Quark–antiquark Potential from the Quantum Spectral Curve

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based on
arXiv:1601.05679 [N. Gromov, FLM]
arXiv:1510.02098 [N. Gromov, FLM]

GATIS workshop, DESY Hamburg, 29 Feb – 1 Mar 2016

Introduction and motivation

Quantum Spectral Curve (QSC): a finite set of equations for exact spectrum of planar N=4 SYM / strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Gromov,Kazakov,Leurent,Volin 2013

Many results for local operators

Gromov, Kazakov, Leurent, Volin 13,14
Gromov, FLM, Sizov 13,14
Alfimov, Gromov, Kazakov 14
Marboe, Volin 14
Gromov, FLM, Sizov 15
Gromov, FLM, Sizov 15

Cavaglia, Fioravanti, Gromov, Tateo 14;
Gromov, Sizov 14;
Anselmetti, Bombardelli, Cavaglia, Tateo 15
Cavaglia, Cornagliotto, Mattelliano, Tateo 15

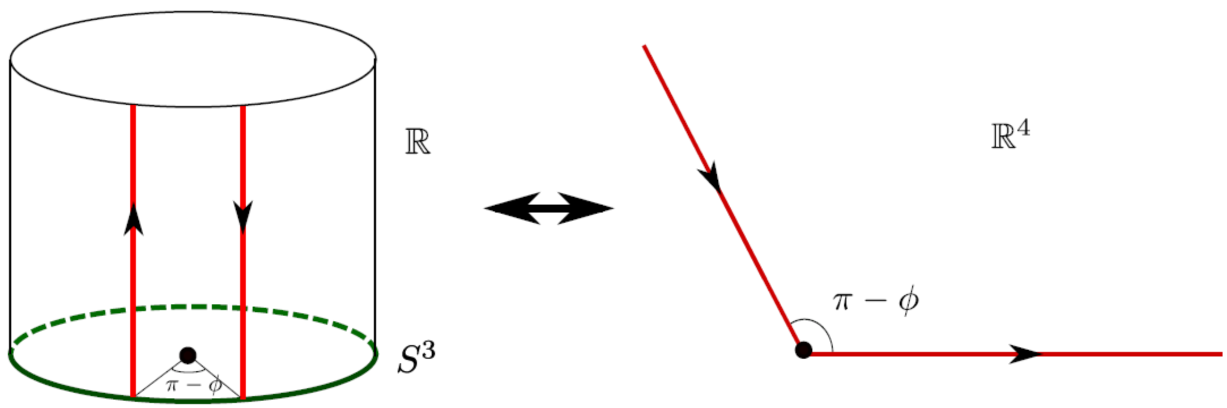
Can we study more general observables?

The quark-antiquark potential

- One of the earliest computations in AdS/CFT
- Many parameters $\Longrightarrow$ interesting scaling limits
- Qualitatively new features in the QSC

Cusped Wilson line in N=4 SYM

$$W = \mathrm{Tr}\, \mathcal{P} \exp \int dt \left[iA \cdot \dot{x} + \vec{\Phi} \cdot \vec{n}\, |\dot{x}| \right]$$



generalized
$q\bar{q}$ potential

$$\langle W \rangle \sim \left(\frac{\Lambda_{IR}}{\Lambda_{UV}} \right)^{\Delta}$$

Parameters:

- Cusp angle ϕ (when $\phi \to \pi$ we get the flat space potential)
- Angle θ between the couplings to scalars on two rays
- ‘t Hooft coupling λ

QSC for the generalized cusp

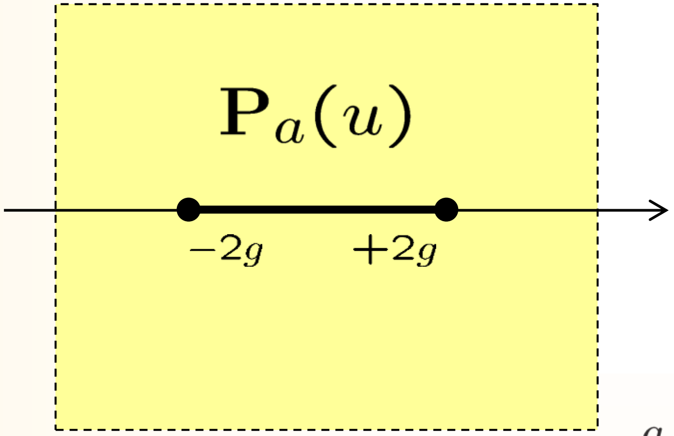
TBA $\longrightarrow$ QSC

Correa,Maldacena,Sever 2012
Drukker 2012

Gromov,FLM 2015

4+4 Q-functions $\mathbf{P}_a,\ \mathbf{Q}_i$

$$\mathbf{P}_a \sim e^{\pm\theta u} \underbrace{A_{a0}\left(1 + A_{a1}/u + A_{a2}/u^2 + A_{a3}/u^3\right.}_{\text{encode } \Delta \text{ and angles}} \left. + \ldots\right)\,, \quad u \to \infty$$



$$g = \frac{\sqrt{\lambda}}{4\pi}$$

$$\mathbf{P}_1 \sim u^{-1/2} e^{+\theta u}$$
$$\mathbf{P}_2 \sim u^{-1/2} e^{-\theta u}$$
$$\ldots$$
$$\mathbf{Q}_1 \sim u^{1/2+\Delta} e^{+\phi u}$$
$$\mathbf{Q}_2 \sim u^{1/2+\Delta} e^{-\phi u}$$
$$\ldots$$

Confirmed by many tests Gromov,FLM 2015

Similar asymptotics were found for deformed N=4 SYM Kazakov, Leurent, Volin 2015

Closing the equations

$\mathbf{P}_a \Longrightarrow \mathbf{Q}_i$ via QQ-relations

E.g. use the 4th order "Baxter" equation Alfimov, Gromov, Kazakov 2014

$$D_0(u)\mathbf{Q}(u+2i) + D_1(u)\mathbf{Q}(u+i) + \ldots + D_4(u)\mathbf{Q}(u-2i) = 0$$

where $D_0 = \det \begin{pmatrix} \mathbf{P}_1^{[+2]} & \mathbf{P}_2^{[+2]} & \mathbf{P}_3^{[+2]} & \mathbf{P}_4^{[+2]} \\ \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3 & \mathbf{P}_4 \\ \mathbf{P}_1^{[-2]} & \mathbf{P}_2^{[-2]} & \mathbf{P}_3^{[-2]} & \mathbf{P}_4^{[-2]} \\ \mathbf{P}_1^{[-4]} & \mathbf{P}_2^{[-4]} & \mathbf{P}_3^{[-4]} & \mathbf{P}_4^{[-4]} \end{pmatrix}, \ \ldots$

To close the system we simply impose Gromov,FLM 2015

$$\tilde{\mathbf{Q}}_1(u) = \mathbf{Q}_1(-u) \quad !$$

Don't need auxuliary functions like $\mu_{ab}, \omega_{ij}, \ldots$

QSC for the quark-antiquark potential

Gromov,FLM 2016

$$\Delta = -\frac{\Omega(\lambda,\theta)}{\pi-\phi}, \quad \phi \to \pi$$

$\mathbf{Q}_1 \sim u^{1/2+\Delta} e^{+\phi u}$ $\Longrightarrow$ expect drastic change in asymptotics

However, $\mathbf{P}_a$ remain finite and encode Ω

We get asymptotics of Q's from the 4th order Baxter equation

$$\mathbf{Q}_1 \sim u^{3/4} e^{\pi u - \sqrt{8\Omega u}} \left(1 + \sum_{n=1}^{\infty} \frac{d_n}{(\sqrt{\Omega u})^n}\right), \quad u \to \infty$$

$$\mathbf{Q}_2 \sim u^{3/4} e^{-\pi u + i\sqrt{8\Omega u}} \left(1 + \sum_{n=1}^{\infty} \frac{d_n}{(-i\sqrt{\Omega u})^n}\right) \quad \text{etc}$$

Asymptotics of a novel type!

Thus we have formulated the QSC directly at $\phi = \pi$

Weak coupling

Weak coupling

$$\Delta = -\frac{\Omega(\lambda)}{\pi - \phi}$$

Cannot get the potential from perturbative Δ
because the limits $\phi \to \pi$ and $\lambda \to 0$ do not commute

Direct N=4 SYM calculation requires effective theory, highly nontrivial

Find qualitatively new features in QSC
compared to all previous weak coupling solutions

Weak coupling from QSC

We parameterize $\mathbf{P}_a$ as e.g.

$$\mathbf{P}_1(u) = C u^{1/2} e^{\theta u} \mathbf{f}(u) \qquad \mathbf{f}(u) = \frac{1}{gx} + \sum_{n=1}^{\infty} \frac{g^{n-1} A_n}{x^{n+1}}$$

Solve the 4th order equation on $\mathbf{Q}_i$ iterativelyin g
via the universal method of Gromov,FLM,Sizov 2015 (see also Marboe,Volin 2014)

Four solutions:

$$\{Q_I, Q_{II}, Q_{III}, Q_{IV}\} = e^{\pm \pi u}\sqrt{u}\ \{1, u, u^2, 4\psi(-iu)u\cos^2\tfrac{\theta}{2} - \tfrac{1}{u}\} + O(g^2)$$

But need their linear combination which gives $\mathbf{Q}_1$
to impose $\tilde{\mathbf{Q}}_1(u) = \mathbf{Q}_1(-u)$ and fix the energy

Scales in the QSC

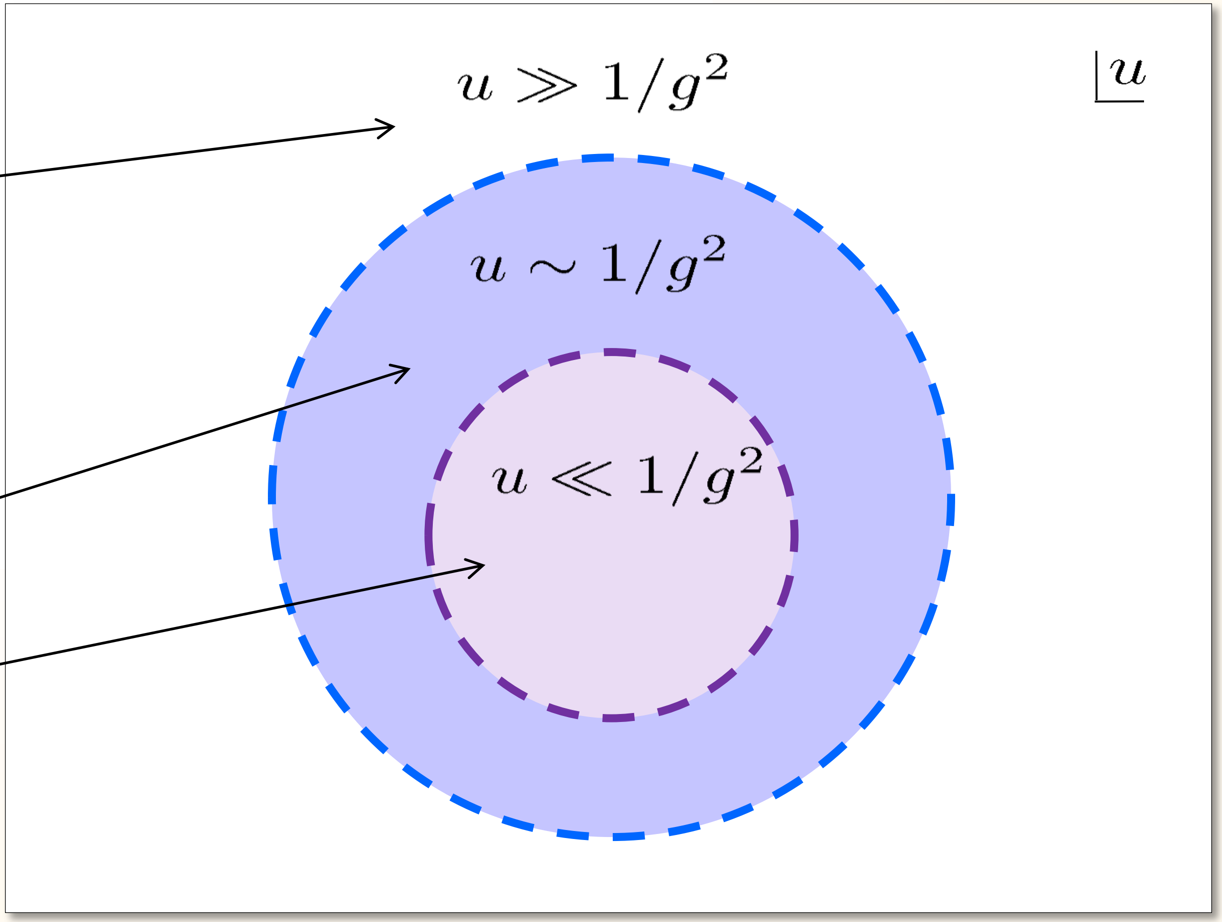
At large u $\quad \mathbf{Q}_1 \sim u^{3/4} e^{\pi u - \sqrt{8\Omega u}} \left(1 + \sum_{n=1}^{\infty} \frac{d_n}{(\sqrt{\Omega u})^n}\right)$

What happens at weak coupling when $\Omega \sim g^2 \to 0$??

scale 1:
large u asymptotics

scale 2:
Ωu is fixed, $g \to 0$

scale 3:
perturbative regime



The intermediate scale

With $v \equiv \Omega u$ fixed, the 4th order equation on $\mathbf{Q}_i$ becomes

$$f^{(4)} + \frac{2f^{(3)}}{v} - \frac{f}{16v^2} + 8\hat{g}^2\frac{f''}{v^2} + \mathcal{O}\left(g^4\right) = 0$$

$\hat{g} \equiv g\cos\frac{\theta}{2}$

$\mathbf{Q} = \sqrt{u}e^{\pm\pi u}f(\Omega u),$

Solution matching the asymptotics of $\mathbf{Q}_1$:

$$f(v) = \sqrt{v}K_1(\sqrt{v}) \simeq \sqrt{\frac{\pi}{2}}\ {}^{1/4}\!\!\sqrt[4]{8\Omega u}\ \ e^{-\sqrt{8\Omega u}}, \quad v \to \infty$$

small v expansion of $f(v)$ $\Longleftrightarrow$ large u expansion of $\mathbf{Q}_1$ in the perturbative regime

This fixes the correct linear combination of the four perturbative solutions $Q_I, Q_{II}, Q_{III}, Q_{IV}$

Weak coupling: results

Gromov,FLM 2016

We have computed the first 7 orders of the expansion

Pefect match with known results!

(first 3 orders and partial data at higher loops)

Ericksson, Semenoff, Szabo, Zarembo 2000; Pineda 2007;
Drukker, Forini 2011; Stahlhofen 2012;
Correa, Henn, Maldacen, Sever 2012;
Bykov, Zarembo 2012; Prausa, Steinhauser 2013;

+

new simple formula for subleading logs to all orders

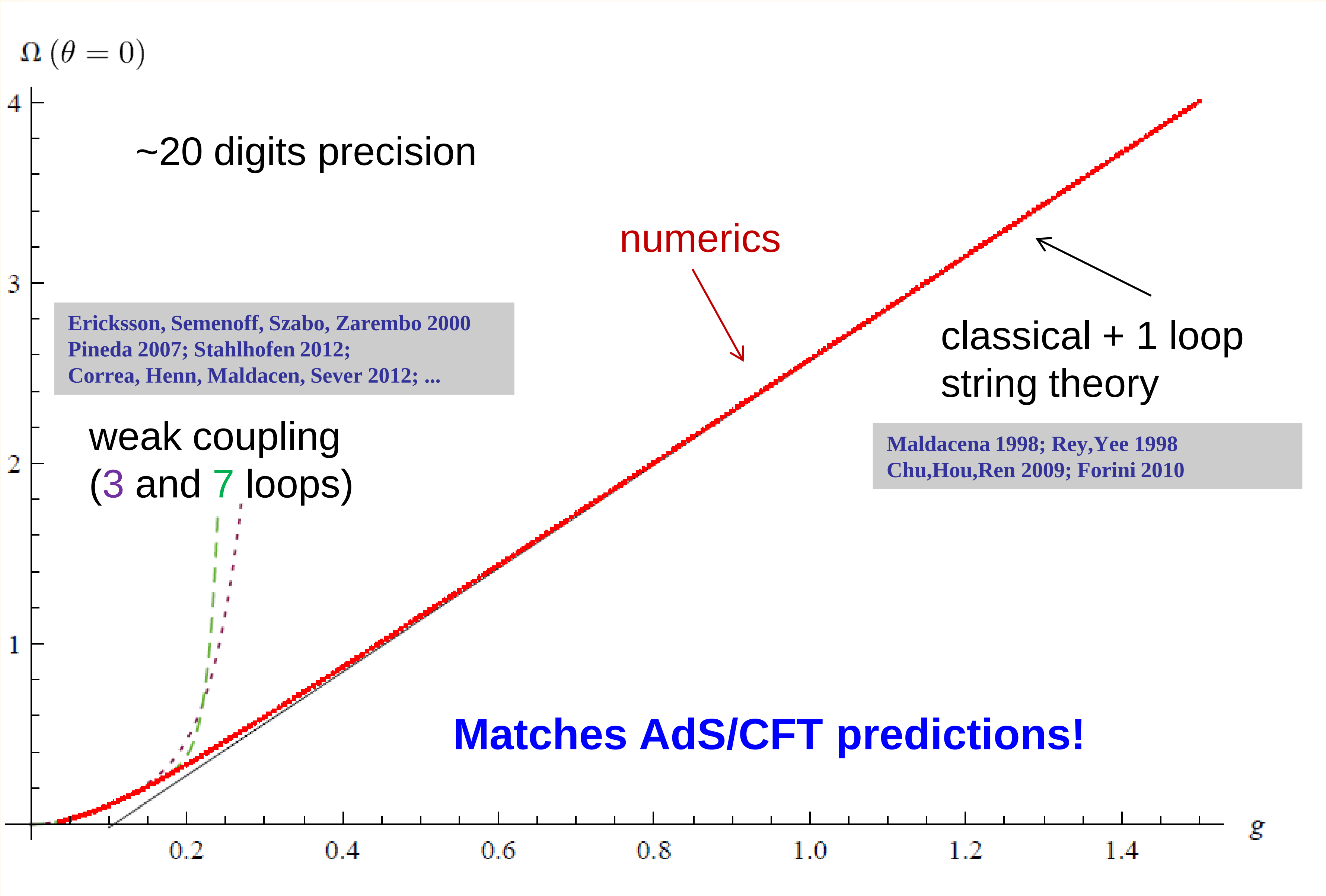
$$
\begin{aligned}
\frac{\Omega}{4\pi} = \hat{g}^2 &+ \\
&\hat{g}^4\left[16L-8\right]+ \\
&\hat{g}^6\left[128L^2+L\left(64+\frac{64\pi^2T}{3}\right)-112-\frac{8\pi^2}{3}+72T\zeta_3\right]+ \\
&\hat{g}^8\Big[\frac{2048L^3}{3}+\frac{1024}{3}\pi^2L^2T+2048L^2+LT\left(768\zeta_3+\frac{2176\pi^2}{3}\right)+\left(-768-\frac{640\pi^2}{3}\right)L \\
&+T^2\left(128\pi^2\zeta_3-760\zeta_5\right)+T\left(384\zeta_3-640\pi^2+\frac{32\pi^4}{9}\right)+\frac{1664\zeta_3}{3}+\frac{1216\pi^2}{9}-1280\Big]+ \\
&\hat{g}^{10}\Big[\frac{8192L^4}{3}+\frac{8192}{3}\pi^2L^3T+\frac{57344L^3}{3}+\frac{2048}{9}\pi^4L^2T^2+L^2T\left(3072\zeta_3+\frac{71680\pi^2}{3}\right) \\
&+\left(20480-\frac{19456\pi^2}{3}\right)L^2+LT^2\left(\frac{8704\pi^2\zeta_3}{3}-6400\zeta_5+\frac{2560\pi^4}{3}\right) \\
&+LT\left(12800\zeta_3-\frac{46592\pi^2}{3}-\frac{6656\pi^4}{45}\right)+L\left(\frac{26624\zeta_3}{3}-26624+\frac{38912\pi^2}{9}\right) \\
&+T^3\left(\frac{1792\pi^4\zeta_3}{45}-\frac{4928\pi^2\zeta_5}{3}+8624\zeta_7\right) \\
&+T^2\left(3392\pi^2\zeta_3+1248\zeta_3^2-4000\zeta_5-\frac{1024\pi^4}{3}-\frac{16\pi^6}{45}\right) \\
&+T\left(896\zeta_3+\frac{3392\pi^2\zeta_3}{3}+1600\zeta_5-\frac{10112\pi^2}{3}+\frac{1408\pi^4}{45}\right) \\
&+6656\zeta_3+\frac{736\pi^4}{45}+\frac{5824\pi^2}{27}-\frac{37888}{3}\Big]\ +\ldots
\end{aligned}
$$

$$
\hat{g}\equiv g\cos\frac{\theta}{2}\ ,\quad T\equiv\frac{1}{\cos^2\frac{\theta}{2}}\ ,\quad L\equiv\log\sqrt{8e^{\gamma}\pi\hat{g}^2}
$$

Numerical solution

Numerical solution

We adapted the efficient algorithm of Gromov,FLM,Sizov 2015



Numerical solution

At strong coupling we get

$$\Omega = 2.8710800436g - 0.3049193819 + \frac{0.0100740}{g} + \frac{0.000381}{g^2} + \dots$$

reproducing the celebrated string theory result

$$\Omega \simeq \frac{\pi(4\pi g + a_1)}{4K\left(\frac{1}{2}\right)^2} = 2.8710800442g - 0.3049193809 \ .$$

Maldacena 1998
Rey,Yee 1998
Chu,Hou,Ren 2009
Forini 2010

At weak coupling we confirm known data and our 7-loop analytic prediction

Ladders limit

Ladders limit

Double scaling limit $\theta \to i\infty, \ g \to 0, \quad \frac{g}{e^{i\theta/2}} = \text{fixed}$

Bethe-Salpeter techniques reduce sum of ladder diagrams to a Schrodinger problem for the ground state

$$-F''(z) - \frac{4\hat{g}^2}{z^2+1} F(z) = -\frac{\Omega^2}{4} F(z)$$

$$\hat{g} \equiv g \cos \frac{\theta}{2} = \text{fixed}$$

Ericksson, Semenoff, Szabo, Zarembo 2000
Correa, Henn, Maldacen, Sever 2012

Captures all orders in $\hat{g}$ including all finite-size effects

Can we get it from the QSC ?

Ladders limit in the QSC

Great simplification as 4th order Baxter equation on $\mathbf{Q}_i$ factorizes

$$-2(2\hat{g}^2 - \Omega u + u^2)q_1(u) + u^2 q_1(u-i) + u^2 q_1(u+i) = 0$$

$$q_1(u) \equiv \mathbf{Q}_1(u) e^{\pm \pi u}/\sqrt{u}$$

(+ another equation with $\Omega \to -\Omega$)

The Schrodinger equation

$$-F''(z) - \frac{4\hat{g}^2}{z^2+1} F(z) = -\frac{\Omega^2}{4} F(z)$$

maps to this Baxter equation after a Mellin-type transform!

$$\frac{q_1(u)}{u} = 2 \int_i^{+\infty} \frac{e^{-\frac{\Omega z}{2}}}{z^2+1} \left(\frac{z+i}{z-i}\right)^{iu} F(z) dz$$

(similar to ODE/IM?)

QSC and Schrodinger problem

$$-2(2\hat{g}^2 - \Omega u + u^2)q_1(u) + u^2 q_1(u-i) + u^2 q_1(u+i) = 0$$

In the QSC we fix Ω from $q_1(u) \simeq \sqrt{\pi/2}\ {}^{1/4}\!\!\sqrt{8\Omega u}\ e^{-\sqrt{8\Omega u}}$ and

$$\frac{\Omega^2(\hat{g})}{8\pi\hat{g}^4} = \lim_{u\to 0} \frac{\bar{q}_1(u)q_1(u)e^{+2\pi u} - \bar{q}_1(0)q_1(0)}{u}$$

which is a nontrivial consequence of $\tilde{\mathbf{Q}}_1(u) = \mathbf{Q}_1(-u)$

This condition in fact follows from normalizability of $F(z)$

To derive it we use a peculiar self-duality property

$$F(k) = \frac{2\hat{g}}{\sqrt{\pi\Omega}} \int\limits_{-\infty}^{\infty} dz \frac{F(z)}{z^2+1} e^{ik\frac{\Omega}{2}z}$$

Conclusions

- QSC formulation allows to explore many regimes:

  weak coupling to 7 loops, numerics, ladders, …

Future

- Similar double scaling limit in gamma-deformed SYM **Gurdogan, Kazakov 2016**
- Guidance for 3pt functions with wrapping from ladders limit?
- Relations with QCD? **Grozin,Henn, Korchemsky,Marquard 2015**
- Other boundary problems and deformations