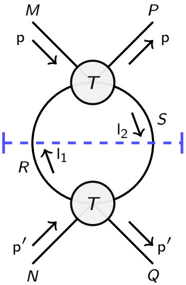
s-channel



t-channel



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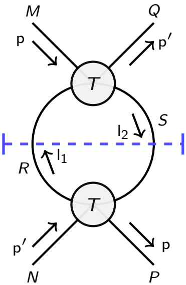

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$T \textcircled{u} T$

The result

$$T^{(1)} = \frac{\theta}{2\pi}\left(T \textcircled{u} T - T \circledS T\right) + \frac{i}{2}\,T \circledS T + \frac{1}{16\pi}\left(\frac{1}{m^2}\,\widetilde{T} \underset{\leftarrow}{\textcircled{t}} T + \frac{1}{m'^2}\,T \underset{\rightarrow}{\textcircled{t}} \widetilde{T}\right)$$

Solution to the sign problem

$$\mathcal{L}_4 = \frac{1}{z^2}\left(-(\eta^2)^2 + \left(i\,\eta_i(\rho^{MN})^i{}_j n^N \eta^j\right)^2\right) = \frac{1}{z^2}\left(-4\,(\eta^2)^2 + 2\left|\eta_i(\rho^N)^{ik} n_N \eta_k\right|^2\right)$$

$$\Sigma_i{}^j = \eta_i\eta^j \qquad \tilde{\Sigma}_j{}^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l$$

$$\Sigma_\pm{}_i^j = \Sigma_i^j \pm \tilde{\Sigma}_i^j$$

$$\mathcal{L}_4 = \frac{1}{z^2}\left(-4\,(\eta^2)^2 \mp 2(\eta^2)^2 \mp \Sigma_\pm{}_i^j \Sigma_\pm{}_j^i\right)$$

$$\mathcal{L}_4 \to \frac{12}{z}\eta^2\phi + 6\phi^2 + \frac{2}{z}\eta_j\phi_i^j\eta^i + \frac{2}{z}(\rho^N)^{ik} n_N \eta_k \phi_i^j (\rho^L)_{jl} n_L \eta^l + \phi_j^i\phi_i^j$$