Form factor remainders:
from $\mathcal{N} = 4$ SYM to QCD

DESY - Nov 29th, 2016
GATIS closing workshop highlights

Based on:

hep-th/ {1406.1443, 1606.08682 + in progress}

In collaboration with:

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Brenda Penante
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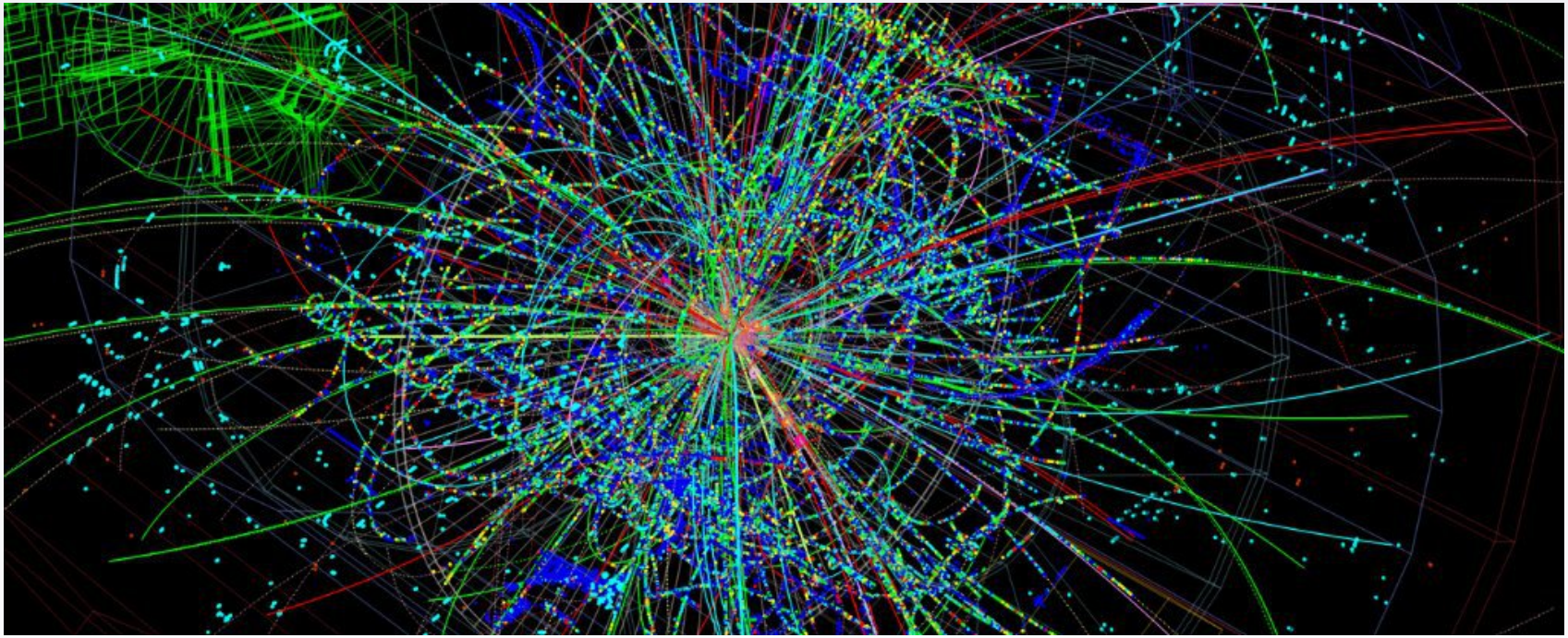
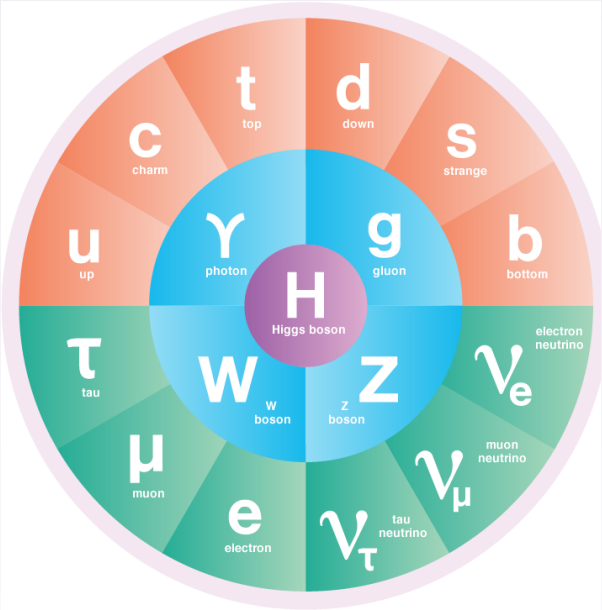
Scattering amplitudes



Approach #1: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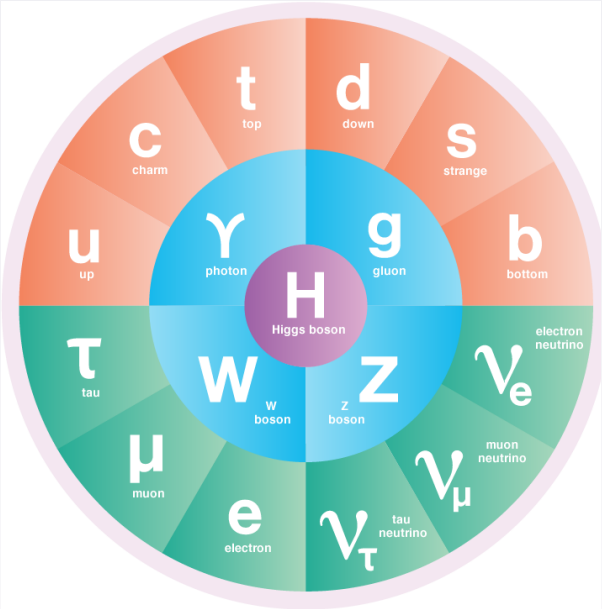

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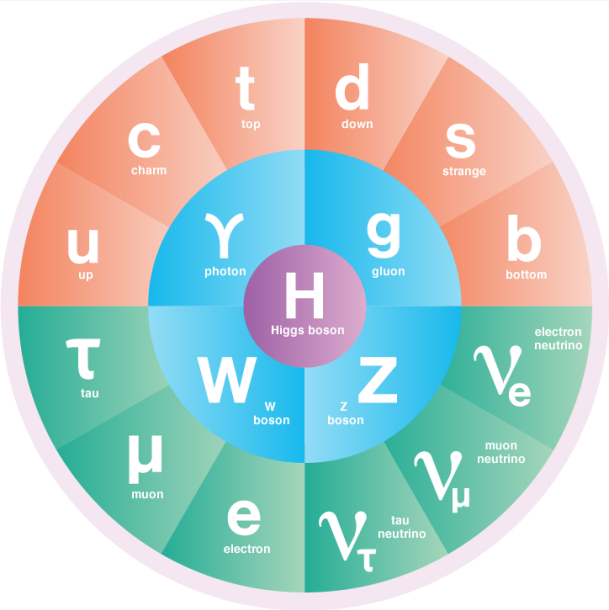
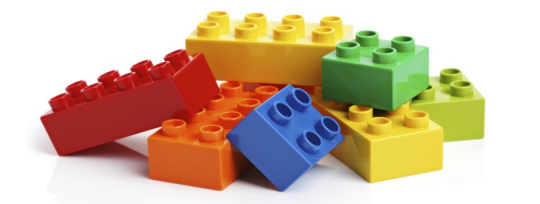
Approach #2:



This talk



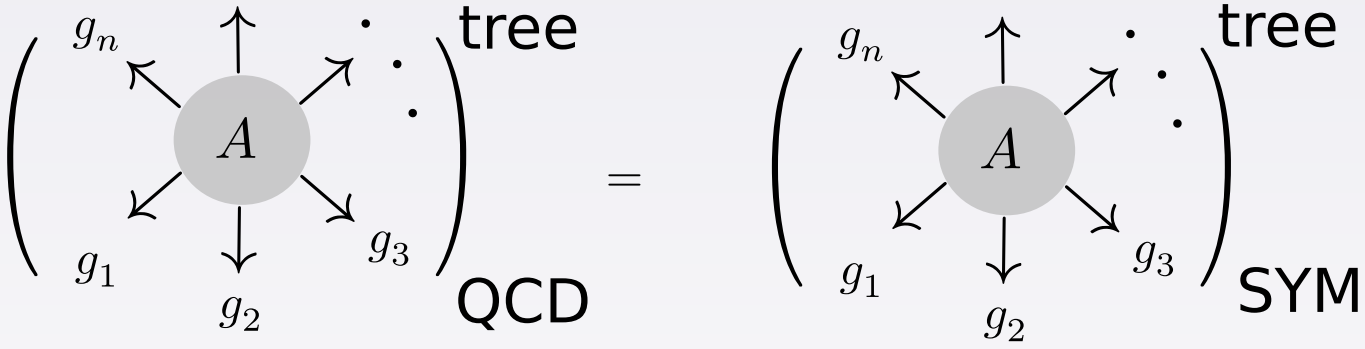
Planar $\mathcal{N} = 4$ super Yang-Mills

Aim:

- Learn as much as possible from our toy model
- Generalise methods to less special quantities
  - Off-shell quantities (form factors, correlation functions)
  - Away from planar limit
- Find links with the SM (in particular QCD)

Overview of tree amplitudes in $\mathcal{N} = 4$ SYM

What about QCD?



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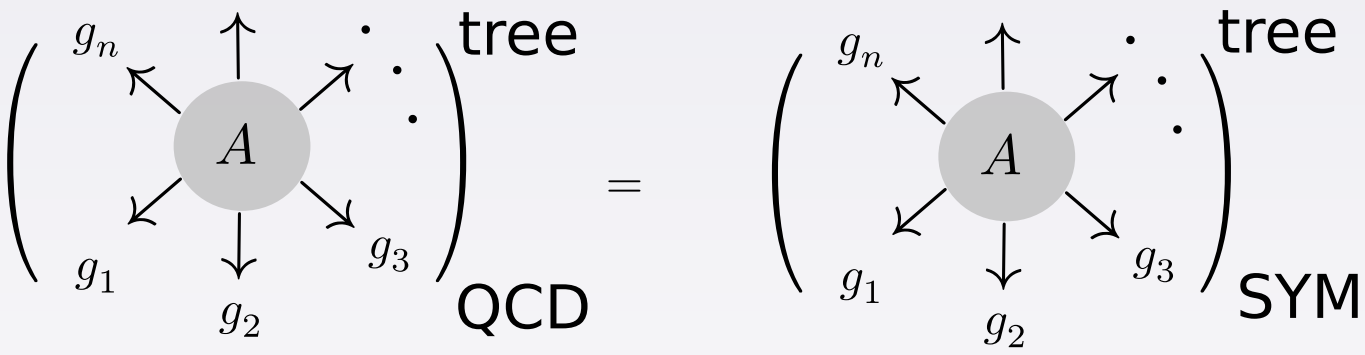
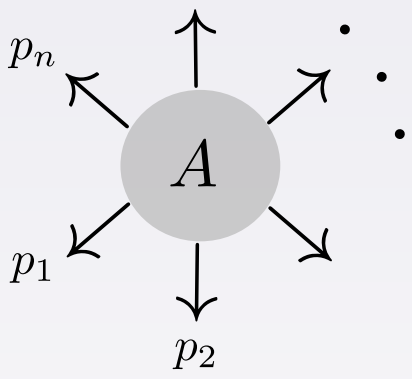


Not extremely useful

- Amplitudes involving states in QCD $\notin \mathcal{N} = 4$ SYM
- Loops

... let's look at a more general observable

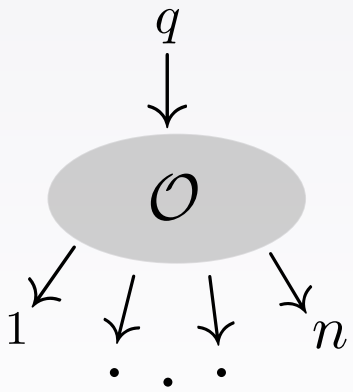
Form factors



$\leftrightarrow A_n = \langle 1, 2, \ldots, n|0\rangle$

vs.

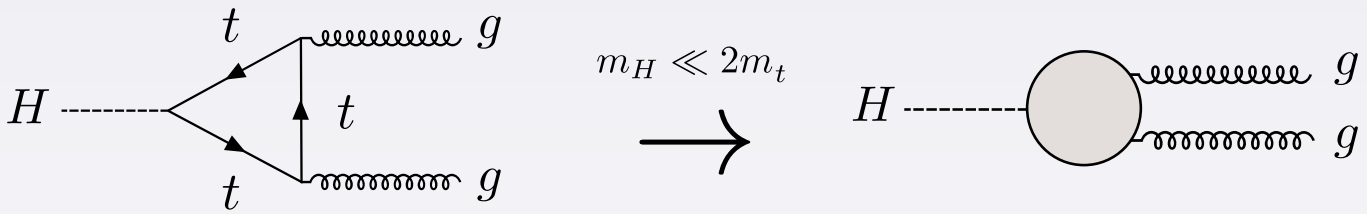
$$F_{\mathcal{O}}(1, \ldots, n; q) \equiv \int d^4x \, e^{-iqx} \langle 1, \ldots, n|\mathcal{O}(x)|0\rangle$$



$$p_1^2 = \cdots = p_n^2 = 0 \rightsquigarrow p_i = \lambda^i \widetilde{\lambda}^i \ , \quad q^2 \neq 0$$

Form factors as effective vertices: Higgs plus multi-gluon

– Gehrmann, Jaquier, Glover, Koukoutsakis –



Effective interaction: $\mathcal{L}_{\text{eff}} = H\,\text{Tr}(F^2)$

$H \to gg\cdots g$ given by form factor:

$$\langle gg\cdots g|\int d^4x\; e^{-iqx}\text{Tr}(F^2)(x)|0\rangle\Big|_{q^2=m_H^2}$$

Meanwhile in $\mathcal{N} = 4$ SYM:

Chiral part of stress tensor multiplet:

$$\mathcal{T} = \mathrm{Tr}(\phi^2) + \cdots + (\theta)^4 \mathcal{L}_{\text{on-shell}}; \quad \mathcal{L}_{\text{on-shell}} = \mathrm{Tr}(F_{\mathrm{SD}}^2)$$

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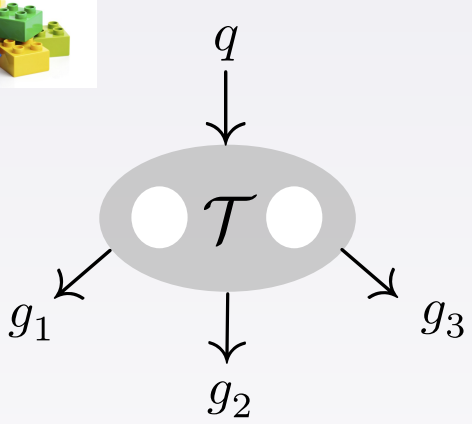

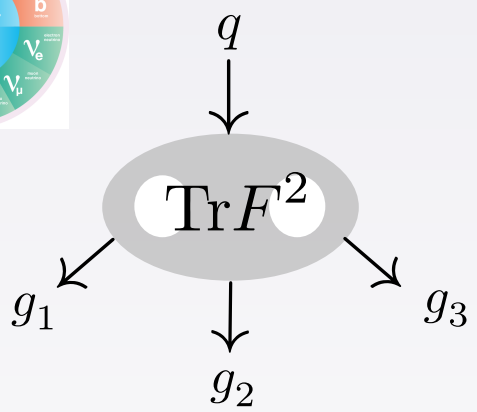

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"Li_4" only

("Maximal transcendentality")

– Brandhuber, Travaglini, Yang –

"Li_4" + "Li_3" + "Li_2"

+ "log" + rational

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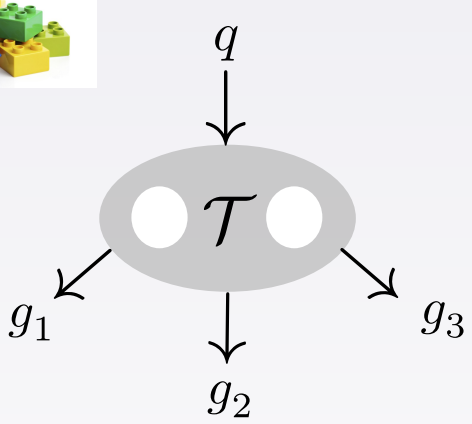

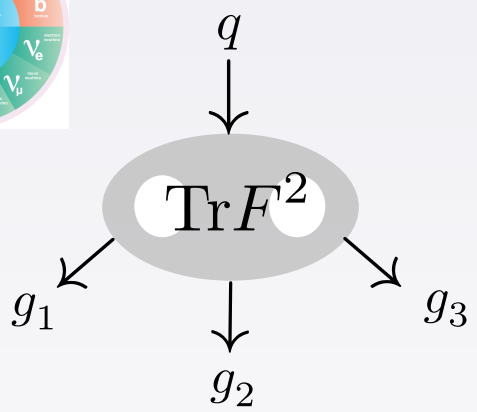

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$$\text{“Li}_4\text{”}\Big|_{\mathrm{QCD}} = \text{“Li}_4\text{”}\Big|_{\mathcal{N}=4\ \mathrm{SYM}}$$

Motivation 1: Is it possible to extend the connection?

The effective Lagrangian approach goes further:
– Neill / Dawson, Lewis, Zeng –

$$\mathcal{L}_{\text{eff}} = H\text{Tr}(F^2) + \frac{1}{m_{\text{top}}^2}\sum_{i=1}^{4} c_i O_i + \mathcal{O}\left(\frac{1}{m_{\text{top}}^4}\right)$$

O_i : dimension 7 operators.
Consider for instance $O_1 = H\text{Tr}(F^3) = H\text{Tr}(F_{\text{SD}}^3 + F_{\text{ASD}}^3)$

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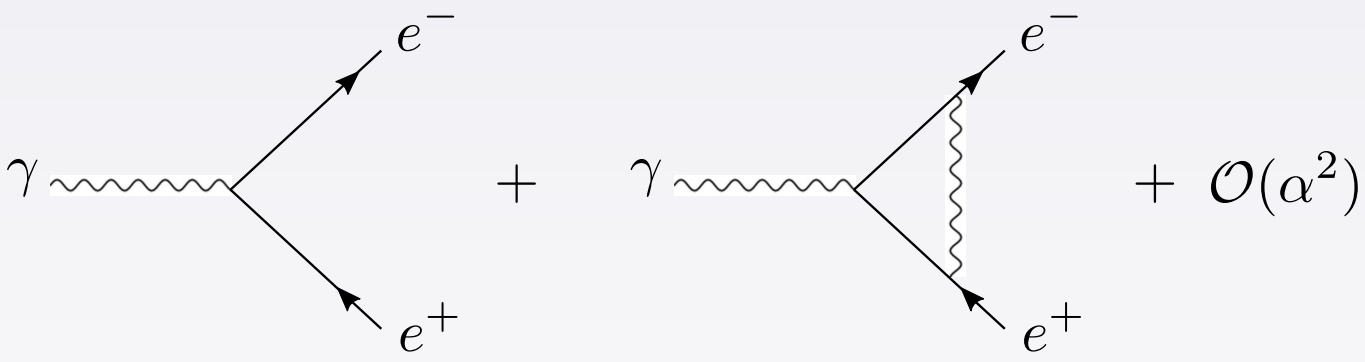
Q: Is there a relation between the max. trans. part of these form factors in $\mathcal{N} = 4$ SYM and QCD? Are they identical?

Q: Can we find universal structures which are invariant among different form factors and theories?

Motivation 2: Hidden simplicity of form factors

QED - electron anomalous magnetic moment:



- At $\mathcal{O}(\alpha^3) \sim 70$ diagrams ± 10 and ± 100, but result is $\mathcal{O}(1)$
  – Schwinger / Cvitanovic, Kinoshita / Laporta, Remiddi –

Motivation 2: Hidden simplicity of form factors

Ex: $\mathrm{Tr}(\phi^2),\ \phi \equiv \phi^{12}$ in $\mathcal{N}=4$ SYM

- MHV amplitudes:
  – Parke, Taylor / Mangano, Parke –

$$A(i^-, j^-, \{g^+\}) \;=\; \frac{\langle i\, j\rangle^4}{\langle 12\rangle \cdots \langle n1\rangle}\delta^{(4)}\Big(\sum p_i\Big)$$

- MHV form factors:

  – Brandhuber, Gurdogan, Mooney, Travaglini, Yang –

$$F_{\mathrm{Tr}(\phi^2)}(i^\phi, j^\phi, \{g^+\}; q) \;=\; \frac{\langle i\, j\rangle^2}{\langle 12\rangle \cdots \langle n1\rangle}\delta^{(4)}\Big(q - \sum p_i\Big)$$

Q: How far can we extend the scope of applicability of *on-shell methods* for form factors?

Outline

Part 1: Very quick review of form factors of protected operators

Part 2: Form factors of non-protected ops in the $SU(2|3)$ sector

- Two-loop remainder function
- Mixing + dilatation operator

Part 3: Work in progress + speculations

Part 1

Overview of form factors of protected operators

Protected operators: no anomalous dimension, no mixing, no renormalisation.

- Chiral part of stress tensor multiplet:

$$\mathcal{T}_2 = \mathrm{Tr}(\phi^2) + \cdots + (\theta)^4 \mathcal{L}_{\text{on-shell}}; \quad \mathcal{L}_{\text{on-shell}} = \mathrm{Tr}(F_{\mathrm{SD}}^2)$$

  - Minimal form factor integrand up to five loops.
  - Non-minimal form factor related to Higgs amplitude at two loops.

– van Neerven / Gehrmann, Henn, Huber / Brandhuber, Travaglini, Yang / Boels, Kniehl, Yang / Yang –

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- More generic half-BPS operators:

$$\mathcal{T}_k = \mathrm{Tr}(\phi^k) + \ldots$$

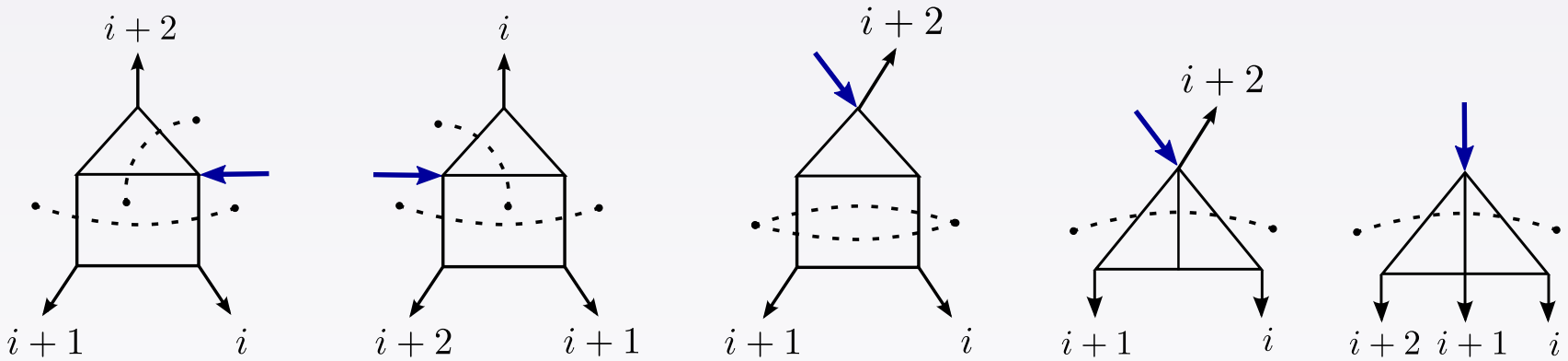
  - Minimal form factors up to two loops, this talk only $k = 3$.

– Bork, Kazakov, Vartanov / Brandhuber, Penante, Travaglni, Wen –

Two-loop minimal form factors of $\mathrm{Tr}(\phi^3)$

Generalised unitarity gives:

$$F_3^{(2)} = -\sum_{i=1}^{3}\Big[I_1(i) + I_2(i) + I_3(i) + I_4(i) - I_5(i)\Big]$$



$I_1(i)$ $I_2(i)$ $I_3(i)$ $I_4(i)$ $I_5(i)$

Reduce to master integrals
[FIRE, LiteRed]

Analylical expressions for all masters
– Gehrmann, Remiddi –

Two-loop minimal form factors of $\mathrm{Tr}(\phi^3)$

L-loop result typically in terms of iterated integrals of weight $\leq 2L$

- Goncharov polylogs

$$G(\{a_1, a_2 \ldots, a_m\}; x) \equiv \int_0^x \frac{dt}{t - a_1} G(\{a_2, \ldots, a_m\}, t)$$

- Classical polylogs:

$$\mathrm{Li}_m(x) \equiv \int_0^x \frac{dt}{t} \mathrm{Li}_{m-1}(t)$$

$$\mathrm{Li}_1(x) \equiv -\log(1-x) = \int_0^x \frac{dt}{1-t} \ .$$

- Functions are related by wild identities

  Ex: DiLog five-term identity

$$\sum_{n=1}^{5} \left[\mathrm{Li}_2(a_n) + \log(a_{n-1}) \log(a_n) \right] = \frac{\pi^2}{6} \,,$$

$$a_1 = x \,, \quad a_2 = \frac{1-x}{1-xy} \,, \quad a_3 = \frac{1-y}{1-xy} \,,$$

$$a_4 = y \,, \quad a_5 = 1 - xy$$

- //`Simplify` doesn't do the job...

Symbols

– Goncharov, Spradlin, Volovich, Vergu –

Map weight-m function $\mapsto$ m-fold tensor product

$$F^{(m)}(x) = \int_0^x d\log[f_1(t_1)] \int_0^{t_1} d\log[f_2(t_2)] \cdots \int_0^{t_{m-1}} d\log[f_m(t_m)]$$

$$\Rightarrow \qquad \mathcal{S}[F^{(m)}(x)] = f_1(x) \otimes f_2(x) \otimes \cdots \otimes f_m(x)$$

Examples:

Function	Symbol
$\log(x)$	x
$\log(x)\log(y)$	$x \otimes y + y \otimes x$
$\mathrm{Li}_n(x)$	$-(1-x) \otimes \underbrace{x \otimes \cdots \otimes x}_{n-1 \text{ times}}$

Properties of symbols

- Symbols behave like "logs"

$$\begin{aligned} \cdots \otimes x\,y \otimes \cdots &= \cdots \otimes x \otimes \cdots + \cdots \otimes y \otimes \cdots \ , \\ \cdots \otimes x^n \otimes \cdots &= n\left(\cdots \otimes x \otimes \cdots\right) \ , \\ \cdots \otimes (\text{constant})\, x \otimes \cdots &= \cdots \otimes x \otimes \cdots \ . \end{aligned}$$

- Identities between transcendental functions become algebraic

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \log(x)\log(1-x) \ = \ \frac{\pi^2}{6}$$

$$\xrightarrow{\mathcal{S}} \ -(1-x)\otimes x - x \otimes (1-x) + x \otimes (1-x) + (1-x)\otimes x \ = \ 0$$

- Some information is lost: $\log\left(e^{i(\theta+2\pi)}\right) \ = \ \log\left(e^{i\theta}\right) + 2\pi i$

Two-loop remainder function for $F_3^{(2)}$

For form factors, a remainder is possible starting at $n = 3$:
– Brandhuber, Travaglini, Yang –

$$\mathcal{R}_3^{(2)} = \mathcal{G}_3^{(2)} - \underbrace{\left[\tfrac{1}{2}(\mathcal{G}_3^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)\mathcal{G}_3^{(1)}(2\epsilon) + C^{(2)}\right]}_{\text{“BDS ansatz”}}$$

$$f^{(2)} = -f_0^{(2)} - f_1^{(2)}\epsilon - f_2^{(2)}\epsilon^2$$

$$\mathcal{G}_3^{(L)} \equiv \frac{F_3^{\mathrm{MHV}(L)}}{F_3^{\mathrm{MHV}(0)}}$$

Finiteness

$$f_1^{(2)} = 2\zeta_3 \sim \Gamma_{\mathrm{coll}}^{(2)}$$

$$f_0^{(2)} = 2\zeta_2 \sim \Gamma_{\mathrm{cusp}}^{(2)}$$

Also: $C^{(2)} = 0,\ f_2 = 2\zeta_4$

Two-loop remainder function $R_3^{(2)}$

- $R_3^{(2)}$ is a combination of thousands of uniform weight four (classical and non-classical) transcendental functions

  Ex: $\mathrm{Li}_4(x)$, $\mathrm{Li}_3(x)\log(y)$, $\mathrm{Li}_2(x)\mathrm{Li}_2(y)$, π^4, etc.

- Arguments depend on the variables

$$u = \frac{s_{12}}{q^2}, \quad v = \frac{s_{23}}{q^2}, \quad w = \frac{s_{31}}{q^2},$$

$$q = p_1 + p_2 + p_3 \quad \Rightarrow \quad u + v + w = 1$$

- Symmetric under permutations of (u, v, w)

- Finite

Simplifying $R_3^{(2)}$

$R_3^{(2)}$ → Very messy

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↓

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↓

$$\mathcal{S}_3^{(2)}(u,v,w) = u \otimes v \otimes \left[\frac{u}{w} \otimes_S \frac{v}{w}\right] + \frac{1}{2} u \otimes \frac{u}{(1-u)^3} \otimes \frac{v}{w} \otimes \frac{v}{w}$$
$$+ \text{ perms}(u,v,w)$$

Simplifying $R_3^{(2)}$

$\mathcal{S}_3^{(2)}$ → Very simple

$\mathcal{S}_3^{(2)}$ → Goncharov's condition $\left\{ \begin{array}{l} \mathcal{S}_{abcd} - \mathcal{S}_{bacd} - \mathcal{S}_{abdc} + \mathcal{S}_{badc} \\ \quad - (a \leftrightarrow c\,,\, b \leftrightarrow d) = 0 \end{array} \right.$

↓

Can be integrated back to classical polylogs

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Can be integrated back to classical polylogs

$$\mathcal{R}_3^{(2)} = -\frac{3}{2}\,\mathrm{Li}_4(u) + \frac{3}{4}\,\mathrm{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{2}\log(w)\mathrm{Li}_3\left(-\frac{u}{v}\right) + \frac{\log^4(u)}{32}$$
$$+\frac{\log^2(u)}{16}\Big[\log^2(v) - 2\log(v)\log(w) + 10\zeta_2\Big]$$
$$-\frac{\log(u)}{4}\Big[\zeta_2\log(v) - 2\zeta_3\Big] + \frac{7}{16}\zeta_4 + \mathrm{perms}(u,v,w)$$

Part 2

Form factors of non-protected operators

Recall: we were interested in form factors of $\mathrm{Tr}(F^3)$.
Idea: Start with scalars and increase complexity

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Overview of form factor computations at two loops:
(Two-loop dilatation operator + finite remainder function)

- $SU(2)$ sector – Loebbert, Nandan, Sieg, Wilhelm, Yang –

$$X := \phi_{14}\,, \quad Y := \phi_{24}$$

- $SU(2|3)$ sector – Brandhuber, Kostacinska, Penante, Travaglini, Young –

$$X := \phi_{12}\,, \quad Y := \phi_{23}\,, \quad Z := \phi_{31}\,, \quad \psi_\alpha := \psi_{123,\alpha}$$

- $SL(2)$ sector – Loebbert, Sieg, Wilhelm, Yang –

$$X := \phi_{12}\,, \quad D^+ := D^\mu \tau_\mu$$

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Form factors in the $SU(2|3)$ sector

- Building blocks:

$$X := \phi_{12}\,, \quad Y := \phi_{23}\,, \quad Z := \phi_{31}\,, \quad \psi_\alpha := \psi_{123,\alpha}$$

- Study operators with bare dimension 3:

$$\mathcal{O}_B := \mathrm{Tr}(X[Y,Z]) \quad \text{and} \quad \mathcal{O}_F = \frac{1}{2}\mathrm{Tr}(\psi^\alpha \psi_\alpha)$$

- $\mathcal{O}_B$ is related to $\mathrm{Tr}(F^3_{\mathrm{ASD}})$ via SUSY, seems like a good place to start.

Mixed form factors up to $\mathcal{O}(g^4)$

$$\mathcal{O}_B := \mathrm{Tr}(X[Y,Z]) \quad \text{and} \quad \mathcal{O}_F = \frac{1}{2}\mathrm{Tr}(\psi^\alpha \psi_\alpha)$$

- Under renormalisation:

$$\begin{pmatrix} \mathcal{O}_F^{\mathrm{ren}} \\ \mathcal{O}_B^{\mathrm{ren}} \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_F{}^F & \mathcal{Z}_F{}^B \\ \mathcal{Z}_B{}^F & \mathcal{Z}_B{}^B \end{pmatrix} \begin{pmatrix} \mathcal{O}_F \\ \mathcal{O}_B \end{pmatrix}$$

- Study form factors up to $\mathcal{O}(g^4)$

$$\langle \bar{\psi}\bar{\psi}|\mathcal{O}_F|0\rangle \quad \langle \bar{\psi}\bar{\psi}|\mathcal{O}_B|0\rangle$$

$$\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_F|0\rangle \quad \langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$$

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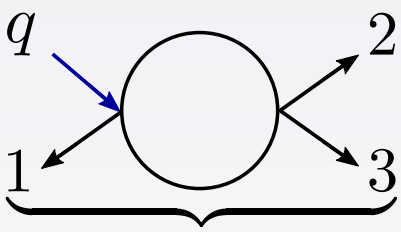
$$\langle \bar{\psi}\bar{\psi}|\mathcal{O}_F|0\rangle \quad \langle \bar{\psi}\bar{\psi}|\mathcal{O}_B|0\rangle$$

$$\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_F|0\rangle \quad \langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$$

$\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$ at one loop

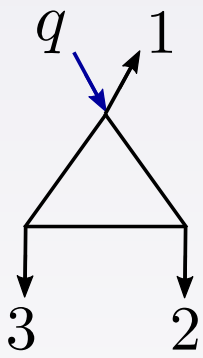
Unitarity cuts:

$F^{(1)}_{\mathcal{O}_B}(1^{\bar{X}}, 2^{\bar{Y}}, 3^{\bar{Z}}; q) \;=\; 2\,i \times$



UV divergent

$+i\,s_{23} \times$



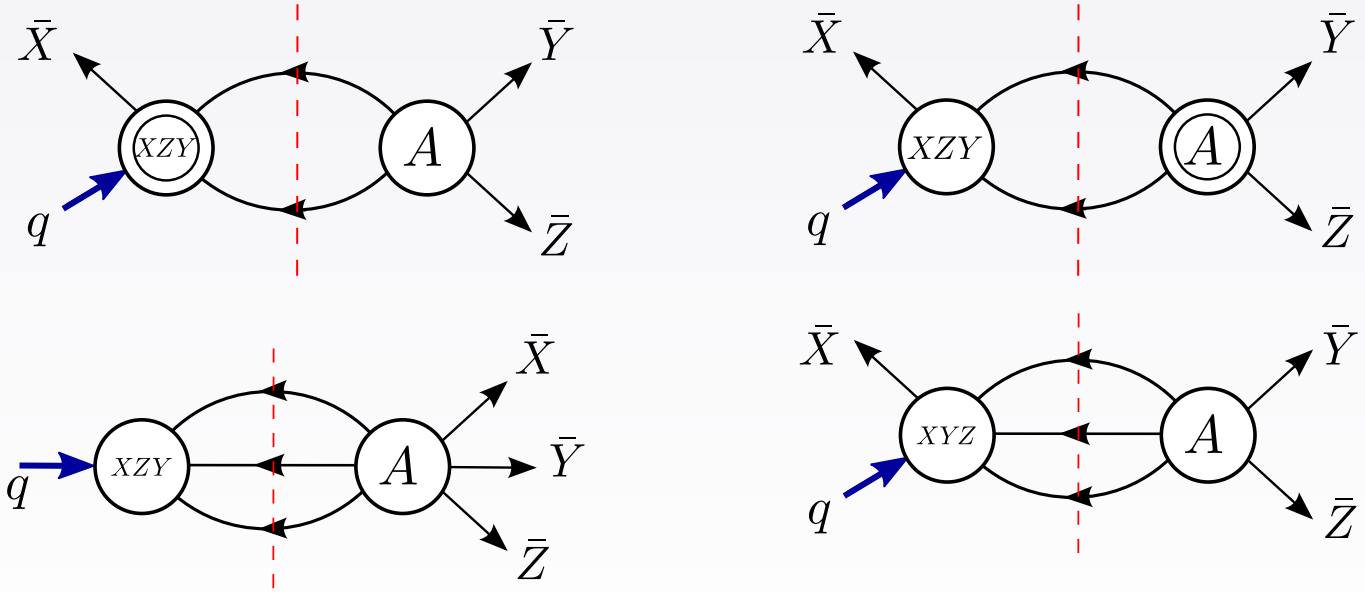
$+\,\text{cyclic}(1,2,3)$

- One-loop anomalous dimension: $\gamma^{(1)} = 12\left(\dfrac{g^2 N}{(4\pi)^2}\right)$

$\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$ at two loops

$$\underbrace{\mathrm{Tr}X[Y,Z]}_{\mathcal{O}_B} = \underbrace{\mathrm{Tr}X\{Y,Z\}}_{\text{Protected, }\mathcal{O}_{\mathrm{BPS}}} \underbrace{-2\mathrm{Tr}XZY}_{\mathcal{O}_{\mathrm{offset}}}$$

- For $\mathcal{O}_{\mathrm{BPS}}$, same integrand as $\mathrm{Tr}(\phi^3)$ ✓
- Compute only $\mathcal{O}_{\mathrm{offset}}$ using cuts as before:



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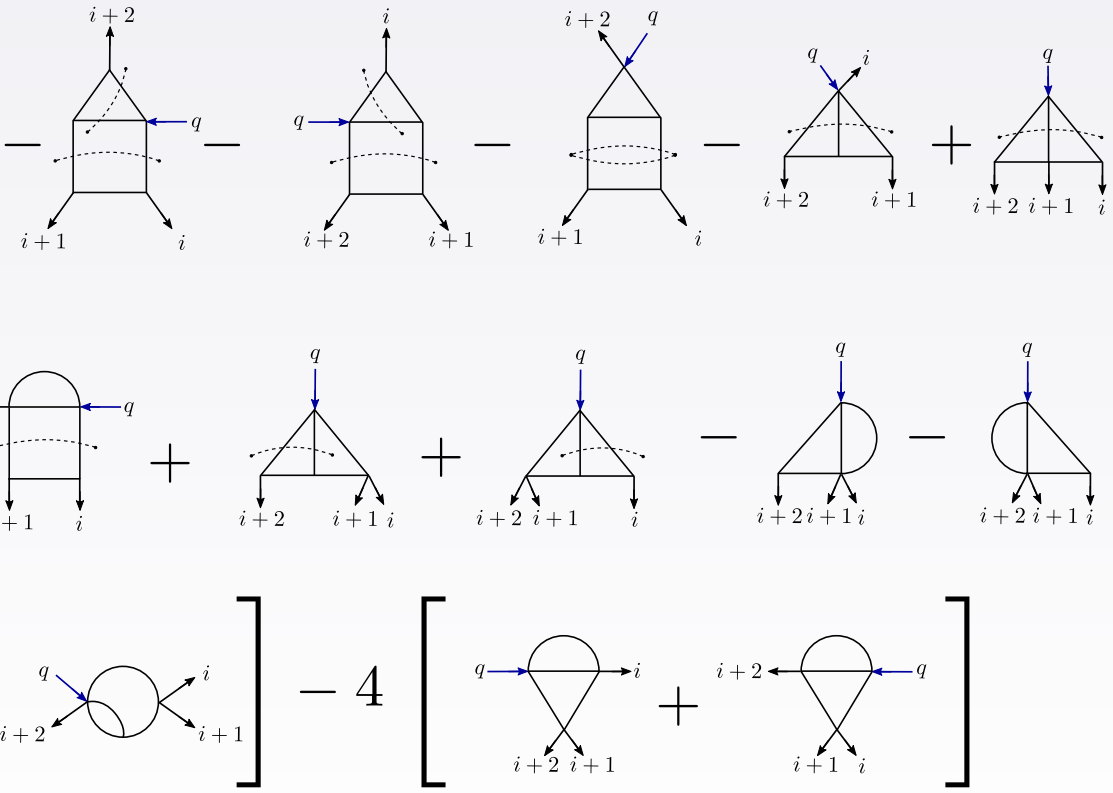
$F^{(2)}_{\mathrm{Tr}(X[Y,Z])}(1^{\bar{X}}, 2^{\bar{Y}}, 3^{\bar{Z}}; q) = \sum_{i=1}^{3}$



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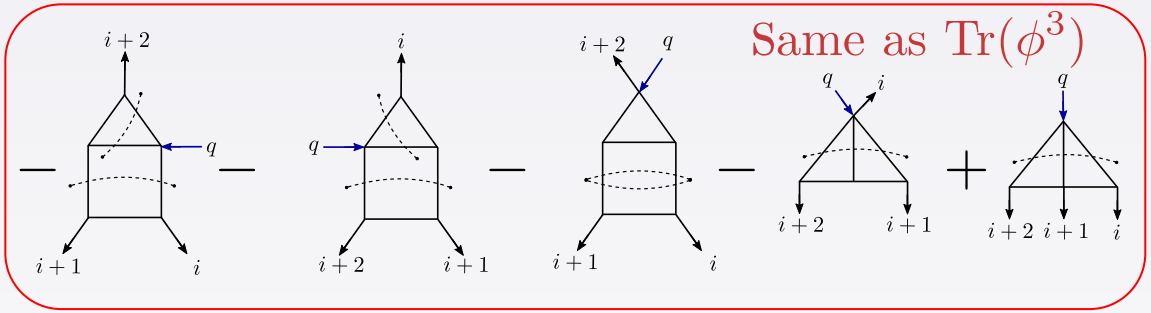
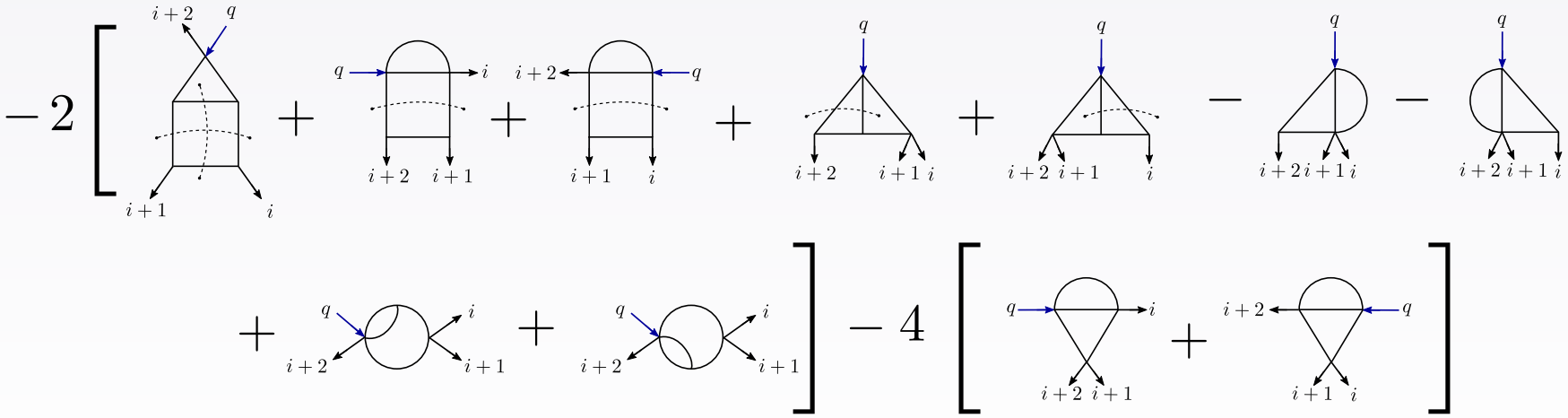




$\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$ two-loop remainder

Generic remainder of operator $\mathcal{O}$:

$$\mathcal{R}^{(2)}_{\mathcal{O}} \equiv F^{(2)}_{\mathcal{O}}(\epsilon) - \frac{1}{2}\big(F^{(1)}_{\mathcal{O}}(\epsilon)\big)^2 - f^{(2)}(\epsilon)\, F^{(1)}_{\mathcal{O}}(2\epsilon) - C^{(2)}$$

Using $F_{\mathcal{O}_B} = F_{\mathcal{O}_{\text{BPS}}} + F_{\mathcal{O}_{\text{offset}}}$:

$$\mathcal{R}^{(2)}_{\mathcal{O}_B} = \underbrace{\mathcal{R}^{(2)}_{\text{BPS}}}_{\text{known}\ \checkmark} + \mathcal{R}^{(2)}_{\text{non-BPS}}$$

$$\mathcal{R}^{(2)}_{\text{non-BPS}} = F^{(2)}_{\mathcal{O}_{\text{offset}}}(\epsilon) - F^{(1)}_{\mathcal{O}_{\text{offset}}}\left(\frac{1}{2}F^{(1)}_{\mathcal{O}_{\text{offset}}} + F^{(1)}_{\text{BPS}}\right)(\epsilon) - f^{(2)}(\epsilon)\, F^{(1)}_{\mathcal{O}_{\text{offset}}}(2\epsilon)$$

$\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$ two-loop remainder

- $\mathcal{R}_{\text{non-BPS}}$ has transcendentality ≤ 3
  $\rightsquigarrow$ Same as observed in $SU(2)$ and $SL(2)$ sectors

  Goncharov: At transcendentality ≤ 3 only classical polylogs needed

  $\Rightarrow$ Maximally transcendental part universal $= \mathcal{R}_{\text{BPS}}$

- UV divergence

$$\mathcal{R}^{(2)}_{\text{non-BPS}} = \frac{c_{\text{UV}}}{\epsilon} + \sum_{i=0}^{3} \mathcal{R}^{(2)}_{\text{non-BPS};3-i}, \quad c_{\text{UV}} = 18$$

$\langle \bar{X}\bar{Y}\bar{Z}|\mathcal{O}_B|0\rangle$ two-loop remainder

- Lower transcendentality parts of remainder:

$$
\begin{aligned}
\mathcal{R}^{(2)}_{\text{non-BPS};3} &= 2\Big[\mathrm{Li}_3(u) + \mathrm{Li}_3(1-u)\Big] - \frac{1}{2}\log^2(u)\log\frac{vw}{(1-u)^2} \\
&\quad + \frac{2}{3}\log(u)\log(v)\log(w) + \frac{2}{3}\zeta_3 + \text{perms}\,(u,v,w) \\
\mathcal{R}^{(2)}_{\text{non-BPS};2} &= -12\Big[\mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w)\Big] - 2\log^2(uvw) + 36\,\zeta_2 \\
\mathcal{R}^{(2)}_{\text{non-BPS};1} &= -12\,\log(uvw) \\
\mathcal{R}^{(2)}_{\text{non-BPS};0} &= 126
\end{aligned}
$$

Connection with remainder densities in the $SU(2)$ sector

– Loebbert, Nandan, Sieg, Wilhelm, Yang –

- $X = \phi_{14},\ Y = \phi_{24}, \quad \mathcal{O} = \mathrm{Tr}(XYXY\cdots YYX)$
- Basis of remainder densities:

$$\underbrace{\left(R_i^{(2)}\right)^{XXX}_{XXX}}_{\text{BPS}},\ \underbrace{\left(R_i^{(2)}\right)^{XYX}_{XXY},\ \left(R_i^{(2)}\right)^{YXX}_{XXY}}_{\text{transcendentality } \leq 3}$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};3} = -\sum_{S_3}\left(R_i^{(2)}\right)^{XYX}_{XXY}\Big|_3 \ + \ 6\,\zeta_3\ ,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};2} = -\sum_{S_3}\left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_2 \ + \ 5\pi^2\ ,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};1} = -\sum_{S_3}\left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_1\ ,$$

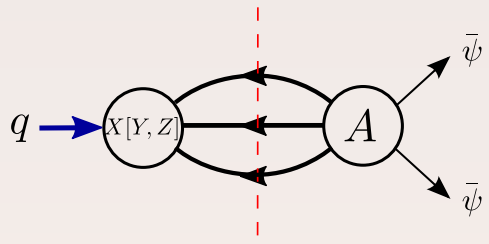
$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};0} = -\sum_{S_3}\left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_0$$

The other terms in the form factor matrix are simpler

Off-diagonal terms non-vanishing only at loop level
(length-changing interactions)

SUSY $Q_3^\alpha Q_{3\alpha}$ relates
$\mathrm{Tr}(\phi_{12}^2)$ to $\mathcal{O}_{\mathrm{BPS}'} = \mathcal{O}_F + g\mathcal{O}_B$
Recycle BPS computation

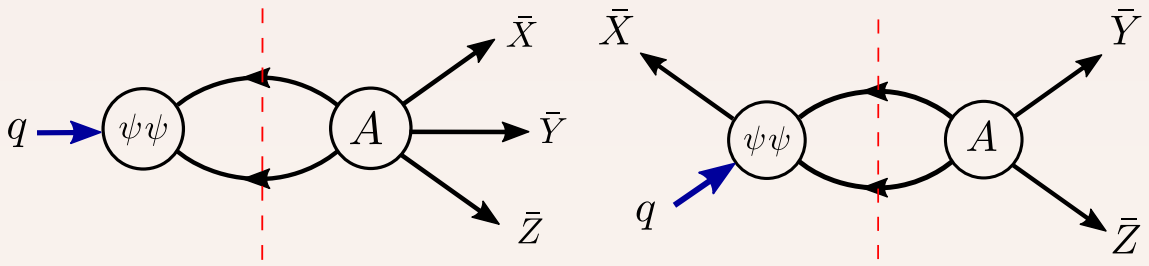
Sub-minimal



$\langle\bar\psi\bar\psi|\mathcal{O}_F|0\rangle$ $\langle\bar\psi\bar\psi|\mathcal{O}_B|0\rangle$

$\langle\bar X\bar Y\bar Z|\mathcal{O}_F|0\rangle$ $\langle\bar X\bar Y\bar Z|\mathcal{O}_B|0\rangle$

Next-to-minimal



Dilatation Operator

- From UV divergences obtain two-loop the dilatation operator:
  – Beisert –

$$\mathcal{O}^{\text{ren}} = \mathcal{Z} \cdot \mathcal{O}^{\text{bare}} \quad \Rightarrow \quad \delta\mathfrak{D} = -\mu_R \frac{\partial}{\partial \mu_R} \log \mathcal{Z}$$

$$\mathcal{Z} = \mathbb{1} + \delta\mathcal{Z} \qquad g(\mu_R) \sim g\mu_R^{-\epsilon}$$

- Eigenstates – Eden, Jarczak, Sokatchev, Stanev / Eden –

$$\begin{cases} \mathcal{O}_F + g\mathcal{O}_B, & \gamma = 0 \\ \mathcal{O}_B - \dfrac{gN}{8\pi^2}\mathcal{O}_F, & \gamma = 12\left(\frac{g^2N}{(4\pi)^2}\right) - 48\left(\frac{g^2N}{(4\pi)^2}\right)^2 + \mathcal{O}(g^6) \end{cases}$$

Summary

- Form factors of protected operators, two-loop remainder very simple and of uniform transcendentality.
- Form factors of

$$\mathcal{O}_B = \mathrm{Tr}(X[Y,Z]) \text{ and } \mathcal{O}_F = \frac{1}{2}\mathrm{Tr}(\psi^\alpha \psi_\alpha)$$

  with external states $\langle \bar{X}\bar{Y}\bar{Z}|$ and $\langle \bar{\psi}\bar{\psi}|$ up to $\mathcal{O}(g^4)$

- Two-loop remainder function of $F_{\mathcal{O}_B}(\bar{X},\bar{Y},\bar{Z};q)$ can be decomposed into BPS part (maximal transc.) and a correction which is of transc. ≤ 3
  - Maximally transcendental part universal
  - Lower transcendentality factors related to remainder densities in $SU(2)$ sector
- UV-divergences, renormalisation and dilatation operator

Work in progress and outlook

- $SL(2)$ computation of Loebbert, Sieg, Wilhelm and Yang already signaled the complexity due to allowing operators with derivatives. Also concept of "hidden uniform transcendentality".
- Partial results for $F_{\mathrm{Tr}(F^3)}(g, g, g; q)$ using unitarity and color-kinematics duality.
- Analogous computation in QCD for comparison.
- Universal structures also for lower transcendentality?
- Two loop dilatation operator in $\mathcal{N} = 4$ SYM not completely known, perhaps on-shell methods can be instrumental.

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Thank you for your attention!