

Form factor remainders: from $\mathcal{N} = 4$ SYM to QCD

DESY - Nov 29th, 2016
GATIS closing workshop highlights

Based on:

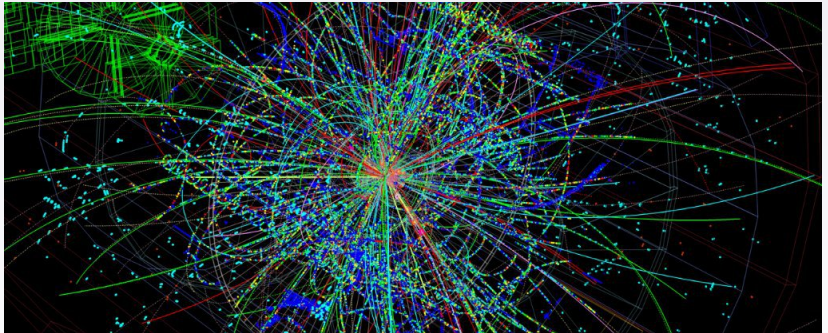
hep-th/ {1406.1443, 1606.08682 + in progress}

In collaboration with:

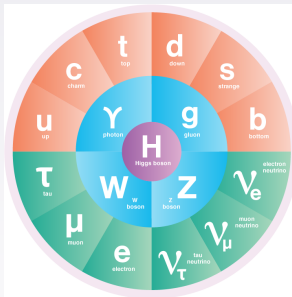
Andreas Brandhuber, Martyna Kostacinska,
Gabriele Travaglini, Congkao Wen and Donovan Young

Brenda Penante
Humboldt University of Berlin

Scattering amplitudes

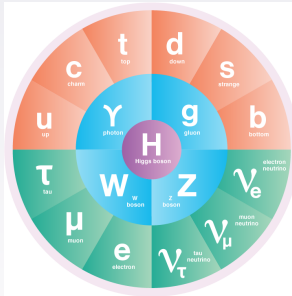


Approach #1:



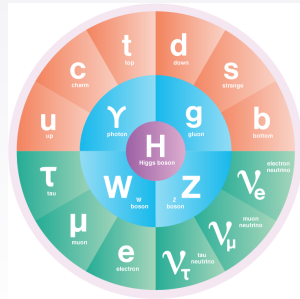
?

Approach #1:



?

Approach #2:



This talk



Planar $\mathcal{N} = 4$ super Yang-Mills

Aim:

- Learn as much as possible from our toy model
- Generalise methods to less special quantities
 - Off-shell quantities (form factors, correlation functions)
 - Away from planar limit
- Find links with the SM (in particular QCD)

Overview of tree amplitudes in $\mathcal{N} = 4$ SYM

What about QCD?

$$\left(\begin{array}{c} g_n \nearrow \\ \uparrow \\ \text{A} \\ \searrow \\ g_1 \end{array} \begin{array}{c} \nearrow \cdot \\ \cdot \\ \searrow \cdot \\ \cdot \end{array} \right)_{\text{QCD}}^{\text{tree}} = \left(\begin{array}{c} g_n \nearrow \\ \uparrow \\ \text{A} \\ \searrow \\ g_1 \end{array} \begin{array}{c} \nearrow \cdot \\ \cdot \\ \searrow \cdot \\ \cdot \end{array} \right)_{\text{SYM}}^{\text{tree}}$$

The diagram shows an equality between two tree-level amplitudes. On the left, the amplitude is labeled 'QCD' and features a central grey circle labeled 'A'. Five external lines are shown: an incoming line from the bottom labeled g_1 , an outgoing line to the top labeled g_n , an outgoing line to the right labeled g_3 , and two unlabeled outgoing lines (one to the top-right and one to the bottom-right). On the right, the amplitude is labeled 'SYM' and has the same central circle 'A' and external line configuration. An equals sign is placed between the two expressions.

Overview of tree amplitudes in $\mathcal{N} = 4$ SYM

What about QCD?

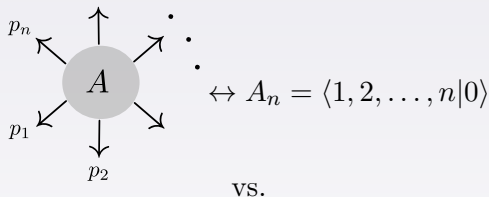
$$\left(\begin{array}{c} g_n \\ \uparrow \\ \text{tree} \\ \swarrow \quad \downarrow \quad \searrow \\ g_1 \quad A \quad g_3 \\ \downarrow \\ g_2 \end{array} \right)_{\text{QCD}} = \left(\begin{array}{c} g_n \\ \uparrow \\ \text{tree} \\ \swarrow \quad \downarrow \quad \searrow \\ g_1 \quad A \quad g_3 \\ \downarrow \\ g_2 \end{array} \right)_{\text{SYM}}$$

Not extremely useful

- Amplitudes involving states in QCD $\notin \mathcal{N} = 4$ SYM
- Loops

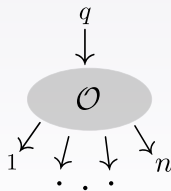
... let's look at a more general observable

Form factors



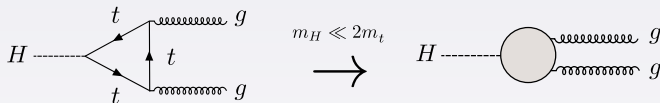
$$F_{\mathcal{O}}(1, \dots, n; q) \equiv \int d^4x \, e^{-iqx} \langle 1, \dots, n | \mathcal{O}(x) | 0 \rangle$$

$$p_1^2 = \dots = p_n^2 = 0 \rightsquigarrow p_i = \lambda^i \tilde{\lambda}^i, \quad q^2 \neq 0$$



Form factors as effective vertices: Higgs plus multi-gluon

– Gehrman, Jaquier, Glover, Koukoutsakis –



Effective interaction: $\mathcal{L}_{\text{eff}} = H\text{Tr}(F^2)$

$H \rightarrow gg \cdots g$ given by form factor:

$$\langle gg \cdots g | \int d^4x e^{-i\textcolor{brown}{q}x} \text{Tr}(F^2)(x) | 0 \rangle \Big|_{\textcolor{brown}{q}^2 = m_H^2}$$

Meanwhile in $\mathcal{N} = 4$ SYM:

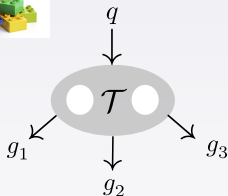
Chiral part of stress tensor multiplet:

$$\mathcal{T} = \text{Tr}(\phi^2) + \cdots + (\theta)^4 \mathcal{L}_{\text{on-shell}}; \quad \mathcal{L}_{\text{on-shell}} = \text{Tr}(F_{\text{SD}}^2)$$

Meanwhile in $\mathcal{N} = 4$ SYM:

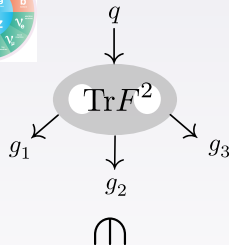
Chiral part of stress tensor multiplet:

$$\mathcal{T} = \text{Tr}(\phi^2) + \cdots + (\theta)^4 \mathcal{L}_{\text{on-shell}}; \quad \mathcal{L}_{\text{on-shell}} = \text{Tr}(F_{\text{SD}}^2)$$



“Li₄” only

(“Maximal transcendentality”)
– Brandhuber, Travaglini, Yang –



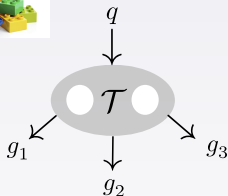
“Li₄” + “Li₃” + “Li₂”

+ “log” + rational
– Gehrmann, Jaquier,
Glover, Koukoutsakis –

Meanwhile in $\mathcal{N} = 4$ SYM:

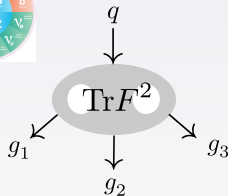
Chiral part of stress tensor multiplet:

$$\mathcal{T} = \text{Tr}(\phi^2) + \cdots + (\theta)^4 \mathcal{L}_{\text{on-shell}}; \quad \mathcal{L}_{\text{on-shell}} = \text{Tr}(F_{\text{SD}}^2)$$



“Li₄” only

(“Maximal transcendentality”)
– Brandhuber, Travaglini, Yang –



“Li₄” + “Li₃” + “Li₂”

+ “log” + rational
– Gehrmann, Jaquier,
Glover, Koukoutsakis –

$$\text{“Li}_4\text{”}|_{\text{QCD}} = \text{“Li}_4\text{”}|_{\mathcal{N}=4 \text{ SYM}}$$

Motivation 1: Is it possible to extend the connection?

The effective Lagrangian approach goes further:

– Neill / Dawson, Lewis, Zeng –

$$\mathcal{L}_{\text{eff}} = H\text{Tr}(F^2) + \frac{1}{m_{\text{top}}^2} \sum_{i=1}^4 c_i O_i + \mathcal{O}\left(\frac{1}{m_{\text{top}}^4}\right)$$

O_i : dimension 7 operators.

Consider for instance $O_1 = H\text{Tr}(F^3) = H\text{Tr}(F_{\text{SD}}^3 + F_{\text{ASD}}^3)$

Motivation 1: Is it possible to extend the connection?

The effective Lagrangian approach goes further:

– Neill / Dawson, Lewis, Zeng –

$$\mathcal{L}_{\text{eff}} = H\text{Tr}(F^2) + \frac{1}{m_{\text{top}}^2} \sum_{i=1}^4 c_i O_i + \mathcal{O}\left(\frac{1}{m_{\text{top}}^4}\right)$$

O_i : dimension 7 operators.

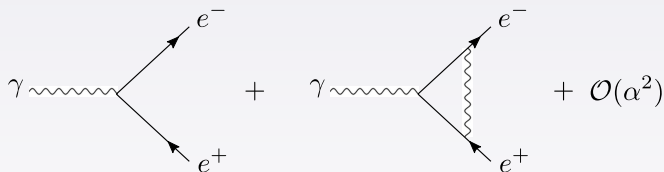
Consider for instance $O_1 = H\text{Tr}(F^3) = H\text{Tr}(F_{\text{SD}}^3 + F_{\text{ASD}}^3)$

Q: Is there a relation between the max. trans. part of these form factors in $\mathcal{N} = 4$ SYM and QCD? Are they identical?

Q: Can we find universal structures which are invariant among different form factors and theories?

Motivation 2: Hidden simplicity of form factors

QED - electron anomalous magnetic moment:



- At $\mathcal{O}(\alpha^3) \sim 70$ diagrams ± 10 and ± 100 , but result is $\mathcal{O}(1)$
 - Schwinger / Cvitanovic, Kinoshita / Laporta, Remiddi –

Motivation 2: Hidden simplicity of form factors

Ex: $\text{Tr}(\phi^2)$, $\phi \equiv \phi^{12}$ in $\mathcal{N} = 4$ SYM

■ MHV amplitudes:

– Parke, Taylor / Mangano, Parke –

$$A(i^-, j^-, \{g^+\}) = \frac{\langle i j \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle} \delta^{(4)}\left(\sum p_i\right)$$

■ MHV form factors:

– Brandhuber, Gurdogan, Mooney, Travaglini, Yang –

$$F_{\text{Tr}(\phi^2)}(i^\phi, j^\phi, \{g^+\}; q) = \frac{\langle i j \rangle^2}{\langle 12 \rangle \cdots \langle n1 \rangle} \delta^{(4)}\left(q - \sum p_i\right)$$

Q: How far can we extend the scope of applicability of *on-shell methods* for form factors?

Part 1: Very quick review of form factors of protected operators

Part 2: Form factors of non-protected ops in the $SU(2|3)$ sector

- Two-loop remainder function
- Mixing + dilatation operator

Part 3: Work in progress + speculations

Part 1

Overview of form factors of protected operators

Protected operators: no anomalous dimension, no mixing, no renormalisation.

- Chiral part of stress tensor multiplet:

$$\mathcal{T}_2 = \text{Tr}(\phi^2) + \cdots + (\theta)^4 \mathcal{L}_{\text{on-shell}}; \quad \mathcal{L}_{\text{on-shell}} = \text{Tr}(F_{\text{SD}}^2)$$

- Minimal form factor integrand up to five loops.
- Non-minimal form factor related to Higgs amplitude at two loops.

– van Neerven / Gehrmann, Henn, Huber / Brandhuber, Travaglini, Yang / Boels, Kniehl, Yang / Yang –

Overview of form factors of protected operators

Protected operators: no anomalous dimension, no mixing, no renormalisation.

- Chiral part of stress tensor multiplet:

$$\mathcal{T}_2 = \text{Tr}(\phi^2) + \dots + (\theta)^4 \mathcal{L}_{\text{on-shell}}; \quad \mathcal{L}_{\text{on-shell}} = \text{Tr}(F_{\text{SD}}^2)$$

- Minimal form factor integrand up to five loops.
- Non-minimal form factor related to Higgs amplitude at two loops.

– van Neerven / Gehrmann, Henn, Huber / Brandhuber, Travaglini, Yang / Boels, Kniehl, Yang / Yang –

- More generic half-BPS operators:

$$\mathcal{T}_k = \text{Tr}(\phi^k) + \dots$$

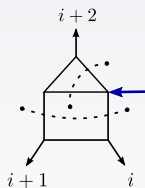
- Minimal form factors up to two loops, this talk only $k = 3$.

– Bork, Kazakov, Vartanov / Brandhuber, Penante, Travaglini, Wen –

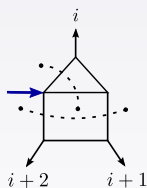
Two-loop minimal form factors of $\text{Tr}(\phi^3)$

Generalised unitarity gives:

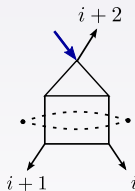
$$F_3^{(2)} = - \sum_{i=1}^3 \left[I_1(i) + I_2(i) + I_3(i) + I_4(i) - I_5(i) \right]$$



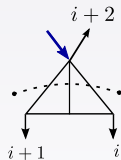
$I_1(i)$



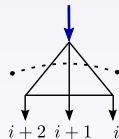
$I_2(i)$



$I_3(i)$



$I_4(i)$



$I_5(i)$

Reduce to master
integrals
[FIRE, LiteRed]

Analytical expressions
for all masters
– Gehrmann, Remiddi –

Two-loop minimal form factors of $\text{Tr}(\phi^3)$

L -loop result typically in terms of iterated integrals of weight $\leq 2L$

- Goncharov polylogs

$$G(\{a_1, a_2, \dots, a_m\}; x) \equiv \int_0^x \frac{dt}{t - a_1} G(\{a_2, \dots, a_m\}, t)$$

- Classical polylogs:

$$\text{Li}_m(x) \equiv \int_0^x \frac{dt}{t} \text{Li}_{m-1}(t)$$

$$\text{Li}_1(x) \equiv -\log(1 - x) = \int_0^x \frac{dt}{1 - t} .$$

- Functions are related by wild identities

Ex: DiLog five-term identity

$$\sum_{n=1}^5 \left[\text{Li}_2(a_n) + \log(a_{n-1}) \log(a_n) \right] = \frac{\pi^2}{6} ,$$
$$a_1 = x , \quad a_2 = \frac{1-x}{1-xy} , \quad a_3 = \frac{1-y}{1-xy} ,$$
$$a_4 = y , \quad a_5 = 1-xy$$

- //Simplify doesn't do the job...

– Goncharov, Spradlin, Volovich, Vergu –

Map weight- m function $\mapsto m$ -fold tensor product

$$F^{(m)}(x) = \int_0^x d\log[f_1(t_1)] \int_0^{t_1} d\log[f_2(t_2)] \cdots \int_0^{t_{m-1}} d\log[f_m(t_m)]$$
$$\Rightarrow \mathcal{S}[F^{(m)}(x)] = f_1(x) \otimes f_2(x) \otimes \cdots \otimes f_m(x)$$

Examples:

| Function | Symbol |
|-------------------|--|
| $\log(x)$ | x |
| $\log(x) \log(y)$ | $x \otimes y + y \otimes x$ |
| $\text{Li}_n(x)$ | $-(1-x) \otimes \underbrace{x \otimes \cdots \otimes x}_{n-1 \text{ times}}$ |

Properties of symbols

- Symbols behave like “logs”

$$\cdots \otimes x y \otimes \cdots = \cdots \otimes x \otimes \cdots + \cdots \otimes y \otimes \cdots ,$$

$$\cdots \otimes x^n \otimes \cdots = n (\cdots \otimes x \otimes \cdots) ,$$

$$\cdots \otimes (\text{constant}) x \otimes \cdots = \cdots \otimes x \otimes \cdots .$$

- Identities between transcendental functions become **algebraic**

$$\text{Li}_2(x) + \text{Li}_2(1-x) + \log(x) \log(1-x) = \frac{\pi^2}{6}$$

$$\xrightarrow{\mathcal{S}} -(1-x) \otimes x - x \otimes (1-x) + x \otimes (1-x) + (1-x) \otimes x = 0$$

- Some information is lost: $\log(e^{i(\theta+2\pi)}) = \log(e^{i\theta}) + 2\pi i$

Two-loop remainder function for $F_3^{(2)}$

For form factors, a remainder is possible starting at $n = 3$:

– Brandhuber, Travaglini, Yang –

$$\mathcal{R}_3^{(2)} = \mathcal{G}_3^{(2)} - \underbrace{\left[\frac{1}{2}(\mathcal{G}_3^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)\mathcal{G}_3^{(1)}(2\epsilon) + C^{(2)} \right]}_{\text{“BDS ansatz”}}$$

$$f^{(2)} = -f_0^{(2)} - f_1^{(2)}\epsilon - f_2^{(2)}\epsilon^2$$

$$\mathcal{G}_3^{(L)} \equiv \frac{F_3^{\text{MHV}(L)}}{F_3^{\text{MHV}(0)}}$$

Finiteness

$$f_1^{(2)} = 2\zeta_3 \sim \Gamma_{\text{coll}}^{(2)}$$

$$f_0^{(2)} = 2\zeta_2 \sim \Gamma_{\text{cusp}}^{(2)}$$

Also: $C^{(2)} = 0$, $f_2 = 2\zeta_4$

Two-loop remainder function $R_3^{(2)}$

- $R_3^{(2)}$ is a combination of thousands of **uniform weight four** (classical and non-classical) transcendental functions

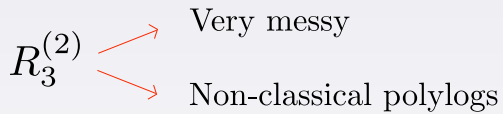
Ex: $\text{Li}_4(x)$, $\text{Li}_3(x) \log(y)$, $\text{Li}_2(x)\text{Li}_2(y)$, π^4 , etc.

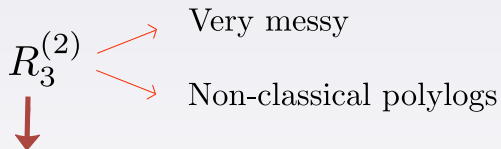
- Arguments depend on the variables

$$u = \frac{s_{12}}{q^2}, \quad v = \frac{s_{23}}{q^2}, \quad w = \frac{s_{31}}{q^2},$$

$$q = p_1 + p_2 + p_3 \quad \Rightarrow \quad u + v + w = 1$$

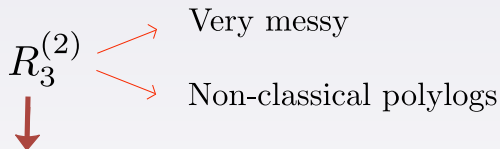
- Symmetric under permutations of (u, v, w)
- Finite






Symbol \longrightarrow Transc. m function $\rightarrow m$ -fold \otimes product

Simplifying $R_3^{(2)}$



Symbol  Transc. m function $\rightarrow m$ -fold \otimes product




$$\mathcal{S}_3^{(2)}(u, v, w) = \textcolor{red}{u} \otimes v \otimes \left[\frac{u}{w} \otimes_S \frac{v}{w} \right] + \frac{1}{2} \textcolor{red}{u} \otimes \frac{u}{(1-u)^3} \otimes \frac{v}{w} \otimes \frac{v}{w} \\ + \text{ perms}(u, v, w)$$

$$\mathcal{S}_3^{(2)} \begin{cases} \rightarrow \text{Very simple} \\ \rightarrow \text{Goncharov's condition} \end{cases} \left\{ \begin{array}{l} \mathcal{S}_{abcd} - \mathcal{S}_{bacd} - \mathcal{S}_{abdc} + \mathcal{S}_{badc} \\ - (a \leftrightarrow c, b \leftrightarrow d) = 0 \end{array} \right.$$

Can be integrated back to **classical** polylogs

Simplifying $R_3^{(2)}$

$$\mathcal{S}_3^{(2)} \begin{cases} \rightarrow \text{Very simple} \\ \rightarrow \text{Goncharov's condition} \end{cases} \left\{ \begin{array}{l} \mathcal{S}_{abcd} - \mathcal{S}_{bacd} - \mathcal{S}_{abdc} + \mathcal{S}_{badc} \\ - (a \leftrightarrow c, b \leftrightarrow d) = 0 \end{array} \right.$$


Can be integrated back to **classical** polylogs

$$\begin{aligned} \mathcal{R}_3^{(2)} = & -\frac{3}{2} \text{Li}_4(u) + \frac{3}{4} \text{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{2} \log(w) \text{Li}_3\left(-\frac{u}{v}\right) + \frac{\log^4(u)}{32} \\ & + \frac{\log^2(u)}{16} \left[\log^2(v) - 2 \log(v) \log(w) + 10 \zeta_2 \right] \\ & - \frac{\log(u)}{4} \left[\zeta_2 \log(v) - 2 \zeta_3 \right] + \frac{7}{16} \zeta_4 + \text{perms}(u, v, w) \end{aligned}$$

Part 2

Form factors of non-protected operators

Recall: we were interested in form factors of $\text{Tr}(F^3)$.

Idea: Start with scalars and increase complexity

Form factors of non-protected operators

Recall: we were interested in form factors of $\text{Tr}(F^3)$.

Idea: Start with scalars and increase complexity

Overview of form factor computations at two loops:
(Two-loop dilatation operator + finite remainder function)

- $SU(2)$ sector – [Loebbert, Nandan, Sieg, Wilhelm, Yang](#) –

$$X := \phi_{14}, \quad Y := \phi_{24}$$

- $SU(2|3)$ sector – [Brandhuber, Kostacinska, Penante, Travaglini, Young](#) –

$$X := \phi_{12}, \quad Y := \phi_{23}, \quad Z := \phi_{31}, \quad \psi_\alpha := \psi_{123,\alpha}$$

- $SL(2)$ sector – [Loebbert, Sieg, Wilhelm, Yang](#) –

$$X := \phi_{12}, \quad D^+ := D^\mu \tau_\mu$$

Form factors of non-protected operators

Recall: we were interested in form factors of $\text{Tr}(F^3)$.

Idea: Start with scalars and increase complexity

Overview of form factor computations at two loops:
(Two-loop dilatation operator + finite remainder function)

- $SU(2)$ sector – Loebbert, Nandan, Sieg, Wilhelm, Yang –

$$X := \phi_{14}, \quad Y := \phi_{24}$$

- $SU(2|3)$ sector – Brandhuber, Kostacinska, Penante, Travaglini, Young –

$$X := \phi_{12}, \quad Y := \phi_{23}, \quad Z := \phi_{31}, \quad \psi_\alpha := \psi_{123,\alpha}$$

- $SL(2)$ sector – Loebbert, Sieg, Wilhelm, Yang –

$$X := \phi_{12}, \quad D^+ := D^\mu \tau_\mu$$

- Building blocks:

$$X := \phi_{12}, \quad Y := \phi_{23}, \quad Z := \phi_{31}, \quad \psi_\alpha := \psi_{123,\alpha}$$

- Study operators with bare dimension 3:

$$\mathcal{O}_B := \text{Tr}(X[Y, Z]) \quad \text{and} \quad \mathcal{O}_F = \frac{1}{2} \text{Tr}(\psi^\alpha \psi_\alpha)$$

- \mathcal{O}_B is related to $\text{Tr}(F_{\text{ASD}}^3)$ via SUSY, seems like a good place to start.

Mixed form factors up to $\mathcal{O}(g^4)$

$$\mathcal{O}_B := \text{Tr}(X[Y, Z]) \quad \text{and} \quad \mathcal{O}_F = \frac{1}{2} \text{Tr}(\psi^\alpha \psi_\alpha)$$

- Under renormalisation:

$$\begin{pmatrix} \mathcal{O}_F^{\text{ren}} \\ \mathcal{O}_B^{\text{ren}} \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_F^F & \mathcal{Z}_F^B \\ \mathcal{Z}_B^F & \mathcal{Z}_B^B \end{pmatrix} \begin{pmatrix} \mathcal{O}_F \\ \mathcal{O}_B \end{pmatrix}$$

- Study form factors up to $\mathcal{O}(g^4)$

$$\begin{aligned} \langle \bar{\psi} \bar{\psi} | \mathcal{O}_F | 0 \rangle & \quad \langle \bar{\psi} \bar{\psi} | \mathcal{O}_B | 0 \rangle \\ \langle \bar{X} \bar{Y} \bar{Z} | \mathcal{O}_F | 0 \rangle & \quad \langle \bar{X} \bar{Y} \bar{Z} | \mathcal{O}_B | 0 \rangle \end{aligned}$$

Mixed form factors up to $\mathcal{O}(g^4)$

$$\mathcal{O}_B := \text{Tr}(X[Y, Z]) \quad \text{and} \quad \mathcal{O}_F = \frac{1}{2} \text{Tr}(\psi^\alpha \psi_\alpha)$$

- Under renormalisation:

$$\begin{pmatrix} \mathcal{O}_F^{\text{ren}} \\ \mathcal{O}_B^{\text{ren}} \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_F^F & \mathcal{Z}_F^B \\ \mathcal{Z}_B^F & \mathcal{Z}_B^B \end{pmatrix} \begin{pmatrix} \mathcal{O}_F \\ \mathcal{O}_B \end{pmatrix}$$

- Study form factors up to $\mathcal{O}(g^4)$

$$\begin{aligned} \langle \bar{\psi} \psi | \mathcal{O}_F | 0 \rangle & \quad \langle \bar{\psi} \psi | \mathcal{O}_B | 0 \rangle \\ \langle \bar{X} \bar{Y} \bar{Z} | \mathcal{O}_F | 0 \rangle & \quad \langle \bar{X} \bar{Y} \bar{Z} | \mathcal{O}_B | 0 \rangle \end{aligned}$$

$\langle \bar{X}\bar{Y}\bar{Z} | \mathcal{O}_B | 0 \rangle$ at one loop

Unitarity cuts:

$$F_{\mathcal{O}_B}^{(1)}(1^{\bar{X}}, 2^{\bar{Y}}, 3^{\bar{Z}}; q) = 2i \times \underbrace{\text{Diagram 1}}_{\text{UV divergent}} + i s_{23} \times \text{Diagram 2} + \text{cyclic}(1, 2, 3)$$

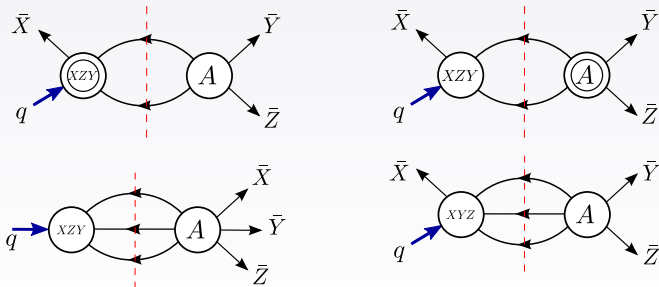
The first diagram is a circle with an incoming blue arrow labeled q and two outgoing arrows labeled 1 and 3. The second diagram is a triangle with an incoming blue arrow labeled q and three outgoing arrows labeled 1, 2, and 3.

- One-loop anomalous dimension: $\gamma^{(1)} = 12 \left(\frac{g^2 N}{(4\pi)^2} \right)$

$\langle \bar{X}\bar{Y}\bar{Z} | \mathcal{O}_B | 0 \rangle$ at two loops

$$\underbrace{\text{Tr} X[Y, Z]}_{\mathcal{O}_B} = \underbrace{\text{Tr} X\{Y, Z\}}_{\text{Protected, } \mathcal{O}_{\text{BPS}}} \underbrace{- 2\text{Tr} XZY}_{\mathcal{O}_{\text{offset}}}$$

- For \mathcal{O}_{BPS} , same integrand as $\text{Tr}(\phi^3)$ ✓
- Compute only $\mathcal{O}_{\text{offset}}$ using cuts as before:



$\langle \bar{X}\bar{Y}\bar{Z} | \mathcal{O}_B | 0 \rangle$ at two loops

$$\underbrace{\text{Tr} X[Y, Z]}_{\mathcal{O}_B} = \underbrace{\text{Tr} X\{Y, Z\}}_{\text{Protected, } \mathcal{O}_{\text{BPS}}} \underbrace{- 2\text{Tr} XZY}_{\mathcal{O}_{\text{offset}}}$$

$$F_{\text{Tr}(X[Y, Z])}^{(2)}(1^{\bar{X}}, 2^{\bar{Y}}, 3^{\bar{Z}}; q) = \sum_{i=1}^3 - \text{diagram}_1 - \text{diagram}_2 - \text{diagram}_3 - \text{diagram}_4 + \text{diagram}_5$$

$$- 2 \left[\text{diagram}_6 + \text{diagram}_7 + \text{diagram}_8 + \text{diagram}_9 + \text{diagram}_{10} - \text{diagram}_{11} - \text{diagram}_{12} \right]$$

$$+ \text{diagram}_{13} + \text{diagram}_{14} - 4 \left[\text{diagram}_{15} + \text{diagram}_{16} \right]$$

The diagrams represent two-loop Feynman diagrams for the process $\langle \bar{X}\bar{Y}\bar{Z} | \mathcal{O}_B | 0 \rangle$. Each diagram shows external legs with momenta $i, i+1, i+2$ and a loop momentum q . The diagrams are arranged in a sum, with some terms multiplied by coefficients like -2 or -4. The diagrams include various internal line configurations, including dashed lines and solid lines, and some have additional labels like $i+2$ or $i+1$ on the internal lines.

$\langle \bar{X}\bar{Y}\bar{Z} | \mathcal{O}_B | 0 \rangle$ at two loops

$$\underbrace{\text{Tr} X[Y, Z]}_{\mathcal{O}_B} = \underbrace{\text{Tr} X\{Y, Z\}}_{\text{Protected, } \mathcal{O}_{\text{BPS}}} \underbrace{- 2\text{Tr} XZY}_{\mathcal{O}_{\text{offset}}}$$

$$F_{\text{Tr}(X[Y, Z])}^{(2)}(1^{\bar{X}}, 2^{\bar{Y}}, 3^{\bar{Z}}; q) = \sum_{i=1}^3 - \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \\ \text{Diagram 9} \\ \text{Diagram 10} \\ \text{Diagram 11} \\ \text{Diagram 12} \\ \text{Diagram 13} \\ \text{Diagram 14} \\ \text{Diagram 15} \\ \text{Diagram 16} \\ \text{Diagram 17} \\ \text{Diagram 18} \\ \text{Diagram 19} \\ \text{Diagram 20} \\ \text{Diagram 21} \\ \text{Diagram 22} \\ \text{Diagram 23} \\ \text{Diagram 24} \\ \text{Diagram 25} \\ \text{Diagram 26} \\ \text{Diagram 27} \\ \text{Diagram 28} \\ \text{Diagram 29} \\ \text{Diagram 30} \\ \text{Diagram 31} \\ \text{Diagram 32} \\ \text{Diagram 33} \\ \text{Diagram 34} \\ \text{Diagram 35} \\ \text{Diagram 36} \\ \text{Diagram 37} \\ \text{Diagram 38} \\ \text{Diagram 39} \\ \text{Diagram 40} \\ \text{Diagram 41} \\ \text{Diagram 42} \\ \text{Diagram 43} \\ \text{Diagram 44} \\ \text{Diagram 45} \\ \text{Diagram 46} \\ \text{Diagram 47} \\ \text{Diagram 48} \\ \text{Diagram 49} \\ \text{Diagram 50} \\ \text{Diagram 51} \\ \text{Diagram 52} \\ \text{Diagram 53} \\ \text{Diagram 54} \\ \text{Diagram 55} \\ \text{Diagram 56} \\ \text{Diagram 57} \\ \text{Diagram 58} \\ \text{Diagram 59} \\ \text{Diagram 60} \\ \text{Diagram 61} \\ \text{Diagram 62} \\ \text{Diagram 63} \\ \text{Diagram 64} \\ \text{Diagram 65} \\ \text{Diagram 66} \\ \text{Diagram 67} \\ \text{Diagram 68} \\ \text{Diagram 69} \\ \text{Diagram 70} \\ \text{Diagram 71} \\ \text{Diagram 72} \\ \text{Diagram 73} \\ \text{Diagram 74} \\ \text{Diagram 75} \\ \text{Diagram 76} \\ \text{Diagram 77} \\ \text{Diagram 78} \\ \text{Diagram 79} \\ \text{Diagram 80} \\ \text{Diagram 81} \\ \text{Diagram 82} \\ \text{Diagram 83} \\ \text{Diagram 84} \\ \text{Diagram 85} \\ \text{Diagram 86} \\ \text{Diagram 87} \\ \text{Diagram 88} \\ \text{Diagram 89} \\ \text{Diagram 90} \\ \text{Diagram 91} \\ \text{Diagram 92} \\ \text{Diagram 93} \\ \text{Diagram 94} \\ \text{Diagram 95} \\ \text{Diagram 96} \\ \text{Diagram 97} \\ \text{Diagram 98} \\ \text{Diagram 99} \\ \text{Diagram 100} \end{array} \right] - 4 \left[\begin{array}{c} \text{Diagram 101} \\ \text{Diagram 102} \end{array} \right]$$

Same as $\text{Tr}(\phi^3)$

Generic remainder of operator \mathcal{O} :

$$\mathcal{R}_{\mathcal{O}}^{(2)} \equiv F_{\mathcal{O}}^{(2)}(\epsilon) - \frac{1}{2} (F_{\mathcal{O}}^{(1)}(\epsilon))^2 - f^{(2)}(\epsilon) F_{\mathcal{O}}^{(1)}(2\epsilon) - C^{(2)}$$

Using $F_{\mathcal{O}_B} = F_{\mathcal{O}_{\text{BPS}}} + F_{\mathcal{O}_{\text{offset}}}$:

$$\mathcal{R}_{\mathcal{O}_B}^{(2)} = \underbrace{\mathcal{R}_{\text{BPS}}^{(2)}}_{\text{known } \checkmark} + \mathcal{R}_{\text{non-BPS}}^{(2)}$$

$$\mathcal{R}_{\text{non-BPS}}^{(2)} = F_{\mathcal{O}_{\text{offset}}}^{(2)}(\epsilon) - F_{\mathcal{O}_{\text{offset}}}^{(1)} \left(\frac{1}{2} F_{\mathcal{O}_{\text{offset}}}^{(1)} + F_{\text{BPS}}^{(1)} \right)(\epsilon) - f^{(2)}(\epsilon) F_{\mathcal{O}_{\text{offset}}}^{(1)}(2\epsilon)$$

- $\mathcal{R}_{\text{non-BPS}}$ has transcendentality ≤ 3
 \rightsquigarrow Same as observed in $SU(2)$ and $SL(2)$ sectors

Goncharov: At transcendentality ≤ 3 only classical polylogs needed

\Rightarrow Maximally transcendental part universal = \mathcal{R}_{BPS}

- UV divergence

$$\mathcal{R}_{\text{non-BPS}}^{(2)} = \frac{c_{\text{UV}}}{\epsilon} + \sum_{i=0}^3 \mathcal{R}_{\text{non-BPS};3-i}^{(2)}, \quad c_{\text{UV}} = 18$$

- Lower transcendentality parts of remainder:

$$\mathcal{R}_{\text{non-BPS};\mathbf{3}}^{(2)} = 2 \left[\text{Li}_3(u) + \text{Li}_3(1-u) \right] - \frac{1}{2} \log^2(u) \log \frac{vw}{(1-u)^2} \\ + \frac{2}{3} \log(u) \log(v) \log(w) + \frac{2}{3} \zeta_3 + \text{perms}(u, v, w)$$

$$\mathcal{R}_{\text{non-BPS};\mathbf{2}}^{(2)} = -12 \left[\text{Li}_2(1-u) + \text{Li}_2(1-v) + \text{Li}_2(1-w) \right] - 2 \log^2(uvw) + 36 \zeta_2$$

$$\mathcal{R}_{\text{non-BPS};\mathbf{1}}^{(2)} = -12 \log(uvw)$$

$$\mathcal{R}_{\text{non-BPS};\mathbf{0}}^{(2)} = 126$$

Connection with remainder densities in the $SU(2)$ sector

– Loebbert, Nandan, Sieg, Wilhelm, Yang –

- $X = \phi_{14}, Y = \phi_{24}, \quad \mathcal{O} = \text{Tr}(XYXY \cdots YYX)$
- Basis of remainder densities:

$$\underbrace{(R_i^{(2)})_{XXX}^{XXX}}_{\text{BPS}}, \quad \underbrace{(R_i^{(2)})_{XXY}^{YXY}, (R_i^{(2)})_{XXY}^{YXX}}_{\text{transcendentality } \leq 3}$$

$$\frac{1}{2} \mathcal{R}_{\text{non-BPS};3}^{(2)} = - \sum_{S_3} (R_i^{(2)})_{XXY}^{YXY} \Big|_3 + 6 \zeta_3 ,$$

$$\frac{1}{2} \mathcal{R}_{\text{non-BPS};2}^{(2)} = - \sum_{S_3} \left[(R_i^{(2)})_{XXY}^{YXY} - (R_i^{(2)})_{XXY}^{YXX} \right] \Big|_2 + 5\pi^2 ,$$

$$\frac{1}{2} \mathcal{R}_{\text{non-BPS};1}^{(2)} = - \sum_{S_3} \left[(R_i^{(2)})_{XXY}^{YXY} - (R_i^{(2)})_{XXY}^{YXX} \right] \Big|_1 ,$$

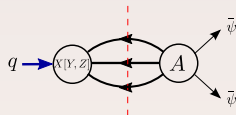
$$\frac{1}{2} \mathcal{R}_{\text{non-BPS};0}^{(2)} = - \sum_{S_3} \left[(R_i^{(2)})_{XXY}^{YXY} - (R_i^{(2)})_{XXY}^{YXX} \right] \Big|_0$$

The other terms in the form factor matrix are simpler

Off-diagonal terms non-vanishing only at loop level
(length-changing interactions)

SUSY $Q_3^\alpha Q_{3\alpha}$ relates
 $\text{Tr}(\phi_{12}^2)$ to $\mathcal{O}_{\text{BPS}'} = \mathcal{O}_F + g\mathcal{O}_B$
Recycle BPS computation

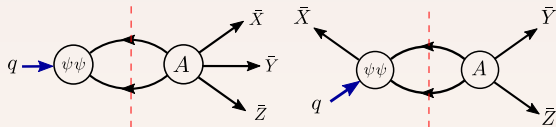
Sub-minimal



$$\langle \bar{\psi} \bar{\psi} | \mathcal{O}_F | 0 \rangle \quad \langle \bar{\psi} \bar{\psi} | \mathcal{O}_B | 0 \rangle$$

$$\langle \bar{X} \bar{Y} \bar{Z} | \mathcal{O}_F | 0 \rangle \quad \langle \bar{X} \bar{Y} \bar{Z} | \mathcal{O}_B | 0 \rangle$$

Next-to-minimal



- From UV divergences obtain two-loop the dilatation operator:
 - Beisert –

$$\mathcal{O}^{\text{ren}} = \mathcal{Z} \cdot \mathcal{O}^{\text{bare}} \quad \Rightarrow \quad \delta \mathfrak{D} = -\mu_R \frac{\partial}{\partial \mu_R} \log \mathcal{Z}$$
$$\mathcal{Z} = \mathbb{1} + \delta \mathcal{Z} \quad g(\mu_R) \sim g \mu_R^{-\epsilon}$$

- Eigenstates – Eden, Jarczак, Sokatchev, Stanev / Eden –

$$\begin{cases} \mathcal{O}_F + g \mathcal{O}_B, & \gamma = 0 \\ \mathcal{O}_B - \frac{gN}{8\pi^2} \mathcal{O}_F, & \gamma = 12 \left(\frac{g^2 N}{(4\pi)^2} \right) - 48 \left(\frac{g^2 N}{(4\pi)^2} \right)^2 + \mathcal{O}(g^6) \end{cases}$$

- Form factors of protected operators, two-loop remainder very simple and of uniform transcendentality.
- Form factors of

$$\mathcal{O}_B = \text{Tr}(X[Y, Z]) \text{ and } \mathcal{O}_F = \frac{1}{2} \text{Tr}(\psi^\alpha \psi_\alpha)$$

with external states $\langle \bar{X} \bar{Y} \bar{Z} |$ and $\langle \bar{\psi} \bar{\psi} |$ up to $\mathcal{O}(g^4)$

- Two-loop remainder function of $F_{\mathcal{O}_B}(\bar{X}, \bar{Y}, \bar{Z}; q)$ can be decomposed into BPS part (maximal transc.) and a correction which is of transc. ≤ 3
 - Maximally transcendental part universal
 - Lower transcendentality factors related to remainder densities in $SU(2)$ sector
- UV-divergences, renormalisation and dilatation operator

Work in progress and outlook

- $SL(2)$ computation of Loebbert, Sieg, Wilhelm and Yang already signaled the complexity due to allowing operators with derivatives. Also concept of “hidden uniform transcendentality”.
- Partial results for $F_{\text{Tr}(F^3)}(g, g, g; q)$ using unitarity and color-kinematics duality.
- Analogous computation in QCD for comparison.
- Universal structures also for lower transcendentality?
- Two loop dilatation operator in $\mathcal{N} = 4$ SYM not completely known, perhaps on-shell methods can be instrumental.

Work in progress and outlook

- $SL(2)$ computation of Loebbert, Sieg, Wilhelm and Yang already signaled the complexity due to allowing operators with derivatives. Also concept of “hidden uniform transcendentality”.
- Partial results for $F_{\text{Tr}(F^3)}(g, g, g; q)$ using unitarity and color-kinematics duality.
- Analogous computation in QCD for comparison.
- Universal structures also for lower transcendentality?
- Two loop dilatation operator in $\mathcal{N} = 4$ SYM not completely known, perhaps on-shell methods can be instrumental.

Thank you for your attention!