Yangian-type symmetries of non-planar on-shell diagrams

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Based on work to appear with David Meidinger

Scattering amplitudes in $\mathcal{N} = 4$ SYM

- Tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM are Yangian invariant: Superconformal + dual superconformal = Yangian

  [Drummond, Henn, Plefka] [Drummond, Henn, Korchemsky, Sokatchev]

- Yangian invariance is broken at loop level
- The all-loop planar integrand can be expressed using the Grassmannian integral formula [Arkani-Hamed et al]
- Grassmannian integral formula is Yangian invariant and fixed by symmetry [Drummond, Ferro]

Scattering amplitudes in $\mathcal{N} = 4$ SYM

- Grassmannian integral formula

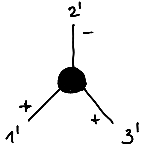
$$\mathcal{G}_{n,k} = \int \frac{d^{k\times n}C}{Vol[GL(k)]} \frac{1}{(1...k)(2...k+1)\cdots(n...n+k-1)} \delta^{4k|4k}(C\cdot\mathcal{W})$$

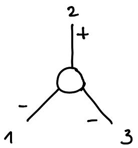
  - n number of particles and k MHV degree
  - $(p\ldots p+k-1)$ minor of the $n\times k$ matrix C
  - Twistor variables $\mathcal{W}_i^a$ with $i = 1,\ldots,n$ and $a = 1,\ldots,8$

- Grassmannian integral formula has interpretation in terms of on-shell diagrams

On-shell diagrams

- All on-shell diagrams can obtained from gluing elementary building blocks $\mathcal{G}_{3,1}$ and $\mathcal{G}_{3,2}$



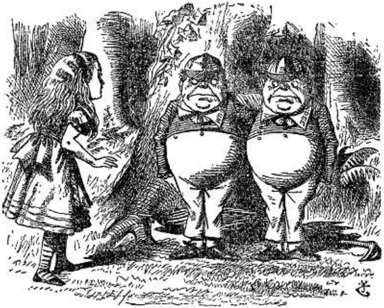


- MHV degree $k = 1, 2$ and number of legs $n = 3$
- Gluing procedure

$$\Delta_{ij} = \int \frac{d\alpha}{\alpha} \delta^{4|4}(\mathcal{W}_i + \alpha \mathcal{W}_j)$$ and integrating over $\mathcal{W}$'s

- Gluing lowers MHV degree and number of external legs

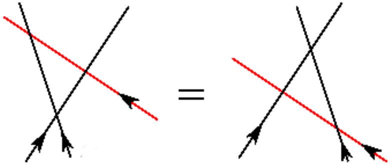
The Yangian



- Infinite-dimensional Hopf algebra
- Appears as a symmetry of exactly solvable models: spin-chains, Hubbard model, 2d IQFT's and planar $\mathcal{N} = 4$ SYM
- Yangian invariance of scattering amplitudes was originally formulated using Drinfeld's 1st realization
- Here RTT-realization (Yang-Baxter)

RTT-realization of $\mathcal{Y}(\mathfrak{gl}(4|4))$

Yang-Baxter equation



$$\mathbf{R}(u-v)\,(\mathcal{M}(u)\otimes\mathbb{I})\,(\mathbb{I}\otimes\mathcal{M}(v)) = (\mathbb{I}\otimes\mathcal{M}(v))\,(\mathcal{M}(u)\otimes\mathbb{I})\,\mathbf{R}(u-v)$$

with $\mathbf{R}(u) = u + (-1)^b e^{ab}\otimes e^{ba}$ and $\mathcal{M}(u) = e^{ab}\otimes\mathcal{M}^{ab}(u)$

- Yangian generators as expansion of operator valued 8×8-matrix

$$\mathcal{M}^{ab}(u) = \delta^{ab} + u^{-1}\left(\mathcal{M}^{[1]}\right)^{ab} + u^{-2}\left(\mathcal{M}^{[2]}\right)^{ab} + \dots$$

Yangian invariance of planar diagrams

In the RTT-realisation Yangian invariance of Grassmannian integral formula is expressed as a set of eigenvalue equations
Chicherin, Derkachov, Kirschner ’14; RF, Kanning, Ko, Staudacher ’14;

$$\mathcal{M}^{ab}(u)\mathcal{G}_{n,k} = (u-1)^k u^{n-k}\delta^{ab}\mathcal{G}_{n,k}$$

involving the spin chain monodromy

$$\mathcal{M}(u) = \mathcal{L}_1(u)\mathcal{L}_2(u)\cdots\mathcal{L}_n(u) \text{ with } \mathcal{L}_i(u) = u + (-1)^b e^{ab}\mathcal{W}_i^b \frac{\partial}{\partial \mathcal{W}_i^b}$$

It follows that Yangian generators act diagonally

$$\left(\mathcal{M}^{[i]}\right)^{ab}\mathcal{G}_{n,k} \sim \delta^{ab}\mathcal{G}_{n,k}$$

Yangian invariance of non-planar diagrams

- Concept of on-shell diagrams naturally generalises to non-planar topologies [Gekhtman,Shapiro,Vainshtein '12] [Franco,Galloni,Mariotti '13] [Franco,Galloni,Penante,Wen '15]
- Relevant for the study of non-planar contributions to amplitudes [Arkani-Hamed,Bourjaily,Cachazo,Postnikov,Trnka '14]

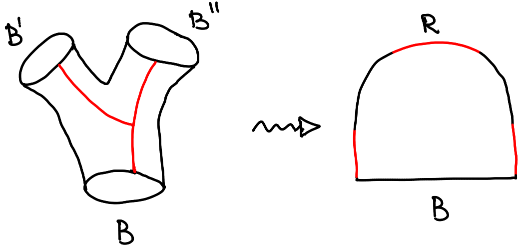
Q: Does Yangian invariance manifest itself in non-planar on shell diagrams?

A: Yes! [RF, Meidinger (to appear)]

- Obtained action of the Yangian on color-ordered boundaries of non-planar on-shell diagrams
- Found that lower levels of the Yangian are broken while higher level invariance may remain intact
- Introduced transfer matrix to derive further Yangian-type symmetries

Decomposition of non-planar diagrams

- All non-planar diagrams $\mathcal{G}_{np}$ can be cut into a planar one



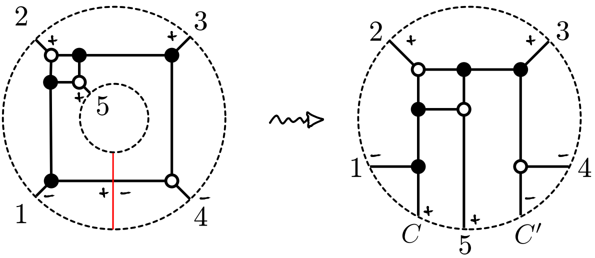
- Planar diagram $\mathcal{G}_p$ has $n_p = n_{np} + 2n_{cut}$ external legs and MHV degree $k_p = k_{np} + n_{cut}$
- Natural decomposition:

$$\mathcal{G}_{np} = \int_{C,C'} \Delta_{C,C'} \mathcal{G}_p$$

- Cut internal edges: $C = \{C_1, \ldots, C_{n_{cut}}\}$ and $C' = \{C'_1, \ldots, C'_{n_{cut}}\}$

Example of a non-planar diagram

5-point MHV diagram on a cylinder



- Non-planar diagram: $n_{np} = 5$, $k_{np} = 2$
- Planar diagram: $n_p = 7$, $k_p = 3$

Yagian invariance of planar decomposition

- Planar diagram $\mathcal{G}_p$ is Yangian invariant

$$\mathcal{M}_{RB}^{ab}(u)\mathcal{G}_p = (u-1)^{k_p} u^{n_p - k_p} \delta^{ab} \mathcal{G}_p$$

- Monodromy can be decomposed as

$$\mathcal{M}_{RB}(u) = \mathcal{M}_R(u)\mathcal{M}_B(u)$$

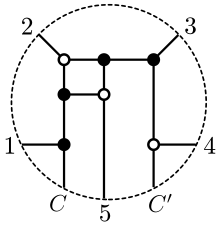
Here we focus on one boundary B

Example:

$B = (1, 2, 3, 4)$ and $R = (C', 5, C)$

$\mathcal{M}_B(u) = \mathcal{L}_1(u)\mathcal{L}_2(u)\mathcal{L}_3(u)\mathcal{L}_4(u)$

$\mathcal{M}_R(u) = \mathcal{L}_{C'}(u)\mathcal{L}_5(u)\mathcal{L}_C(u)$



Yagian invariance of planar decomposition

- Inverse of Lax operators is again Lax operator (unitarity)

$$\mathcal{L}_i^{-1}(u) = \frac{1}{u(1-u-\mathbb{C}_i)}\mathcal{L}_i(1-u-\mathbb{C}_i)$$

- Central charge set to zero when acting on on-shell diagrams
- Rewrite Yangian invariance relation as

$$u^{n_R}(1-u)^{n_R}\mathcal{M}_B^{ab}(u)\mathcal{G}_p = (u-1)^{k_p}u^{n_p-k_p}\mathcal{M}_{\bar{R}}^{ab}(1-u)\mathcal{G}_p$$

- $\bar{R}$ denotes reversed ordered set R

Symmetries of non-planar diagrams

- Integrate over cut lines to obtain action of monodromy on non-planar on-shell diagram

$$\mathcal{M}_B^{ab}(u)\mathcal{G}_{np} = \frac{(-1)^{k_p}u^{n_B-k_p}}{(1-u)^{n_R-k_p}}\int_{C,C'}\Delta_{C,C'}\mathcal{M}_{\bar{R}}^{ab}(1-u)\mathcal{G}_p$$

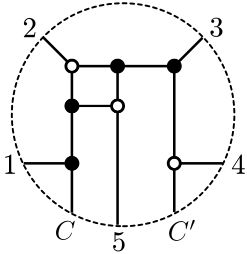
- Yangian generators

$$\mathcal{M}_B^{ab}(u) = u^{n_B}\delta^{ab} + u^{n_B-1}\left(\mathcal{M}_B^{[1]}\right)^{ab} + \ldots + \left(\mathcal{M}_B^{[n_B]}\right)^{ab}$$

- Higher level Yangian generators may still annihilate $\mathcal{G}_{np}$

$$\boxed{\left(\mathcal{M}_B^{[i]}\right)^{ab}\mathcal{G}_{np} = 0 \text{ for } i = k_p+1,\ldots,n_B}$$

Example



$$\Omega = \frac{1}{(12)(23)(34)(41)}\frac{(13)}{(35)(51)}$$

- Cut diagram has $n_B = 4$ $k_p = 3$
- One level annihilates $\mathcal{G}_{np}$

$$\left(\mathcal{M}_B^{[4]}\right)^{ab}\mathcal{G}_{np} = 0$$

with $\left(\mathcal{M}_B^{[4]}\right)^{ab} = (-1)^{ab+c+d+e}(\mathcal{W}_4^a\partial_4^c)(\mathcal{W}_3^c\partial_3^d)(\mathcal{W}_2^d\partial_2^e)(\mathcal{W}_1^e\partial_1^b)$

Transfer matrix identities

- Further identities can be obtained for the transfer matrix

$$\mathcal{T}_B(u) = \mathrm{str}\mathcal{M}_B(u)$$

- Consider Yangian invariance condition for cylinder

$$\mathcal{M}_B^{ab}(u)\mathcal{G}_{cyl} = \frac{(-1)^{k_p} u^{n_B - k_p}}{(1-u)^{n_R - k_p}} \int_{C,C'} \Delta_{C,C'} \mathcal{M}_{\bar{R}}^{ab}(1-u)\mathcal{G}_p$$

- Yields analog of Yangian invariance for transfer matrix

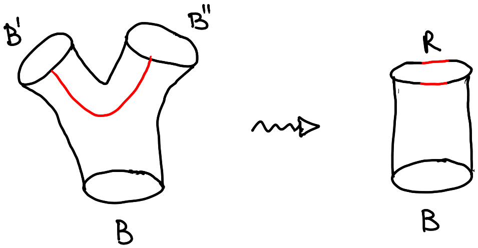
$$u^{k_{cyl} - n_B} \mathcal{T}_B(u)\mathcal{G}_{cyl} = (-1)^{k_{cyl}}(1-u)^{k_{cyl} - n_{B'}} \mathcal{T}_{B'}(u)\mathcal{G}_{cyl}$$

- Follows after integrating by parts and using "unitarity relation"

$$\mathcal{L}_{C_i'}(u)\mathcal{L}_{C_i}(u)\Delta_{C_i' C_i} = u(u-1)\Delta_{C_i' C_i}$$

Decomposition of non-planar diagrams

- All non-planar diagrams $\mathcal{G}_{np}$ can be cut into a cylindrical one



- Decomposition in terms of $\mathcal{G}_{cyl}$:

$$\mathcal{G}_{np} = \int_{C,C'} \Delta_{C,C'} \mathcal{G}_{cyl}$$

Transfer matrix identities

- We obtain action of transfer matrix on a given boundary

$$\mathcal{T}_B(u)\mathcal{G}_{np} = \frac{(-1)^{k_p} u^{n_B-k_p}}{(1-u)^{n_R-k_p}} \int_{C,C'} \Delta_{C,C'} \mathcal{T}_{\bar{R}}(1-u)\mathcal{G}_{cyl}$$

- Expansion of transfer matrix

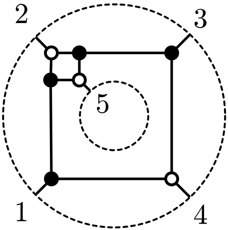
$$\mathcal{T}_B(u) = u^{n_B} + u^{n_B-1}\mathcal{T}_B^{[1]} + \ldots + \mathcal{T}_B^{[n_B]}$$

- Higher level transfer matrix coefficients annihilate $\mathcal{G}_{np}$

$$\boxed{\mathcal{T}_B^{[i]}\mathcal{G}_{np} = 0 \text{ for } i = k_{cyl}+1,\ldots,n_B}$$

- Does not follow from planar cutting as in general $k_{cyl} < k_p$!

Example



$$\Omega = \frac{1}{(12)(23)(34)(41)}\frac{(13)}{(35)(51)}$$

- Special case $B' = (5)$ contains only one element

$$\mathcal{T}_{B'}\mathcal{G}_{cyl} = 0 = \mathcal{T}_B\mathcal{G}_{cyl}$$

- LHS is trivially satisfied as central charge vanishes
- RHS yields constraints $\mathcal{T}_B^{[i]}\mathcal{G}_{cyl} = 0$ for $i = 2, 3, 4$ with

$$\mathcal{T}_B^{[i]} = \sum_{j_1 > \ldots > j_i}(\mathcal{W}_{j_1}^{a_1}\partial_{j_1}^{a_2})(\mathcal{W}_{j_2}^{a_2}\partial_{j_2}^{a_3})\cdots(\mathcal{W}_{j_i}^{a_i}\partial_{j_i}^{a_1})$$

Conclusion

- Identified Yangian-type symmetries in non-planar on-shell diagrams
- While Yangian invariance is broken, some generators may still annihilate non-planar on-shell functions
- Similar statement for coefficients of the transfer matrix

Outlook

- Determine general expression for non-planar on-shell functions
- Investigate how symmetries manifest themselves in non-planar contribution to amplitudes

Thank you!

Drinfeld's 1st realization of $\mathcal{Y}(\mathfrak{sl}(m|n))$

- Infinite tower of Yangian generators is generated by 1st and 2nd level:

$$[J^{(1)}_{\mu}, J^{(1)}_{\nu}] = f_{\mu\nu\rho} J^{(1)}_{\rho}, \qquad [J^{(1)}_{\mu}, J^{(2)}_{\nu}] = f_{\mu\nu\rho} J^{(2)}_{\rho}$$

with Serre relations ($n \neq 2$)

$$[J^{(2)}_{\mu}, [J^{(2)}_{\nu}, J^{(1)}_{\lambda}]] + \text{cyclic} = a_{\mu\nu\lambda\delta\gamma\sigma} \{J^{(1)}_{\delta}, J^{(1)}_{\gamma}, J^{(1)}_{\sigma}\}$$

- Co-product

$$\Delta(J^{(1)}_{\mu}) = J^{(1)}_{\mu} \otimes \mathbb{I} + \mathbb{I} \otimes J^{(1)}_{\mu}$$

$$\Delta(J^{(2)}_{\mu}) = J^{(2)}_{\mu} \otimes \mathbb{I} + \mathbb{I} \otimes J^{(2)}_{\mu} + \tfrac{1}{2} f_{\mu\nu\rho} J^{(1)}_{\nu} \otimes J^{(1)}_{\rho}$$

- Antipode