Integrability and Exact results in $\mathcal{N} = 2$ gauge theories

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DESY Theory

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arXiv:1310.5709
arXiv:1406.3629 with Vladimir Mitev
arXiv:1511.02217 with Vladimir Mitev
work in progress

Motivation: The success story for $\mathcal{N} = 4$ SYM

Possible to compute observables in the **strong coupling regime** and in some cases to even obtain **Exact results** (for any value of the coupling).

- **AdS/CFT** (gravity/sigma model description)
- **Integrability** (The spectral problem is solved) *at large N_c*
- **Localization** (Exact results: e.x. Circular WL) for *any N_c*

Which of these properties/techniques are transferable to **more realistic gauge theories** in 4D with less SUSY?

Localization

Localization works for $\mathcal{N}=2$ theories (Pestun)

The **path integral** localizes to (a Matrix model) an **ordinary integral**!

$$Z_{S^4} = \int [D\Phi]\, e^{-S[\Phi]} = \int da\, |\mathcal{Z}(a)|^2$$

An example of exact observable:

$$W(\lambda) = 2\,\frac{I_1(\sqrt{\lambda})}{\sqrt{\lambda}} = \begin{cases} 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \cdots & \lambda << 1 \\ \sqrt{\frac{2}{\pi}}\lambda^{-\frac{3}{4}} e^{\sqrt{\lambda}} + \cdots & \lambda >> 1 \end{cases}$$

the $\mathcal{N}=4$ SYM **circular Wilson loop** in the planar limit.

- Even if the observable cannot be written in a **closed form**, one can always **expand** both from the weak and from the strong coupling.

Integrability (only in the planar limit)

- **Perturbation theory**: integrable spin chain (Minahan, Zarembo, . . .).
- **Gravity side**: integrable 2D sigma model (Bena, Polchinski, Roiban,. . .)
- The spectral problem is solved $\forall\lambda$ (Gromov, Kazakov, Vieira,. . .).
- Now other observables: scattering amplitudes, correlation functions. . .

Integrability: 2-body problem $\rightarrow$ n-body problem

Exact result $(\forall\lambda)$ is due to **symmetry**:

- The dispersion relation $\Delta - |r| = \sqrt{1 + h(g)\sin^2\left(\frac{p}{2}\right)}$
  and the 2-body **S- matrix**

are fixed due to the $SU(2|2) \subset PSU(2,2|4)$ symmetry (Beisert).

- For $\mathcal{N} = 2$ theories we also have it: $SU(2|2) \subset SU(2,2|2)$!

AdS/CFT

AdS duals only for a sparse set of 4D theories:

- $D3$ branes in **critical string theory**. (e.g. orbifolds)
- Adjoint and bifundamental matter. (Flavors in the **probe** approx.)

It has been argued that:

- $\mathcal{N} = 1$ SQCD in the Seiberg **conformal** window is dual to $6d$ non-critical backgrounds of the form $AdS_5 \times S^1$.
  (Klebanov-Maldacena, Fotopoulos-Niarchos-Prezas, Murthy-Troost,...)
- $\mathcal{N} = 2$ SCQCD is dual to $8d$ *non-critical* string theory in a background with an $AdS_5 \times S^1$ factor (Gadde-EP-Rastelli)

Checked at the level of the **chiral spectrum**.
For non-protected quantities there is nothing to compare with!

Plan of attack

Discover the string from the "**bottom up**".

$$AdS_5 \times S^1 \times \mathcal{M}$$

- Probe the $AdS_5 \times S^1$ factor of the geometry: **purely gluonic sector**
- Probe the compact $S^1 \times \mathcal{M}$ factor: **sectors with quarks**

$$f_1(g^2) = \frac{R^4_{AdS}}{(2\pi\alpha')^2}\,, \quad f_2(g^2) = \frac{R^4_{S^1}}{(2\pi\alpha')^2}\,, \quad f_3(g^2) = \frac{R^4_{\mathcal{M}}}{(2\pi\alpha')^2}$$

using:

- Perturbation theory
- The spin chain description (Symmetry and Integrability)
- Localization

The main statement

The main statement

1. Every $\mathcal{N}=2$ superconformal gauge theory has a **purely gluonic** $SU(2,1|2)$ sector **integrable in the planar limit**

$$H_{\mathcal{N}=2}(g) = H_{\mathcal{N}=4}(\mathbf{g})$$

2. The **Exact Effective coupling** (relative **finite renormalization** of g)

$$\mathbf{g}^2 = f(g^2) = g^2 + g^2\left(Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4}\right)$$

we compute using localization

$$W_{\mathcal{N}=2}\left(g^2\right) = W_{\mathcal{N}=4}\left(\mathbf{g}^2\right)$$

3. AdS/CFT: **effective string tension** $f(g^2) = T^2_{eff} = \left(\frac{R^4}{(2\pi\alpha')^2}\right)_{eff}$

Obtain any observable classically in the factor $AdS_5 \times S^1$ of the geometry by replacing $g^2 \to f(g^2)$.

Outline of the rest of the talk

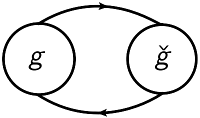
1. Review
2. Integrability of the purely gluonic $SU(2,1|2)$ Sector
3. Localization and the Exact Effective couplings
4. Conclusions and outlook

Review

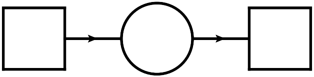
$\mathcal{N} = 2$ SuperConformal QCD (SCQCD)

- ADE classification $\mathcal{N} = 2$ SCFT: finite/affine Dynkin diagrams

One parameter family $\mathcal{N} = 2$ SCFT: product gauge group $SU(N) \times SU(N)$ and **two exactly marginal couplings** g and $\check{g}$ (Gadde-EP-Rastelli)



- For $\check{g} \to 0$ obtain $\mathcal{N} = 2$ SCQCD with $N_f = 2N$



- For $\check{g} = g$ one finds the well-known $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM, with an $AdS_5 \times S^5/\mathbb{Z}_2$ gravity dual (Kachru-Silverstein, Lawrence-Nekrasov-Vafa,. . .)

$\mathcal{N} = 2$ SuperConformal QCD (SCQCD)

$U(1)_r \times SU(2)_R$

$\mathcal{N} = 2$ vector multiplet **adjoint** in $SU(N)$:

$$\begin{array}{ccc} & A_\mu & \\ \lambda^1_\alpha & & \lambda^2_\alpha \\ & \phi & \end{array} \;, \quad \lambda^{\mathcal{I}} = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \,, \quad \mathcal{I} = 1,2$$

$\mathcal{N} = 2$ hypermultiplet **fundamental** in $SU(N)$ and $U(N_f)$:

$$\begin{array}{ccc} & \psi_{\alpha\, i} & \\ q_i & & (\tilde{q})^*_i \\ & \left(\tilde{\psi}_\alpha\right)^\dagger_i & \end{array} \;, \quad Q^{\mathcal{I}} = \begin{pmatrix} q \\ \tilde{q}^* \end{pmatrix} \,, \quad i = 1, \ldots N_f$$

$\boxed{\beta = \frac{g^3_{YM}}{16\pi^2} (N_f - 2N)}$ when $N_f = 2N$ **exactly marginal coupling**!

It should have an AdS dual description with $\lambda \equiv g^2_{YM} N \longrightarrow \left(\frac{R_{AdS}}{\ell_{st}}\right)^4$.

Veneziano expansion

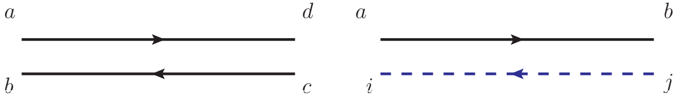
The theory admits a *Veneziano* expansion:

$$N \to \infty \quad \text{and} \quad N_f \to \infty$$

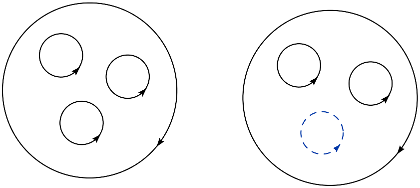
$$\text{with} \quad \frac{N_f}{N} \quad \text{and} \quad \lambda = g_{YM}^2 N \quad \text{kept fixed.}$$

(Veneziano 1976)

Generalized double line notation:



The two diagrams are of the same order $N \sim N_f$



Operators will mix:

$$\mathcal{O} \sim \mathrm{Tr}\left(\phi^{\ell}\bar{\phi}\right) + \mathrm{Tr}\left(\phi^{\ell-1}q_i\bar{q}^i\right)$$

Closed string states $\longrightarrow$ **"generalized single-trace"** operators

$$\mathrm{Tr}\left(\phi^{k_1}\mathcal{M}^{\ell_1}\phi^{k_2}\dots\phi^{k_n}\mathcal{M}^{\ell_n}\right), \quad \mathcal{M}^a{}_b \equiv \sum_{i=1}^{N_f} q^a{}_i\,\bar{q}^i{}_b\,, \quad a, b = 1, \dots, N$$

One parameter family of $\mathcal{N}=2$ SCFT $SU(N)\times SU(N)$

$\mathcal{N}=2$ vector multiplet adjoint in $SU(N)$: $\left(\phi,\ \lambda^{\mathcal{I}}_{\alpha},\ \mathcal{F}_{\alpha\beta}\right)^{a}{}_{b}$

$\mathcal{N}=2$ vector multiplet adjoint in $SU(N)$: $\left(\check{\phi},\ \check{\lambda}^{\mathcal{I}}_{\alpha},\ \check{\mathcal{F}}_{\alpha\beta}\right)^{\check{a}}{}_{\check{b}}$

$\mathcal{N}=2$ hypermultiplet bifundamental in $SU(N)$ and $SU(N)$:

$$\left(Q^{\mathcal{I}},\ \psi_{\alpha},\ \bar{\tilde{\psi}}_{\alpha}\right)^{\hat{\mathcal{I}}\ a}{}_{\check{a}} \qquad \hat{\mathcal{I}}=1,2 \quad SU(2)_L$$

For $\check{g}\to 0$: $\mathcal{N}=2$ SCQCD *plus* a decoupled free vector multiplet.
symmetry enhancement: $SU(2)_L\times SU(N)\to U(N_f=2N)$, $(\hat{\mathcal{I}},\check{a})\to i$

$\mathcal{N} = 4$ SYM

$\forall$ site hosts a "letter" from the single **ultrashort singleton** multiplet:

$$V_F = \mathcal{D}^n \left(X,\ Y,\ Z,\ \bar{X},\ \bar{Y},\ \bar{Z},\ \lambda_\alpha^A,\ \bar{\lambda}_A^{\dot{\alpha}},\ \mathcal{F}_{\alpha\beta},\ \bar{\mathcal{F}}_{\dot{\alpha}\dot{\beta}} \right)$$

The **state space** $\forall$ lattice site is $\mathcal{V}_\ell = V_F$ and the total space is $\otimes_\ell^L \mathcal{V}_\ell$.

$\mathcal{N} = 2$ SCQCD

The **state space** at each lattice site is ∞-dim, spanned by

$$\mathcal{V}_\ell = \{\mathcal{V}\,,\quad \bar{\mathcal{V}}\,,\quad \mathcal{H}\,,\quad \bar{\mathcal{H}}\}$$

$$\mathcal{V} = \mathcal{D}^n \left(\phi\,,\, \lambda_\alpha^{\mathcal{I}}\,,\, \mathcal{F}_{\alpha\beta} \right)^a{}_b \quad , \quad \mathcal{H} = \mathcal{D}^n \left(Q^{\mathcal{I}}\,,\, \psi\,,\, \bar{\tilde{\psi}} \right)^a{}_i$$

The color index structure imposes **restrictions** on the total space $\otimes_\ell^L \mathcal{V}_\ell$:

$$\cdots\ \phi\,\phi\,Q\,\check{\phi}\,\check{\phi}\ \cdots\ \check{\phi}\,\check{\phi}\,\bar{Q}\,\phi\,\phi\ \cdots$$

Elementary excitations come from the vector multiplet

$\mathcal{N} = 4$ SYM

- Choice of **vacuum** $\mathrm{tr} Z^\ell$ with $\Delta - r = 0$ (= *magnon number*).
- $8+8$ **elementary excitations** with $\Delta - r = 1$:

$\lambda^A_\alpha,\, X, \bar{X}, Y, \bar{Y},\, \mathcal{D}_{\alpha\dot{\alpha}}$ with $A = 1, \dots, 4$ the $SU(4)$ index.

- $\Delta - r \geq 2$: $\bar{Z},\ \mathcal{F}_{\alpha\beta},\ \dots$ are **composite states**.

$\mathcal{N} = 2$ SCQCD

- Choice of **vacuum** $\mathrm{tr}\phi^\ell$ with $\Delta - r = 0$
- $4+4$ **elementary excitations** with $\Delta - r = 1$:

$\lambda^{\mathcal{I}}_\alpha$ and $\mathcal{D}_{\alpha\dot{\alpha}}$ with $\mathcal{I} = 1, 2$ the $SU(2)_R$ index.

- $\Delta - r \geq 2$: $\mathcal{M},\ \mathcal{F}_{\alpha\beta}\ \dots$ are **composite states**.

“Regularizing” $\mathcal{N} = 2$ SCQCD by gauging the flavor

- Consider the $SU(N) \times SU(N)$ $\mathcal{N} = 2$ SCFT (SCQCD $\check{g} \to 0$)

  $\check{g}/g$ should be thought of as a **regulator**.

- We **regularize by inserting** $\check{\phi}$**s between the** Q**s** giving the dimeric impurities the possibility to split:

$$\cdots \phi\, \phi\, Q\, \check{\phi}\, \check{\phi} \cdots \check{\phi}\, \check{\phi}\, \bar{Q}\, \phi\, \phi \cdots$$

- Now the Qs can move independently

$$\Delta - r = 1$$

  and can be interpreted as **elementary excitations**!

- Back to $8 + 8$ **elementary excitations** with $\Delta - r = 1$:

  $\boxed{Q^{\mathcal{I}\hat{\mathcal{I}}}, \quad \lambda^{\mathcal{I}}_{\alpha} \text{ and } \mathcal{D}_{\alpha\dot{\alpha}}}$ with $\mathcal{I}$ the $SU(2)_R$ and $\hat{\mathcal{I}}$ the $SU(2)_L$ index.

Beisert's all loop Scattering Matrix

$\mathcal{N} = 4$ SYM

	$SU(2)_{\dot\alpha}$	$SU(2)_R$	$SU(2)_\alpha$	$SU(2)_L$
$SU(2)_{\dot\alpha}$	$\mathcal{L}^{\dot\alpha}{}_{\dot\beta}$	$\mathcal{Q}^{\dot\alpha}{}_{\mathcal{J}}$	$\mathcal{P}^{\dot\alpha}{}_{\beta}$	$\bar{\mathcal{Q}}^{\dot\alpha}{}_{\hat{\mathcal{J}}}$
$SU(2)_R$	$\mathcal{S}^{\mathcal{I}}{}_{\dot\beta}$	$\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}$	$\mathcal{Q}^{\mathcal{I}}{}_{\beta}$	$\mathcal{R}^{\mathcal{I}}{}_{\hat{\mathcal{J}}}$
$SU(2)_\alpha$	$\mathcal{P}^{\alpha}{}_{\dot\beta}$	$\mathcal{Q}^{\alpha}{}_{\mathcal{J}}$	$\mathcal{L}^{\alpha}{}_{\beta}$	$\mathcal{Q}^{\alpha}{}_{\hat{\mathcal{J}}}$
$SU(2)_L$	$\bar{\mathcal{Q}}^{\hat{\mathcal{I}}}{}_{\dot\beta}$	$\mathcal{R}^{\hat{\mathcal{I}}}{}_{\mathcal{J}}$	$\mathcal{S}^{\hat{\mathcal{I}}}{}_{\beta}$	$\mathcal{R}^{\hat{\mathcal{I}}}{}_{\hat{\mathcal{J}}}$

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$SU(2)_R$	$\mathcal{S}^{\mathcal{I}}_{\dot\beta}$	$\mathcal{R}^{\mathcal{I}}_{\mathcal{J}}$	$\lambda^{\dagger\mathcal{I}}_{\beta}$	$\mathcal{X}^{\dagger\mathcal{I}}_{\hat{\mathcal{J}}}$
$SU(2)_\alpha$	$\mathcal{D}^{\alpha}_{\dot\beta}$	$\lambda^{\alpha}_{\mathcal{J}}$	$\mathcal{L}^{\alpha}_{\beta}$	$\mathcal{Q}^{\alpha}_{\hat{\mathcal{J}}}$
$SU(2)_L$	$\lambda^{\hat{\mathcal{I}}}_{\dot\beta}$	$\mathcal{X}^{\hat{\mathcal{I}}}_{\mathcal{J}}$	$\mathcal{S}^{\hat{\mathcal{I}}}_{\beta}$	$\mathcal{R}^{\hat{\mathcal{I}}}_{\hat{\mathcal{J}}}$

- The **broken generators** (Goldstone excitations) and correspond to **gapless magnons**.

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- The **broken generators** (Goldstone excitations) and correspond to **gapless magnons**.
- These magnons transform in the fundamental of $SU(2|2)$

$$\Delta - |r| = 2\,\mathcal{C} = \sqrt{1 + h(g)\sin^2\left(\frac{p}{2}\right)}$$

- The two-body S-matrix is fixed by Beisert's centrally extended $SU(2|2) \times SU(2|2)$ symmetry. (Beisert)

$\mathcal{N} = 2$ all loop Scattering Matrix

	$SU(2)_{\dot\alpha}$	$SU(2)_R$	$SU(2)_\alpha$	$SU(2)_L$
$SU(2)_{\dot\alpha}$	$\mathcal{L}^{\dot\alpha}_{\dot\beta}$	$\mathcal{Q}^{\dot\alpha}_{\mathcal{J}}$	$\mathcal{P}^{\dot\alpha}_{\beta}$	
$SU(2)_R$	$\mathcal{S}^{\mathcal{I}}_{\dot\beta}$	$\mathcal{R}^{\mathcal{I}}_{\mathcal{J}}$	$\mathcal{Q}^{\mathcal{I}}_{\beta}$	
$SU(2)_\alpha$	$\mathcal{P}^{\alpha}_{\dot\beta}$	$\mathcal{Q}^{\alpha}_{\mathcal{J}}$	$\mathcal{L}^{\alpha}_{\beta}$	
$SU(2)_L$				$\mathcal{R}^{\hat{\mathcal{I}}}_{\hat{\mathcal{J}}}$

$\mathcal{N} = 2$ all loop Scattering Matrix

Choice of vacuum $\mathrm{tr}\phi^\ell$:

	$SU(2)_{\dot\alpha}$	$SU(2)_R$	$SU(2)_\alpha$	$SU(2)_L$
$SU(2)_{\dot\alpha}$	$\mathcal{L}^{\dot\alpha}_{\dot\beta}$	$\mathcal{Q}^{\dot\alpha}_{\mathcal{J}}$	$\mathcal{D}^{\dagger\dot\alpha}_{\beta}$	$\psi^{\dagger\dot\alpha}_{\hat{\mathcal{J}}}$
$SU(2)_R$	$\mathcal{S}^{\mathcal{I}}_{\dot\beta}$	$\mathcal{R}^{\mathcal{I}}_{\mathcal{J}}$	$\lambda^{\dagger\mathcal{I}}_{\beta}$	$\bar{Q}^{\mathcal{I}}_{\hat{\mathcal{J}}}$
$SU(2)_\alpha$	$\mathcal{D}^{\alpha}_{\dot\beta}$	$\lambda^{\alpha}_{\mathcal{J}}$	$\mathcal{L}^{\alpha}_{\beta}$	
$SU(2)_L$	$\psi^{\hat{\mathcal{I}}}_{\dot\beta}$	$Q^{\hat{\mathcal{I}}}_{\mathcal{J}}$		$\mathcal{R}^{\hat{\mathcal{I}}}_{\hat{\mathcal{J}}}$

The broken generators → Goldstone excitations → **Gapless magnons**
Non-existing generators→non-Goldstone excitations→**Gapped magnons**

$\mathcal{N} = 2$ all loop Scattering Matrix

Choice of vacuum $\text{tr}\phi^{\ell}$:

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The broken generators → Goldstone excitations → **Gapless magnons**
Non-existing generators→non-Goldstone excitations→**Gapped magnons**

$$2\,\mathcal{C}_{\lambda,\mathcal{D}} = \sqrt{1 + 8\mathbf{g}^2 \sin^2\left(\tfrac{p}{2}\right)} \qquad 2\,\mathcal{C}_{Q,\psi} = \sqrt{1 + 2(\mathbf{g} - \check{\mathbf{g}})^2 + 8\mathbf{g}\check{\mathbf{g}} \sin^2\left(\tfrac{p}{2}\right)}$$

$$\mathbf{g}^2 = f(g,\check{g}) = g^2 + \cdots \qquad (\mathbf{g} - \check{\mathbf{g}})^2 = f_1(g,\check{g}) = (g - \check{g})^2 + \cdots$$

$$\check{\mathbf{g}}^2 = \check{f}(g,\check{g}) = \check{g}^2 + \cdots \qquad \mathbf{g}\check{\mathbf{g}} = f_2(g,\check{g}) = g\check{g} + \cdots$$

The S-matrix of highest weight states in $SU(2)_{\alpha}$ and $SU(2)_L$ is fixed by the centrally extended $SU(2|2)$. (Gadde, Rastelli)

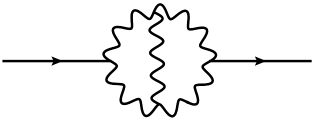
Integrability of the purely gluonic $SU(2,1|2)$ Sector

$$\phi\,,\;\lambda^{\mathcal{I}}_{+}\,,\;\mathcal{D}_{+\dot{\alpha}}$$

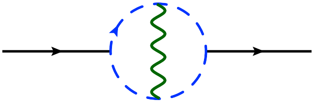
A diagrammatic observation

The **only possible way** to make diagrams with **external fields in the vector mult.** different from the $\mathcal{N} = 4$ ones is to make a loop with hyper's and then in this loop let a **checked vector** propagate!

(EP-Sieg)



The same with $\mathcal{N} = 4$ SYM



Different from $\mathcal{N} = 4$ SYM

but **finite !!**

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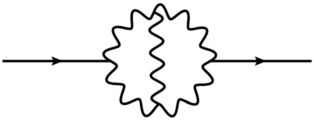

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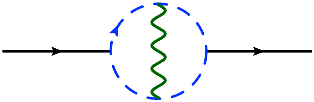
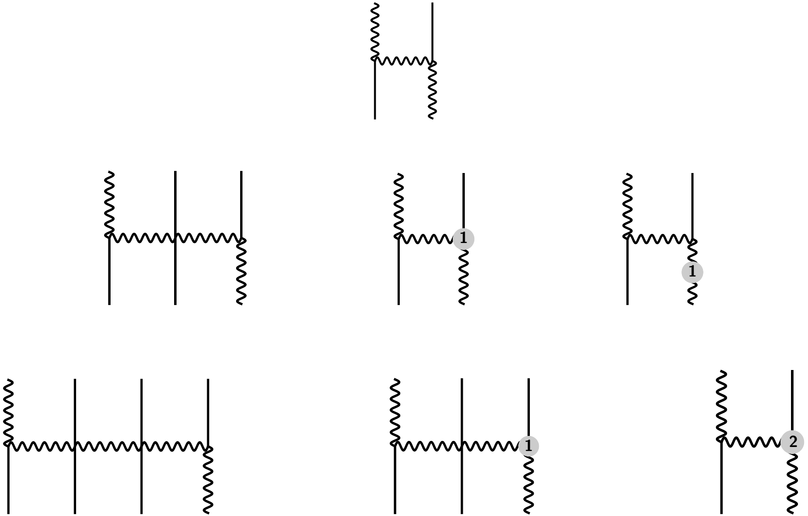


Different from $\mathcal{N} = 4$ SYM but **finite !!**

Novel Regularization prescription: (Arkani-Hamed-Murayama)

For every individual $\mathcal{N} = 2$ diagram subtract its $\mathcal{N} = 4$ counterpart.



$$H^{(3)}_{\mathcal{N}=2}(\lambda) - H^{(3)}_{\mathcal{N}=4}(\lambda) \sim H^{(1)}_{\mathcal{N}=4}(\lambda) \quad \Rightarrow \quad H^{(3)}_{\mathcal{N}=2}(\lambda) = H^{(3)}_{\mathcal{N}=4}(f(\lambda))$$

with $f(\lambda) = \lambda + c\lambda^3$

Operator renormalization in the Background Field Gauge

Background Field Method: $\varphi \to A + Q$

where A the classical background and Q the quantum fluctuation

$$g_{bare} = Z_g\, g_{ren}\,,\; A_{bare} = \sqrt{Z_A}\, A_{ren}\,,\; Q_{bare} = \sqrt{Z_Q}\, Q_{ren}\,,\; \xi_{bare} = Z_\xi\, \xi_{ren}$$

In the Background Field Gauge $\boxed{Z_g \sqrt{Z_A} = 1}$ and $\boxed{Z_Q = Z_\xi}$

Operator renormalization in the Background Field Gauge

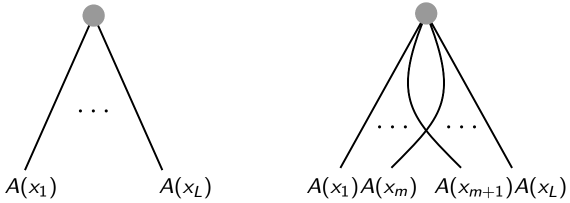
Background Field Method: $\varphi \to A + Q$

where A the classical background and Q the quantum fluctuation

$$g_{bare} = Z_g \, g_{ren} \, , \; A_{bare} = \sqrt{Z_A} \, A_{ren} \, , \; Q_{bare} = \sqrt{Z_Q} \, Q_{ren} \, , \; \xi_{bare} = Z_\xi \, \xi_{ren}$$

In the Background Field Gauge $\boxed{Z_g \sqrt{Z_A} = 1}$ and $\boxed{Z_Q = Z_\xi}$

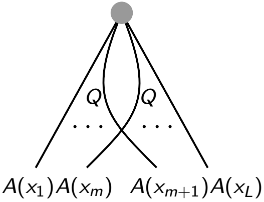
- Compute $\langle \mathcal{O}(y) A(x_1) \cdots A(x_L) \rangle$ for $\mathcal{O} \sim \mathrm{tr}\left(\varphi^L\right)$.



+ *more diagrams*

Wick contract $\mathcal{O}_i^{ren}\left(Q_{ren} \, , \; A_{ren}\right) = \sum_j Z_{ij} \mathcal{O}_j^{bare}\left(Z_Q^{1/2} Q \, , \; Z_A^{1/2} A\right)$

Background Field Method: No Q's outside, no A's inside!



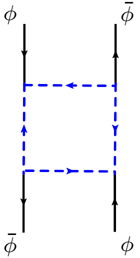
- $\langle QQAA \rangle$ renormalize as $Z_Q^{2/2} Z_A^{2/2} \langle QQAA \rangle$
- The Q propagators as Z_Q^{-1}
- the $\mathcal{O}^{ren}$ has two more $Z_Q^{1/2}$
- **all** Z_Q **will cancel** $\forall$ **individual diagram** (We knew it - **gauge invariance!**)
- Only $Z = Z_g^2 = Z_A^{-1}$, the combinatorics the same as in $\mathcal{N} = 4$:

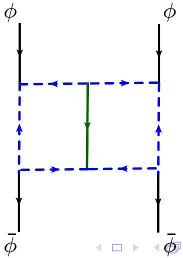
$$\mathrm{H}_{\mathcal{N}=2}(g) = H_{\mathcal{N}=4}(\mathbf{g}) \quad \text{with} \quad \mathbf{g}^2 = f(g^2, \check{g}^2) = g^2 + g^2 \left(Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4}\right)$$

New non-Holomorphic vertices cannot contribute

$$\Gamma = \Gamma_{ren.\ tree} + \Gamma_{new} = \int d^4\theta \mathcal{F}(\mathcal{W}) + c.c. + \int d^4\theta d^4\bar{\theta}\, \mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$$

- $\Gamma_{ren.\ tree}$: **vertex** and **self-energy renormalization** all encoded in $Z = Z_g^2 = Z_A^{-1}$
- Γ_{new} : New non-Holomorphic vertices cannot contribute due to the **non-renormalization theorem** (Fiamberti, Santambrogio, Sieg, Zanon)





Localization
and
Exact Effective couplings

Pestun Localization on the sphere

$\langle \phi \rangle = \text{diag}\,(a_1, \ldots, a_N)$

$$Z_{S^4} = \int [da]\, |Z_{Nek}(a, \epsilon_1 = r^{-1}, \epsilon_2 = r^{-1})|^2$$

$\epsilon_{1,2} = r^{-1}$ omega deformation parameters serve as an **IR regulator**

$$\log\left(Z_{Nek}(a, \epsilon_1, \epsilon_2)\right) \sim -\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}(a)$$

- The **UV divergences** on the sphere are the same as those on $\mathbb{R}^4$.

The circular wilson loop can be computed

$$W(g) = Z_{S^4}^{-1} \int [da] \left(\frac{1}{N} \sum_i e^{2\pi a_i} \right) |Z_{Nek}(a, r^{-1})|^2$$

and is given by a matrix model calculation.

- For $\mathcal{N}=4$ the matrix model is Gaussian (Erickson, Semenoff, Zarembo)

$$W_{\mathcal{N}=4}(g) = \frac{I_1(4\pi g)}{2\pi g}$$

- For $\mathcal{N}=2$ theories we have a more complicated multi-matrix model

$$W_{\mathcal{N}=2}(g,\check{g}) = W_{\mathcal{N}=4}(f(g,\check{g}))$$

$$f(g,\check{g}) = \begin{cases} g^2 + 2\left(\check{g}^2 - g^2\right)\left[6\zeta(3)g^4 - 20\zeta(5)g^4\left(\check{g}^2+3g^2\right)\right] + \mathcal{O}(g^{10}) \\ \frac{2g\check{g}}{g+\check{g}} + \mathcal{O}(1) \end{cases}$$

  - Checked with Feynman diagrams calculation (up to 4-loops)

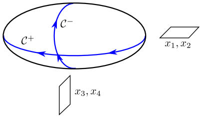
  - Agrees with AdS/CFT (Gadde-EP-Rastelli) (Gadde-Liendo-Rastelli-Yan)

Hama-Hosomichi Localization on the ellipsoid

Deformation parameter $b = \sqrt{\frac{r_1}{r_2}} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$

$$Z_{S_b^4} = \int [da] \left| Z_{Nek}(a, \epsilon_1, \epsilon_2) \right|^2$$

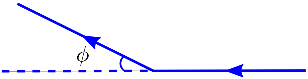
- The **UV divergences** on the ellipsoid are the same as those on $\mathbb{R}^4$.
- Two parameter **IR regularization**



Two supersymmetric wilson loops $\langle W^{\pm}_{\mathcal{N}=4}(g^2; b)\rangle = \frac{I_1(4\pi g b^{\pm 1})}{2\pi g b^{\pm 1}} + \mathcal{O}((b-1)^2)$

$$W^{+}_{\mathcal{N}=2}(g^2; b)\rangle = W^{+}_{\mathcal{N}=4}(f(g^2, b))\rangle$$

Cusp anomalous dimension and Bremsstrahlung function



Analytically continue to Minkowski signature $\phi = i\varphi$:

$$W_\varphi \sim e^{-\Gamma_{\text{cusp}}(\varphi)\log\frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}}} \qquad \text{with } \Lambda_{\text{UV}} \text{ and } \Lambda_{\text{IR}} \text{ the UV and IR cutoff.}$$

- For big φ: light-like cusp anomalous dimension

$$\Gamma_{\text{cusp}}(\varphi) \sim K\varphi$$

leading log behavior of the anomalous dims of finite twist operators

$$\Delta - S \sim K \log S \qquad \text{as} \qquad S \to \infty$$

- For small φ:

$$\Gamma_{\text{cusp}}(\varphi) = B\varphi^2 + \mathcal{O}(\varphi^4)$$

$B \propto$ the energy emitted by an uniformly accelerating probe quark

Follow (Lewkowycz-Maldacena) and (Fiol-Gerchkovitz-Komargodski)

$$B = \pm \frac{1}{4\pi^2} \frac{d}{db} \log \langle W^{\pm}(b) \rangle \Big|_{b=1}$$

- For $\mathcal{N} = 4$

$$B_{\mathcal{N}=4}(g^2) = \frac{g I_2(4g\pi)}{\pi I_1(4g\pi)}$$

- For $\mathcal{N} = 2$ theories we have a more complicated multi-matrix model

$$B_{\mathcal{N}=2}(g, \check{g}) = B_{\mathcal{N}=4}(f(g, \check{g}))$$

$$f(g, \check{g}) = \begin{cases} g^2 + 2\left(\check{g}^2 - g^2\right)\left[6\zeta(3)g^4 - 20\zeta(5)g^4\left(\check{g}^2 + 3g^2\right)\right] + \mathcal{O}(g^{10}) \\ \frac{2g\check{g}}{g+\check{g}} + \mathcal{O}(1) \end{cases}$$

The same up to four-loops and the leading term in strong coupling!

Discrepancies from 5-loops and scheme dependance

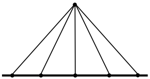
$$\Delta f = f_B - f_W \sim \frac{\partial f}{\partial b} \propto \zeta(2)$$

- Discrepancies depend on **how we cut-off the low energy momenta**

Five loops: order g^{10}

$$f_W^{(5)} = 2\left(g_2^2 - g_1^2\right) g_1^4 \Big[70\zeta(7)\left(g_2^4 + 5g_1^2 g_2^2 + 8g_1^4\right) - 2(6\zeta(3))^2\left(g_2^4 - g_1^2 g_2^2 + 2g_1^4\right) - 40\zeta(2)\zeta(5)g_1^4\Big]$$

We compute on sphere, or on ellipsoid some fields become massive:
scheme dependance due to IR regularization!



$$= -14\zeta(7) - 12\zeta(3)\zeta(4) + 36\zeta(2)\zeta(5)$$

Conclusions

Conclusions and outlook

- $\forall$ observable in the purely gluonic $SU(2,1|2)$ sector $(AdS_5 \times S^1)$
  take the $\mathcal{N}=4$ answer and replace $g^2 \to \mathbf{g}^2 = f(g^2) = \frac{R^4}{(2\pi\alpha')^2}$
  We need more data! (EP-Mitev), (Leoni-Mauri-Santambrogio) and (Fraser)
  $\Gamma_{cusp}\left(\varphi, g^2\right) = \Omega\left(\varphi, K(g^2)\right)$ (Grozin-Henn-Korchemsky-Marquard)
  Use the "**Exact correlation functions**" (Baggio-Niarchos-Papadodimas).

- Similar story for:
  - asymptotically conformal $\mathcal{N}=2$ theories (massive quarks) and
  - $\mathcal{N}=1$ SCFTs in 4D (EP-Roček)
  - theories in 3D: compare ABJ with ABJM (localization powerful in 3D)

Conclusions and Lessons

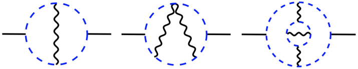
- Lesson: Think of $\mathcal{N} = 4$ **SYM as a regulator!** (A.Hamed-Murayama)
  The **integrable** $\mathcal{N} = 4$ **model** knows all about the **combinatorics**.
  For $\mathcal{N} = 2$: **relative finite renormalization** encoded in $\mathbf{g}^2 = f(g^2)$.

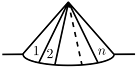
- Even explicit calculation the **Feynman diagrams** is **not so hard**:

  Only calculate the **difference**:

$$\mathbf{g}^2 = f(g^2) = g^2 + g^2 \left(Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4} \right)$$

  Only very particular **finite** integrals:





$$= 2 \binom{2n-1}{n} \zeta(2n-1) \frac{1}{p^2}$$

(Broadhurst)

Thank you!