Ambitwistor Strings

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Based on 1404.6219 and 1406.1462 (Geyer, Lipstein, Mason)

Spinor-Helicity

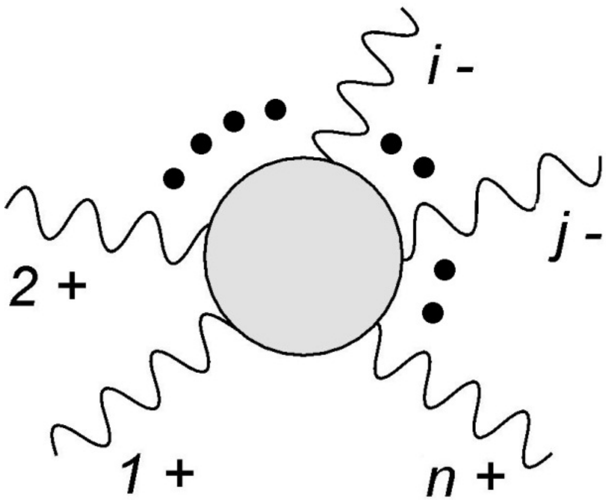
- 4d null momentum:

$$p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

- Expressing amplitudes in terms of these spinors leads to very simple expressions.

MHV Amplitudes

At tree-level:



$$A_n = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

(Parke,Taylor)

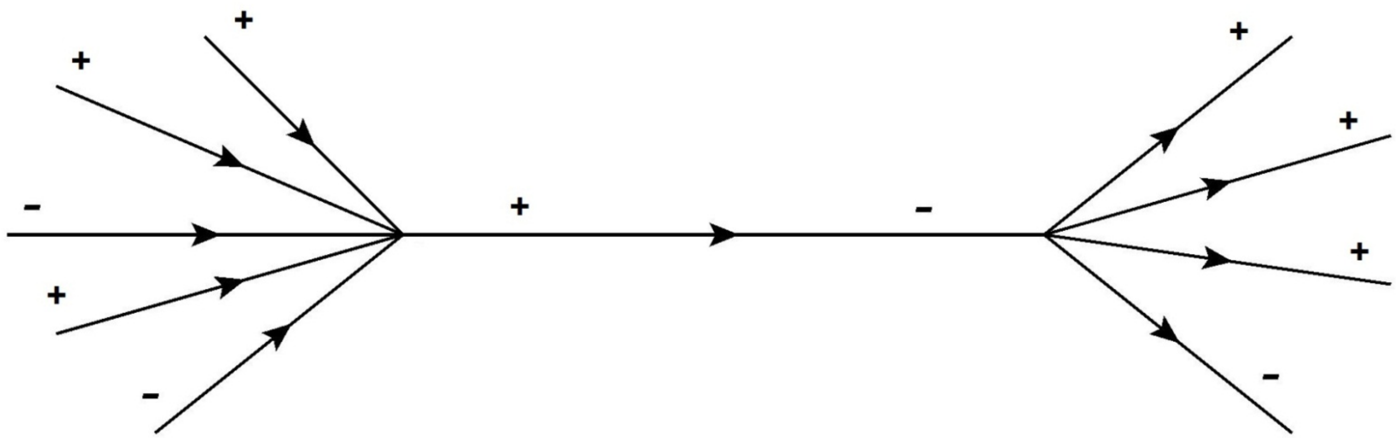
where $\langle ij \rangle = \epsilon_{\alpha\beta} \lambda_i^{\alpha} \lambda_j^{\beta}$

CSW Formalism

- Use tree-level MHV amplitudes as Feynman vertices for constructing tree-level non-MHV amplitudes.

  (Cachazo,Svrcek,Witten)

- Example: NMHV amplitude



Twistor String Theory

- The simplicity of MHV amplitudes and the CSW formalism suggests a deeper mathematical structure.

- Is there a way to reformulate Yang-Mills theory to make this structure manifest?

- Berkovits/Witten: N=4 SYM is equivalent to string theory with target space $CP^{3|4}$

Twistors

- Twistors: (Penrose)

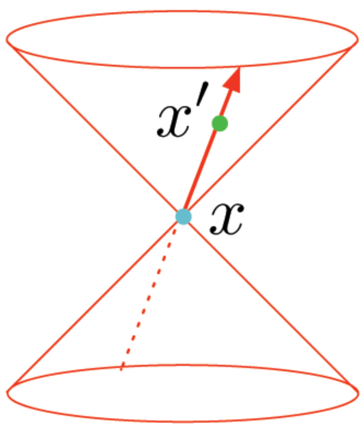
$$\begin{pmatrix} Z^A \\ \chi^a \end{pmatrix}, \quad Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot{\alpha}} \end{pmatrix}$$

- Incidence relations:

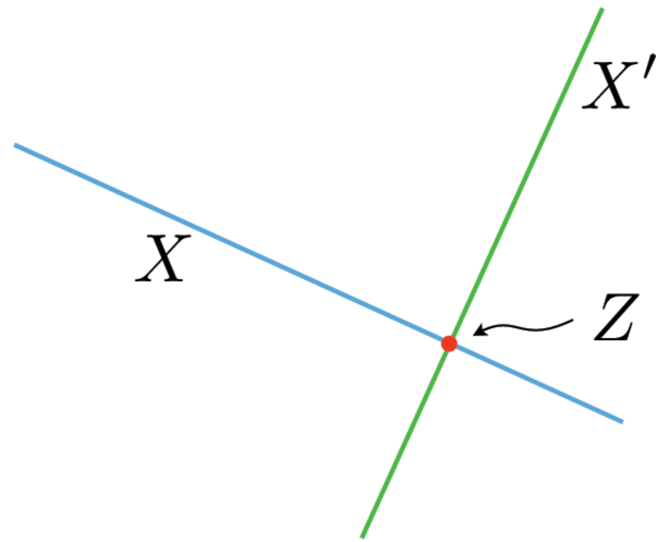
$$\mu^{\dot{\alpha}} = -ix^{\dot{\alpha}\alpha}\lambda_\alpha, \quad \chi^a = -i\theta^{a\alpha}\lambda_\alpha$$

Spacetime vs Twistor Space

Space-time



Twistor Space



Point in spacetime ⟷ CP^1 in twistor space

Point in twistor space ⟷ null ray in spacetime

- Combining insights from AdS/CFT and twistor string theory has lead to powerful techniques for computing amplitudes of N=4 super-Yang-Mills, which have revealed new symmetries, dualities, and mathematical structures.

- Question: Can these ideas be extended to other theories such as gravity or N<4 SYM?

Extension to Gravity

- Hodges formula for tree-level MHV:

$$\mathcal{M}_{\mathrm{n},0} = \langle i, j\rangle^8 \det{}'(\mathrm{H})\, \delta^4\left(\sum_i p_i\right)$$

$$\mathrm{H}_{ij} = \frac{[i,j]}{\langle i,j\rangle} \qquad \text{for } i \neq j, \qquad\qquad \mathrm{H}_{ii} = -\sum_{j\neq i} \frac{[i,j]}{\langle i,j\rangle} \frac{\langle a,j\rangle\langle b,j\rangle}{\langle a,i\rangle\langle b,i\rangle}$$

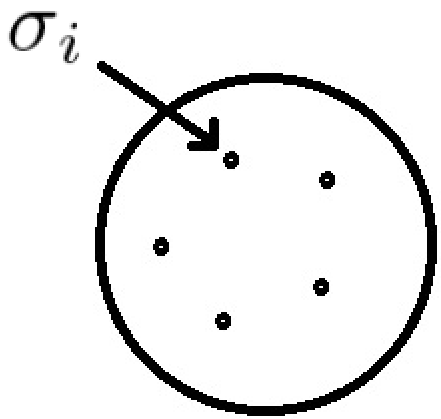
- Skinner: N=8 SUGRA is equivalent to string theory with target space $CP^{3|8}$

Scattering Equations

external momentum

$$\sum_{i \neq j} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0$$

point on 2-sphere



- Gross/Mende: These equations arise from the tensionless limit of string amplitudes

- Cachazo/He/Yuan: They also arise in the amplitudes of massless point particles!

Ambitwistor Strings

- Mason/Skinner: Amplitudes of complexified massless point particles can be computed using a chiral, infinite tension limit of the RNS string:

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu + \frac{e}{2} P_\mu P^\mu + \dots$$

- Correlation functions of vertex operators reproduce the CHY formulae!

- Critical in d=26 (bosonic) and d=10 (superstring)

4d Ambitwistor Space

- Twistors/Dual Twistors:

$$Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot\alpha} \\ \chi^a \end{pmatrix}, \ W_A = \begin{pmatrix} \tilde\mu^\alpha \\ \tilde\lambda_{\dot\alpha} \\ \tilde\chi_a \end{pmatrix}$$

- Incidence Relations:

$$\mu^{\dot\alpha} = i(x^{\alpha\dot\alpha} + i\theta^{a\alpha}\tilde\theta^{\dot\alpha}_a)\lambda_\alpha \qquad \chi^a = \theta^{a\alpha}\lambda_\alpha$$

$$\tilde\mu^{\alpha} = -i(x^{\alpha\dot\alpha} - i\theta^{a\alpha}\tilde\theta^{\dot\alpha}_a)\tilde\lambda_{\dot\alpha} \qquad \tilde\chi_a = \tilde\theta^{\dot\alpha}_a\tilde\lambda_{\dot\alpha}$$

4d Ambitwistor Strings

- Action for YM:

$$S = \int_\Sigma W \cdot \bar{\partial} Z - Z \cdot \bar{\partial} W ...$$

- Action for gravity (N=2 worldsheet susy):

$$S = \int_\Sigma W \cdot \bar{\partial} Z - Z \cdot \bar{\partial} W + \rho \bar{\partial} \tilde{\rho} + \tilde{\rho} \bar{\partial} \rho ...$$

Vertex Operators

- (super)Yang-Mills:

h= -1 : $\widetilde{\mathcal{V}}_a = \int \frac{\mathrm{d}s_a}{s_a} \bar{\delta}^2(\tilde{\lambda}_a - s_a\tilde{\lambda})\mathrm{e}^{is_a\left(\langle\tilde{\mu}\,\lambda_a\rangle + \tilde{\chi}_I\,\eta_a^I\right)} J \cdot t_a$

h= +1 : $\mathcal{V}_a = \int \frac{\mathrm{d}s_a}{s_a} \bar{\delta}^{2|\mathcal{N}}(\lambda_a - s_a\lambda|\eta_a - s_a\chi)\mathrm{e}^{is_a[\mu\,\tilde{\lambda}_a]} J \cdot t_a$

where “a” labels external particles and J is current algebra

- (super)gravity:

$$\mathcal{V}_h = \int \left[W, \frac{\partial h}{\partial Z}\right] + \left[\tilde{\rho}, \frac{\partial}{\partial Z}\right] \rho \cdot \frac{\partial h}{\partial Z}$$

$$\widetilde{\mathcal{V}}_{\tilde{h}} = \int \left\langle Z, \frac{\partial \tilde{h}}{\partial W} \right\rangle + \left\langle \rho, \frac{\partial}{\partial W} \right\rangle \tilde{\rho} \cdot \frac{\partial \tilde{h}}{\partial W}$$

where

$$h_a = \int \frac{\mathrm{d}s_a}{s_a^3} \bar{\delta}^{2|\mathcal{N}}(\lambda_a - s_a \lambda | \eta_a - s_a \chi) \mathrm{e}^{i s_a [\mu \, \tilde{\lambda}_a]}$$

$$\tilde{h}_a = \int \frac{\mathrm{d}s_a}{s_a^3} \bar{\delta}^2(\tilde{\lambda}_a - s_a \tilde{\lambda}) \mathrm{e}^{i s_a \left(\langle \tilde{\mu} \, \lambda_a \rangle + \tilde{\chi}_I \eta_a^I\right)} .$$

$$\langle Z_1 Z_2 \rangle \equiv \langle \lambda_1 \lambda_2 \rangle \, , \quad [W_1 W_2] \equiv \left[\tilde{\lambda}_1 \tilde{\lambda}_2\right]$$

Correlation Functions

- Consider N^{k-2}MHV amplitude:

$$\mathcal{A} = \left\langle \widetilde{\mathcal{V}}_1 \ldots \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} \ldots \mathcal{V}_n \right\rangle$$

- Bringing exponentials into the action gives

$$\int_\Sigma \sum_{i=1}^{k} is_i(\langle\tilde{\mu}\lambda_i\rangle+\tilde{\chi}\cdot\eta_i)\bar{\delta}(\sigma-\sigma_i)+\sum_{p=k+1}^{n} is_p[\mu\,\tilde{\lambda}_p]\bar{\delta}(\sigma-\sigma_p)$$

- Equations of motion:

$$\bar{\partial}_\sigma Z = \bar{\partial}\,(\lambda, \mu, \chi) = \sum_{i=1}^{k} s_i\,(\lambda_i, 0, \eta_i)\,\bar{\delta}\,(\sigma - \sigma_i)\,,$$

$$\bar{\partial}_\sigma W = \bar{\partial}\left(\tilde{\mu}, \tilde{\lambda}, \tilde{\chi}\right) = \sum_{p=k+1}^{n} s_p\left(0, \tilde{\lambda}_p, 0\right)\bar{\delta}(\sigma - \sigma_p)$$

- Solution:

$$Z(\sigma) = (\lambda, \mu, \chi) = \sum_{i=1}^{k} \frac{s_i\,(\lambda_i, 0, \eta_i)}{\sigma - \sigma_i}$$

$$W(\sigma) = \left(\tilde{\mu}, \tilde{\lambda}, \tilde{\chi}\right) = \sum_{p=k+1}^{n} \frac{s_p\left(0, \tilde{\lambda}_p, 0\right)}{\sigma - \sigma_p}$$

- Scattering equations (refined by helicity):

$$[\tilde{\lambda}_i\,\tilde{\lambda}(\sigma_i)] = 0\,,\ i = 1\ldots k\,, \quad \langle\lambda_p\,\lambda(\sigma_p)\rangle = 0,\ p = k{+}1\ldots n$$

Amplitudes

- Geyer/Lipstein/Mason: 4d ambitwistor string theory gives rise to new tree-level formulae for gauge and gravity amplitudes with any amount of susy!

$$\mathcal{A} = \int \frac{1}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \prod_{i=1}^{k} \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i))$$
$$\prod_{p=k+1}^{n} \bar{\delta}^{2|\mathcal{N}}(\lambda_p - \lambda(\sigma_p), \eta_p - \chi(\sigma_p))$$

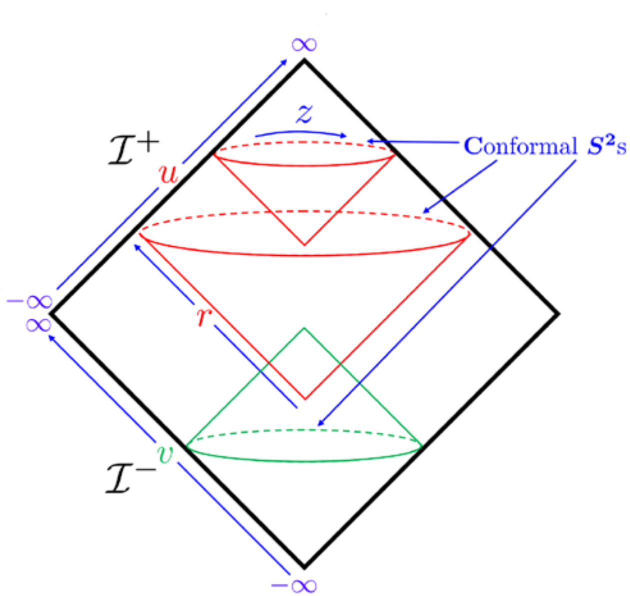
$$\mathcal{M} = \int \frac{\prod_{a=1}^{n} \mathrm{d}^2\sigma_a}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \det{}'(\mathcal{H}) \prod_{i=1}^{k} \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i))$$
$$\prod_{p=k+1}^{n} \bar{\delta}^{2|\mathcal{N}}(\lambda_p - \lambda(\sigma_p), \eta_p - \chi(\sigma_p))$$

- Much simpler than previous formulae.

BMS Symmetry

- Strominger conjectured that diag(BMS^+ x BMS^-) is a symmetry of the 4d gravitational S-matrix:

$$\langle out| B^+\mathcal{S} - \mathcal{S}B^- |in\rangle = 0$$



Soft Limits

- The Ward-identities associated with BMS symmetry correspond to soft graviton theorems:

$$\lim_{k_{n+1}\to 0} \mathcal{M}_{n+1} = \left(S^{(0)} + S^{(1)} \right) \mathcal{M}_n$$

where

$$S^{(0)} = \sum_{a=1}^{n} \frac{(\epsilon \cdot k_a)^2}{s \cdot k_a} \longleftarrow \text{supertranslations}$$

$$S^{(1)} = \frac{\epsilon_{\mu\nu} k_a^{\mu} s_{\lambda} J_a^{\lambda\nu}}{s \cdot k_a} \longleftarrow \text{superrotations}$$

(Weinberg, White, Cachazo/Strominger)

Soft Limits and BMS

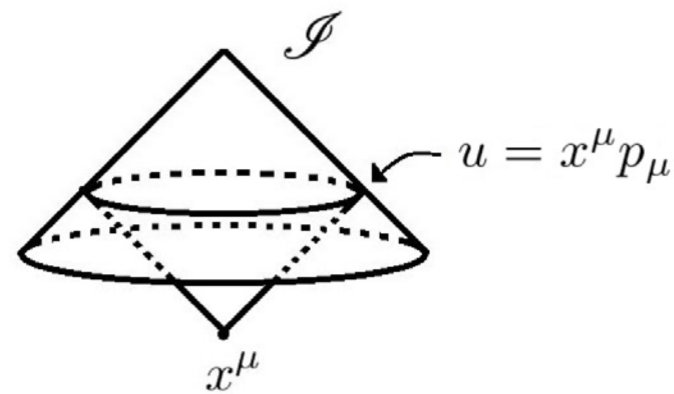
- Ambitwistor string theory makes the relation between BMS symmetry and soft limits transparent, and implies an extension to gravity and Yang-Mills theory in arbitrary dimensions!
  (Adamo/Casali/Skinner, Geyer/Lipstein/Mason)

- Key idea: BMS generators correspond to leading and subleading terms in the Taylor expansion of soft vertex operators

Ambitwistor Space vs Null Infinity

- A null geodesic through the point x^μ with tangent vector P_μ reaches null infinity at

$$(u, p_\mu) = w(x^\nu P_\nu, P_\mu)$$



- Ambitwistor space can be described using coordinates

$$(u, p_\mu, w, q^\mu)$$

where $q^\mu = wx^\mu$

- Diffeomorphisms of null infinity lift to Hamiltonian actions of ambitwistor space.

- Translations: $\delta x^\mu = a^\mu$

$$H_a = wa \cdot p \longrightarrow H_f = wf(p)$$

- Rotations: $\delta x^\mu = r^\mu{}_\nu x^\nu$, $r_{\mu\nu} = - r_{\nu\mu}$

$$H_r = q^{[\mu}p^{\nu]}r_{\mu\nu} \longrightarrow H_r = q^{[\mu}p^{\nu]}r_{\mu\nu}(p)$$

Ambitwistor Strings at Null Infinity

- Action:

$$S = \int_{\Sigma} w\bar{\partial}u + p_{\mu}\bar{\partial}q^{\mu} + \sum_{r=1}^{2} \Psi_r \cdot \bar{\partial}\Psi_r + \ldots$$

- Vertex Operators:

$$\mathcal{V} \;=\; \int_{\Sigma} \bar{\delta}(k \cdot p)\, w\, \mathrm{e}^{ik\cdot q/w} \prod_{r=1}^{2} \epsilon_{r\mu}(p^{\mu} + i\Psi_r^{\mu}\Psi_r \cdot k)$$

- Equations of motion:

$$p(\sigma) = \sum_i \frac{k_i}{\sigma - \sigma_i}$$

From Soft Limits to BMS

- Taylor expanding in soft momentum s gives

$$\mathcal{V}_s \;=\; \mathcal{V}_s^0 + \mathcal{V}_s^1 + \mathcal{V}_s^2 + \mathcal{V}_s^3 + \ldots .$$

where

$$\mathcal{V}_s^0 = \frac{1}{2\pi i} \oint w \frac{(\epsilon \cdot p)^2}{s \cdot p}$$

$$\mathcal{V}_s^1 = \frac{1}{2\pi} \oint \frac{\epsilon \cdot p}{s \cdot p} \epsilon^\mu s^\nu J_{\mu\nu}$$

$$J_{\mu\nu} = p_{[\mu} q_{\nu]} + w \sum_{r=1}^{2} \Psi_{r\mu} \Psi_{r\nu}$$

From BMS to Soft Limits

- OPEs:

$$p_\mu(\sigma)q^\nu(\sigma') = \frac{\delta_\mu^\nu \, \mathrm{d}\sigma}{\sigma - \sigma'} + \ldots\,, \quad \Psi^\mu(\sigma)\Psi_\nu(\sigma') = \frac{\delta_\nu^\mu}{\sigma - \sigma'} + \ldots$$

- Correlator with supertranslation generator:

$$\left\langle \mathcal{V}_1 \ldots \mathcal{V}_n \mathcal{V}_s^0 \right\rangle = \left(\sum_{a=1}^{n} \frac{(\epsilon \cdot k_a)^2}{s \cdot k_a} \right) \langle \mathcal{V}_1 \ldots \mathcal{V}_n \rangle$$

- Correlator with superrotation generator:

$$\left\langle \mathcal{V}_1 \ldots \mathcal{V}_n \mathcal{V}_s^1 \right\rangle = \sum_{a=1}^{n} \frac{\epsilon_{\mu\nu} k_a^\mu s_\lambda J_a^{\lambda\nu}}{s \cdot k_a} \langle \mathcal{V}_1 \ldots \mathcal{V}_n \rangle$$

Summary

- Complexified massless point-particles can be formulated as ambitwistor strings.

- 4d ambitwistor string theory gives rise to new tree-level formulae for gauge and gravity amplitudes with any amount of susy.

- Ambitwistor string theory provides new insight into BMS symmetries and their relationship to soft limits.

Future Directions

- loops
- massive particles
- ABJM/BLG
- AdS/dS background

Thank You