Harmony of conformal blocks

Evgeny Sobko

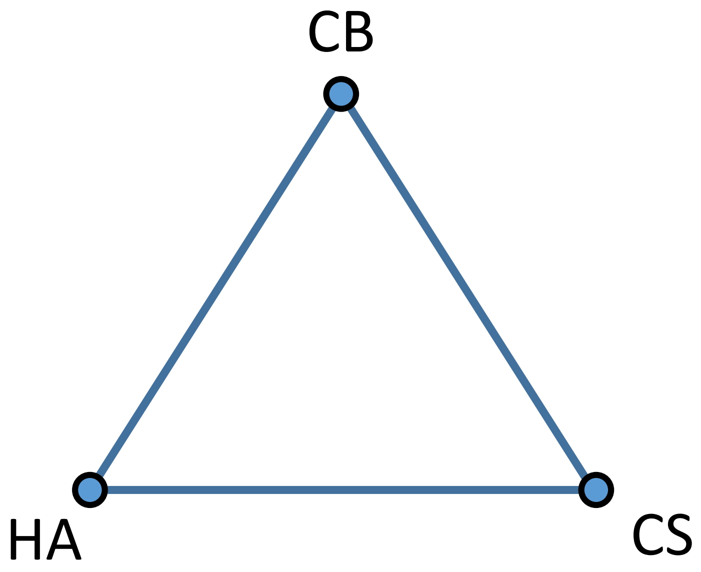
NORDITA
The Nordic Institute for Theoretical Physics

V.Schomerus, E.S. & M.Isachenkov [1611.xxxxx]

+ V.Schomerus , E.S. [work in progress]

GATIS Closing Workshop , 2016

Plan



CFT = Self-consisting CFT data

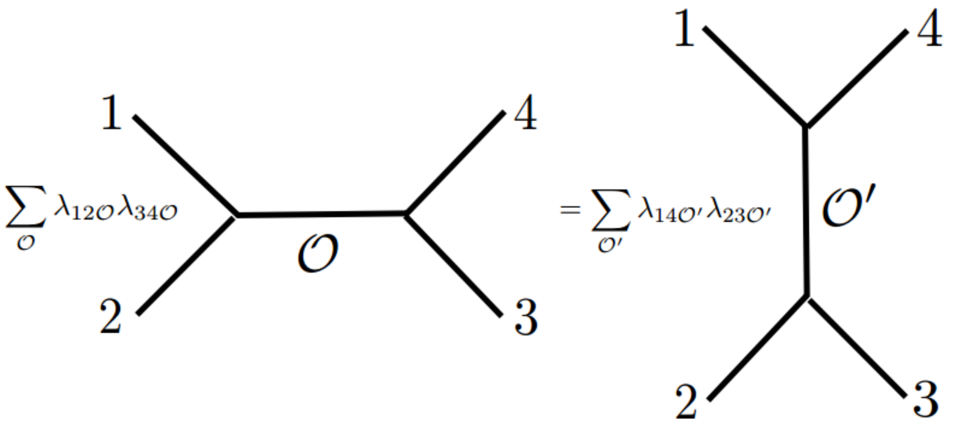
Ferrara, Grillo, Gato ’73
Polyakov ’74
Mack ‘77

- CFT data:

  - Primaries $\{\mathcal{O}_{\Delta,\mu}\}$ + descendants $\{P\mathcal{O}_{\Delta,\mu}, PP\mathcal{O}_{\Delta,\mu}, \ldots\}$

  - OPE: $\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{12}, \partial_2)\mathcal{O}_k(x_2)$

- Self-consistency = crossing symmetry:



- Conformal group in $\mathbb{R}^d$ is $G = SO(1, d+1)$
- Subgroup $$K = \underset{\Delta}{SO(1,1)} \times \underset{\mu}{SO(d)} \subset G$$
- Primaries $\longleftrightarrow$ π_μ^Δ - reps of G induced from (Δ, μ) reps of K
- 2pt correlators (sc.): $$< \mathcal{O}_i(x)\mathcal{O}_j^\dagger(y) >= \frac{\delta_{ij} t_i}{|x-y|^{2\Delta_i}}$$
- 3pt correlators (sc.): $$< \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3) >= \frac{C_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{13}|^{\Delta_i-\Delta_j+\Delta_k}|x_{23}|^{-\Delta_i+\Delta_j+\Delta_k}}$$

general reps:

G-invariants

$$< \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3) >= \sum_{a=1}^{N_3} \lambda_{ijk}^a t^a(x_1, x_2, x_3)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \; v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- 4-point correlation function:

$$< \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\mathcal{O}_l(x_4) >= \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{14}}{x_{24}}\right)^{\Delta_2-\Delta_1} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_3-\Delta_4} \sum_{I=1}^{N_4} g^I(u,v) t^I$$

- Decomposition over CPWs:

$$< \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\mathcal{O}_l(x_4) >= \sum_{\mathcal{O}} \sum_{a,b} \lambda^a_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}} \lambda^b_{\mathcal{O}^\dagger\mathcal{O}_k\mathcal{O}_l} W^{ab}_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\mathcal{O}_l,\mathcal{O}}(x_1,x_2,x_3,x_4)$$

- CPW:

$$W^{ab}_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\mathcal{O}_l,\mathcal{O}} = \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{14}}{x_{24}}\right)^{\Delta_2-\Delta_1} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_3-\Delta_4} \sum_I g^{I,ab}_{(\Delta,\mu)}(u,v) t^I$$

- Decomposition $g^I(u,v)$ over conformal blocks:

$$g^I(u,v) = \sum_{\mathcal{O}} \sum_{a,b} \lambda^a_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}} \lambda^b_{\mathcal{O}_k\mathcal{O}_l\mathcal{O}} g^{I,ab}_{\Delta,\mu}(u,v)$$

Casimir in scalar case

F.A.Dolan, H.Osborn

- Eigenproblem for Casimir :

$$D_\epsilon^2 G(z,\bar z) = \frac{1}{2} C_{\Delta,l} G(z,\bar z)$$

$$z\bar z = u,$$
$$(1-z)(1-\bar z) = v$$

where

$$C_{\Delta,l} = \Delta(\Delta - d) + l(l+d-2)$$

$$D_\epsilon^2 := D^2 + \overline{D}^2 + \epsilon\left[\frac{z\bar z}{\bar z - z}\left(\bar\partial - \partial\right) + \left(z^2\partial - \bar z^2\bar\partial\right)\right]$$

$$\epsilon = d-2$$

$$D^2 = z^2(1-z)\partial^2 - (a+b+1)z^2\partial \;\; - abz.$$

$$2a = \Delta_2 - \Delta_1$$
$$2b = \Delta_3 - \Delta_4$$

plus b.c. at $z, \bar z \to 0$:

$$G_{\Delta,l}(z,\bar z) \sim (z\bar z)^{\frac{1}{2}(\Delta - l)}(z+\bar z)^l + \ldots$$

Scalar Casimir as C-S hamiltonian

V.Schomerus,
M.Isachenkov
1602.01858

Changing variables :

$$z=-\frac{1}{\sinh^2\frac{x}{2}},\quad \bar{z}=-\frac{1}{\sinh^2\frac{y}{2}},$$

$$\psi(x,y)=\frac{(z-1)^{\frac{a+b}{2}+\frac{1}{4}}}{z^{\frac{1+\epsilon}{2}}}\frac{(\bar{z}-1)^{\frac{a+b}{2}+\frac{1}{4}}}{\bar{z}^{\frac{1+\epsilon}{2}}}|z-\bar{z}|^{\frac{\epsilon}{2}}G(z,\bar{z})$$

One gets Casimir operator in the form of BC2 C-S :

$$D_\epsilon^2\to-(H_{CS}^{(a,b,\epsilon)}+\frac{d^2-2d+2}{4})=-(-\partial_x^2-\partial_y^2+V_{C.S.}^{(a,b,\epsilon)}+\frac{d^2-2d+2}{4}),$$

$$V_{C.S.}^{(a,b,\epsilon)}=V_{PT}^{(a,b)}(x)+V_{PT}^{(a,b)}(y)+\frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x-y}{2}}+\frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x+y}{2}},$$

$$V_{PT}^{(a,b)}(x)=\frac{(a+b)^2-\frac{1}{4}}{\sinh^2 x}-\frac{ab}{\sinh^2\frac{x}{2}}$$

Emergence of (Super)Integrable C-S for scalar blocks – is just an exception or general feature of all conformal blocks in any CFT?

What is the natural framework to think about it?

Hint : many integrable QMs come as a radial part of Laplacian on the proper coset.

Idea : let's try to reformulate Casimir eigenproblem as Harmonic analysis on the proper bundle.

Harmonic analysis approach to CBs

- Just notations: $K = SO(1,1) \times SO(d) = d \times R$ $\quad d(\lambda) = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}$

- Functions $g^I(u,v)$ live in $(\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4)^G$

- Tensor product $\pi_1 \otimes \pi_2$ can be realized on the space of equivariant f-s:

$$\Gamma^{(\pi_1,\pi_2)}_{K\backslash G} = \left\{ f : g \to V_{\mu_1} \otimes V_{\mu_2} \;\middle|\; \begin{array}{ll} f(d(\lambda)g) = e^{\lambda(\Delta_2 - \Delta_1)} f(g) & \text{for } \; d(\lambda) \in D \subset G \\ f(rg) = \mu_1(r) \otimes \mu_2(r) f(g) & \text{for } r \in R \subset G \end{array} \right\}$$

- G-invariant tensor product of 4 irreps where $g^I(u,v)$ lives:

$$\left(\Gamma_{K\backslash G}^{(\Delta_1,\mu_1;\Delta_2,\mu_2)} \otimes \Gamma_{G/K}^{(\Delta_3,\mu_3;\Delta_4,\mu_4)}\right)^G \cong \Gamma_{G/\!/K}^{(a,\mu_1\otimes\mu_2;b,\mu_3\otimes\mu_4)}$$

Double factor : $G/\!/K = K\setminus G/K = (K\setminus G\times G/K)/G$

Equivariant f-s on the double-factor:

$$\Gamma_{G/\!/K}^{(\mathcal{L}\mathcal{R})} = \{\, f : G \to V_{\mathcal{L}}\otimes V_{\mathcal{R}} \mid f(k_l g k_r^{-1}) = [\mathcal{L}(k_l)\otimes\mathcal{R}(k_r)]f(g)\,\}$$

$$\mathcal{L}(d(\lambda)R) = e^{2a\lambda}\mu_1(R)\otimes\mu_2(R), \quad \mathcal{R}(d(\lambda)R) = e^{2b\lambda}\mu_3(R)\otimes\mu_4(R)$$

- Decomposition over CBs corresponds to:

$$\Gamma_{G/\!/K}^{(a,\mu_1\otimes\mu_2;b,\mu_3\otimes\mu_4)} = \sum_{\mathcal{O}_\alpha}\Gamma_{G/\!/K}^{(a,\mu_1\otimes\mu_2;b,\mu_3\otimes\mu_4),\mathcal{O}_\alpha}$$

KAK decomposition of G

- Cartan involution $\theta = diag\,(-1,-1,1,\ldots,1), \quad \theta^2 = 1$ acts on Lie algebra as

$$\theta : \mathfrak{g} \to \mathfrak{g}, \quad \theta(g) = \theta g \theta$$

- It splits $\mathfrak{g}$ in a direct sum $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{p}$ such that $\theta(\mathfrak{k}) = \mathfrak{k}, \theta(\mathfrak{p}) = -\mathfrak{p}$ and

$$\mathfrak{k} = \mathrm{Lie}(K) \quad , \quad [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \quad , \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \quad , \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$$

- $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. $A = exp\ \mathfrak{a}$

- KAK decomposition : $G = KAK$

More details on KAK

- Generators of $G = SO(1, d+1)$: $M_{ab} = -M_{ba}$ and $a, b = 0, 1, 2, \ldots d+1$

$$\mathfrak{k} = \mathrm{Lie}(K) = \{M_{12}, M_{\mu\nu}\}\,, \quad \mu, \nu \in (2, .., d+1)$$

$$\mathfrak{a} = \{M_{02}, M_{13}\}$$

- Example. d=2 : $G = SO(1,3)$ $\qquad M = K(\psi_l, \phi_l) A(\tau_1, \tau_2) K(\psi_r, \phi_r)$

$$K(\psi, \phi) = \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad A(\tau_1, \tau_2) = \begin{pmatrix} \cosh\frac{\tau 1}{2} & 0 & \sinh\frac{\tau 1}{2} & 0 \\ 0 & \cos\frac{\tau 2}{2} & 0 & -\sin\frac{\tau 2}{2} \\ \sinh\frac{\tau 1}{2} & 0 & \cosh\frac{\tau 1}{2} & 0 \\ 0 & \sin\frac{\tau 2}{2} & 0 & \cos\frac{\tau 2}{2} \end{pmatrix}$$

- In $d \geqslant 4$ we have nontrivial stabilizer B of $K \times K$ action. The fixing gives:

$$G = KA(K/B), \quad B = \mathrm{SO}(d-2)$$

Laplace-Beltrami operator

- The metric on G is induced by Killing form:

$$g_{\alpha\beta}(x) = -2\ \mathrm{tr}\ h^{-1}\partial_\alpha h\ h^{-1}\partial_\beta h, \quad h \in G$$

- Metric in 2d:

$$g_{\alpha\beta}dx^\alpha dx^\beta = 4(d^2\phi_l + d^2\phi_r - d^2\psi_l - d^2\psi_r) - d^2\tau_1 + d^2\tau_2$$

$$-8\sinh\frac{\tau_1}{2}\sin\frac{\tau_2}{2}(d\psi_l d\phi_r + d\psi_r d\phi_l) + 8\cosh\frac{\tau_1}{2}\cos\frac{\tau_2}{2}(d\phi_l d\phi_r - d\psi_l d\psi_r)$$

- L-B operator:

$$\Delta_{LB} = \sum_{\alpha,\beta} |\det(g_{\alpha\beta})|^{-\frac{1}{2}}\partial_\alpha g^{\alpha\beta}|\det(g_{\alpha\beta})|^{\frac{1}{2}}\partial_\beta$$

- We extend action of Δ_{LB} on $V_{\mathcal{L}} \otimes V_{\mathcal{R}}$ - valued functions just as $\Delta_{LB}\, 1_{\mathcal{L}} \otimes 1_{\mathcal{R}}$

Harmonic analysis on $\Gamma^{(\mathcal{L}\mathcal{R})}_{G//K}$

- Sections of $\Gamma^{(\mathcal{L}\mathcal{R})}_{G//K}$ form a subspace in $L^2(G, V_{\mathcal{L}} \otimes V_{\mathcal{R}}; d\mu_G)$
- Sections of $\Gamma^{(\mathcal{L}\mathcal{R})}_{G//K}$: $f(g) = f(k_l a k_r^{-1}) = [\mathcal{L}(k_l) \otimes \mathcal{R}(k_r)] f(a)$ are defined by their

restriction to maximal torus A : $f(a) = f_A(\tau_1, \tau_2)$

- Reduction of an operator $\mathcal{O}$ on $L^2(G, V_{\mathcal{L}} \otimes V_{\mathcal{R}}; d\mu_G)$

to $\mathcal{O}^A$ acting on $L^2(A, V_{\mathcal{L}} \otimes V_{\mathcal{R}}; d\mu_A)$:

$$\int d\mu_A < f_A, \mathcal{O}^A g_A >= \frac{e^{-4b\psi_l - 4a\psi_r}}{\int d''\mu_G} \int d'\mu_G < [\mathcal{L}(k_l) \otimes \mathcal{R}(k_r^{-1})] f_A, \mathcal{O}[\mathcal{L}(k_l) \otimes \mathcal{R}(k_r^{-1})] g_A >$$

$$f(k_l a k_r) = [\mathcal{L}(k_l) \otimes \mathcal{R}(k_r^{-1})] f_A[\tau_1, \tau_2] = e^{2b\psi_l + 2a\psi_r} [(\mu_1 \otimes \mu_2)(r_l) \otimes (\mu_3 \otimes \mu_4)(r_r^{-1})] f_A[\tau_1, \tau_2] \qquad k = d(\psi) r$$

From HA to QM

- Making above described reduction for Δ_{LB} we get Δ_{LB}^{A}

- In order to get ordinary QM we make transformation of space of functions $f(x) \to \frac{f}{\sqrt{m}}$ where $d\mu_A = m(\tau_1, \tau_2) d\tau_1 d\tau_2$

- It leads to the following transformation for LB :

$$H = \sqrt{m} \Delta_{\mathrm{LB}}^{A} \frac{1}{\sqrt{m}}$$

Examples. Scalar case

- In scalar case we have just

$$[\mathcal{L}(k_l)\otimes\mathcal{R}(k_r^{-1})]g[\tau_1,\tau_2]=e^{2b\psi_l+2a\psi_r}g[\tau_1,\tau_2]$$

- And in d=2 it leads to

$$\int d\mu_A f_A(\tau_1,\tau_2),\Delta^A g_A(\tau_1,\tau_2)=$$

$$\frac{e^{-4b\psi_l-4a\psi_r}}{\int 8d\phi_l d\phi_r}\int 8(\cosh(\tau_1)-\cos(\tau_2))d\tau_1 d\tau_2 d\phi_l d\phi_r e^{2b\psi_l+2a\psi_r}f_A(\tau_1,\tau_2)\Delta e^{2b\psi_l+2a\psi_r}g_A(\tau_1,\tau_2)$$

Gauge transformation + new variables $\tau_1=x+y,\ \tau_2=i(x-y)$ gives us

$$H=\sqrt{m}\Delta^A\frac{1}{\sqrt{m}}=\frac{1}{2}(H_{C.S}^{(a,b,0)}+\frac{1}{2})\qquad \text{where } m=\cosh(\tau_1)-\cos(\tau_2)$$

- In d=3:

$$H=\frac{1}{2}(H_{C.S}^{(a,b,1)}+\frac{5}{4})$$

Idea of “seed CBs” in one slide

- Many 3pt tensor structures are related by differential operators:

$$< \mathcal{O}_1\mathcal{O}_2\mathcal{O}_r >^a = D_{12}^{aa'} < \mathcal{O}_1'\mathcal{O}_2'\mathcal{O}_r >^{a'}$$

- Combining it with shadow formalism:

$$W^{ab}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\mathcal{O}_r}(x_i) \sim \int d^d y_1 d^d y_2 < \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_r(y_1) >^a \Pi(y_1,y_2) < \mathcal{O}_r^\dagger(y_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) >^b$$

- We get:

$$W^{ab}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\mathcal{O}_r} = D_{12}^{aa'} D_{34}^{bb'} W^{a'b'}_{\mathcal{O}_1'\mathcal{O}_2'\mathcal{O}_3'\mathcal{O}_4'\mathcal{O}_r}$$

- Seed CBs - the min set of CBs which is enough to reconstruct all others.
- In 3D there are only two:

$$< \phi_1\phi_2\phi_3\phi_4 >, \qquad < \psi_1\phi_2\phi_3\psi_4 >$$

$$< \psi_1 \phi_2 \phi_3 \psi_4 >$$

- For left and right pair we have $\mu_l = 0 \otimes \frac{1}{2} = \frac{1}{2},\ \mu_r = 0 \otimes \frac{1}{2} = \frac{1}{2}$

- We use the standard spin-1/2 representation of SO(3):

$$\mu_{l,r}\left[\begin{pmatrix} \cos\phi_1 & -\sin\phi_1 & 0 \\ \sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi_2 & -\sin\phi_2 & 0 \\ \sin\phi_2 & \cos\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} \cos\frac{\theta}{2}e^{i\frac{\phi_1+\phi_2}{2}} & i\sin\frac{\theta}{2}e^{i\frac{\phi_1-\phi_2}{2}} \\ i\sin\frac{\theta}{2}e^{-i\frac{\phi_1-\phi_2}{2}} & \cos\frac{\theta}{2}e^{-i\frac{\phi_1+\phi_2}{2}} \end{pmatrix}$$

- Section:

$$\mathbf{u} = [\mathcal{L}(l_l) \otimes \mathcal{R}(k_r^{-1})]u(\tau_1, \tau_2) = e^{2b\psi_l + 2a\psi_r}[\mu_l(r_l) \otimes \mu_r(r_r)]u(\tau_1, \tau_2), \qquad k = d(\psi)r$$

$$u = (u_1, \ldots, u_4)^{\mathrm{T}}$$

- Reduction:

$$\int d\mu_A < v, \Delta^A u >= \frac{e^{-4b\psi_l - 4a\psi_r}}{\int d''\mu_G} \int d'\mu_G < \mathbf{v}, \Delta\mathbf{u} >$$

$$\tilde{H} = \sqrt{m}\Delta^A \frac{1}{\sqrt{m}}, \quad m = (\cosh\tau_1 - \cos\tau_2)\sinh\frac{\tau_1}{2}\sin\frac{\tau_2}{2}$$

- Matrix $\tilde{H}$ has a block-diagonal form : $\tilde{H}_1 = \begin{pmatrix} \tilde{H}_{22} & \tilde{H}_{23} \\ \tilde{H}_{32} & \tilde{H}_{33} \end{pmatrix}, \tilde{H}_2 = \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{14} \\ \tilde{H}_{41} & \tilde{H}_{44} \end{pmatrix}$

- On the last step we do simple rotation:

$$H_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tilde{H}_1 \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$H_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tilde{H}_2 \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

- The result has a form of matrix version of CS:

$$H_1 = \frac{1}{2}\left(\begin{pmatrix} H_{CS}^{(a,b,1)} + \frac{5}{4} & 0 \\ 0 & H_{CS}^{(a,b,1)} + \frac{5}{4} \end{pmatrix} + \right.$$

$$\left.\begin{pmatrix} -\frac{1}{16}\left(\frac{1}{\cosh^2\frac{x}{2}} + \frac{1}{\cosh^2\frac{y}{2}} - \frac{2}{\sinh^2\frac{x-y}{4}} - \frac{2}{\sinh^2\frac{x+y}{4}}\right) & \frac{a+b}{4}\left(\frac{1}{\cosh^2\frac{x}{2}} - \frac{1}{\cosh^2\frac{y}{2}}\right) \\ \frac{a+b}{4}\left(\frac{1}{\cosh^2\frac{x}{2}} - \frac{1}{\cosh^2\frac{y}{2}}\right) & -\frac{1}{16}\left(\frac{1}{\cosh^2\frac{x}{2}} + \frac{1}{\cosh^2\frac{y}{2}} + \frac{2}{\cosh^2\frac{x-y}{4}} + \frac{2}{\cosh^2\frac{x+y}{4}}\right) \end{pmatrix}\right)$$

$$H_2 = \frac{1}{2}\left(\begin{pmatrix} H_{CS}^{(a,b,1)} + \frac{5}{4} & 0 \\ 0 & H_{CS}^{(a,b,1)} + \frac{5}{4} \end{pmatrix} + \right.$$

$$\left.\begin{pmatrix} \frac{1}{16}\left(\frac{1}{\sinh^2\frac{x}{2}} + \frac{1}{\sinh^2\frac{y}{2}} + \frac{2}{\sinh^2\frac{x+y}{4}} - \frac{2}{\cosh^2\frac{x-y}{4}}\right) & \frac{b-a}{4}\left(\frac{1}{\sinh^2\frac{x}{2}} - \frac{1}{\sinh^2\frac{y}{2}}\right) \\ \frac{b-a}{4}\left(\frac{1}{\sinh^2\frac{x}{2}} - \frac{1}{\sinh^2\frac{y}{2}}\right) & \frac{1}{16}\left(\frac{1}{\sinh^2\frac{x}{2}} + \frac{1}{\sinh^2\frac{y}{2}} + \frac{2}{\sinh^2\frac{x-y}{4}} - \frac{2}{\cosh^2\frac{x+y}{4}}\right) \end{pmatrix}\right)$$

- One can check that it coincides with:

Iliesiu, Kos, Poland, Pufu,
Simmons-Duffin, Yacoby '15

$$\left[\begin{pmatrix} \mathcal{L}_D^+ & \mathcal{L}_A^+ \\ \mathcal{L}_A^+ & \mathcal{L}_D^+ \end{pmatrix} + \begin{pmatrix} 0 & \frac{4r(\Delta_{12}+\Delta_{43})}{1+r^2-2r\eta} \\ 0 & \frac{4r(\eta-2r+r^2\eta)(\Delta_{12}+\Delta_{43})}{(1+r^2-2r\eta)^2} \end{pmatrix}\right] \begin{pmatrix} g^1_{\Delta,\ell} \\ g^2_{\Delta,\ell} \end{pmatrix} = C_{\Delta,\ell} \begin{pmatrix} g^1_{\Delta,\ell} \\ g^2_{\Delta,\ell} \end{pmatrix},$$
$$\left[\begin{pmatrix} \mathcal{L}_D^- & \mathcal{L}_A^- \\ \mathcal{L}_A^- & \mathcal{L}_D^- \end{pmatrix} + \begin{pmatrix} \frac{4r(\eta+2r+r^2\eta)\Delta_{43}}{(1+r^2+2r\eta)^2} & -\frac{4r\Delta_{12}}{1+r^2+2r\eta} \\ -\frac{4r\Delta_{43}}{1+r^2+2r\eta} & \frac{4r(\eta+2r+r^2\eta)\Delta_{12}}{(1+r^2+2r\eta)^2} \end{pmatrix}\right] \begin{pmatrix} g^3_{\Delta,\ell} \\ g^4_{\Delta,\ell} \end{pmatrix} = C_{\Delta,\ell} \begin{pmatrix} g^3_{\Delta,\ell} \\ g^4_{\Delta,\ell} \end{pmatrix}, \tag{A.10}$$

where

$$\begin{aligned}
\mathcal{L}_D^\pm &= r^2\partial_r^2 + (\eta^2-1)\partial_\eta^2 \\
&+ \left(\frac{4r^2\eta(1-r^2)(\Delta_{12}+\Delta_{43})}{(1+r^2-2r\eta)(1+r^2+2r\eta)} - \frac{r(1+3r^2)}{1-r^2} - \frac{r(1-r^2)(1+r^2\mp 2r\eta)}{(1+r^2+2r\eta)(1+r^2-2r\eta)}\right)\partial_r \\
&+ \left(\frac{4\,(\eta^2-1)\,(r^3+r)\,(\Delta_{12}+\Delta_{43})}{(1+r^2+2\eta r)\,(1+r^2-2\eta r)} + \frac{[3\eta(1+r^2)\pm 2r(4\eta^2-1)]\,(1+r^2\mp 2r\eta)}{(1+r^2+2r\eta)(1+r^2-2r\eta)}\right)\partial_\eta \\
&+ \left(\frac{3}{4} - \frac{4r\Delta_{12}\Delta_{43}\,(\eta+2r+r^2\eta)}{(1+r^2+2r\eta)^2}\right), \\
\mathcal{L}_A^\pm &= \frac{2r^2}{1-r^2}\partial_r \pm \partial_\eta\,,
\end{aligned} \tag{A.11}$$

Outline

We developed a universal approach to general spinning conformal blocks through the harmonic analysis of certain bundles over a double coset of the conformal group. The resulting Casimir equations are given by a matrix version of the Calogero-Sutherland Hamiltonian that describes the scattering of spinning particles in the external potential.

The approach was illustrated in several examples including fermionic seed blocks in 3D CFT where they take a very simple form.

Future

- Work out all details in d=4 and higher. Ideally, to prove all theorems with mathematical level of rigorous.
- SUSY
- Boundary CFT
- Understand (super)Integrability of these matrix CS models.
- Integrability -> theory of solutions -> modern theory of special functions
- With “CB-HA-CS” dictionary in hands one can play the game of translating results from one science into another and see what it means
- Why only 4-points? Something about dynamics?
- …

Thank you!