On a Four-Dimensional Formulation of Dimensionally Regulated Amplitudes

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August 26, 2014

This talk is based on the paper

A.R. Fazio, P. Mastrolia, E. Mirabella, W.J. Torres,
"On a four-dimensional formulation of dimensionally regulated amplitudes"

arXiv:14044783 [hep-ph]

and references therein.

Outline

The NLO computations of hard processes.

Four Dimensional Feynman Rules.

Generalized Internal lines.

Four point one-loop massless color ordered amplitudes.

The all helicity-plus four gluons planar amplitude with a gluonic loop.

The effective coupling of Higgs to gluons in NLO amplitudes.

Generalized Open Loop.

Conclusions and perspectives.

The NLO computations of hard processes

A hard partonic cross section $2 \to m$ at NLO is constituted by

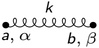
$$\sigma^{NLO} = \int_m d\sigma^B + \int_m \left(d\sigma^V + \int_1 d\sigma^A \right) + \int_{m+1} \left(d\sigma^R - d\sigma^A \right)$$

- $d\sigma^B$ is the Born exclusive regularization scheme indepedent cross section ($\bar{\Sigma}\, A^B A^{B*}$).
- $d\sigma^V$ is the virtual correction ($\bar{\Sigma}\, \Re[A^B A^{V*}]$). It involves loop diagrams whose UV (ultraviolet divergent) part is made finite in a given **renormalization scheme** and therefore those divergencies are **regularization scheme** independent.
- $d\sigma^R$ is the real corrections, affected (together with $d\sigma^V$) by soft and collinear divergencies.
- $d\sigma^A$ and $\int_1 d\sigma^A$ are unintegrated and integrated counterterms (allowing to compute real emission of massless particles in 4 dimensions).

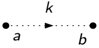
Four Dimensional Feynman Rules for gauge theories one-loop dimensionally regularized diagrams

The external legs are treated as usual four dimensional states

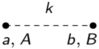
- Loop causal propagators in Feynman-'t Hooft gauge



$$= -i\,\delta^{ab}\,\frac{g^{\alpha\beta}}{k^2-\mu^2+i\varepsilon} \quad \text{(gluon)},$$



$$= i\,\delta^{ab}\,\frac{1}{k^2-\mu^2+i\varepsilon} \quad \text{(ghost)},$$



$$= -i\,\delta^{ab}\,\frac{G^{AB}}{k^2-\mu^2+i\varepsilon} \quad \text{(scalar)},$$

The scalars come from a dimensional reduction of the $D = 4 - 2\epsilon$ dimensional gluons vector fields.

In $D = 4 - 2\epsilon$ dimensions we perform the decomposition of the loop momentum $\bar{\ell}^\alpha$ in a 4-dimensional part ℓ^α and in its orthogonal complement the -2ϵ-dimensional **fixed** vector μ^α

$$\bar{\ell}^\alpha = \ell^\alpha + \mu^\alpha \quad \mu^\alpha \mu_\alpha = -\mu^2$$
$$\bar{g}^{\alpha\beta} = g^{\alpha\beta} + \tilde{g}^{\alpha\beta} \quad \tilde{g}^{\alpha\beta} \to G^{AB} \quad \mu^\alpha \to i\mu Q^A \tag{2}$$

The Euclidean metric G^{AB} and the vector Q^A satisfy the consistency conditions

$$G^{AB} G^{BC} = G^{AC}, \qquad G^{AA} = 0, \qquad G^{AB} = G^{BA}$$
$$Q^A G^{AB} = Q^B, \qquad Q^A Q^A = 1$$

preventing the extra-dimensional dependence of the numerators of the integrands of Feynman diagrams at one-loop.

- Fermion propagator in a loop
  Dirac matrices have the following decomposition

$$\bar{\gamma}^{\alpha} = \gamma^{\alpha} + \tilde{\gamma}^{\alpha}$$

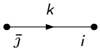
and satisfy in D dimensions the Clifford algebra

$$\{\bar{\gamma}^{\alpha}, \bar{\gamma}^{\beta}\} = 2\bar{g}^{\alpha\beta}.$$

A possible 4-dimensional representation of $\tilde{\gamma}$ matrices is in terms of γ^5

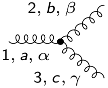
$$\tilde{\gamma}^{\alpha} = \gamma^5 \Gamma^A.$$

By imposing the rule $Q^A \Gamma^A = 1$ in order to recover $\not{\mu}\not{\mu} = -\mu^2$, the fermion propagator suitable for dimensionally regulated amplitudes is

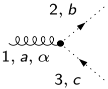


$$= i\delta^{i}_{\bar{\jmath}} \frac{\not{\ell} + m - i\mu\gamma^5}{\ell^2 - m^2 - \mu^2 + i\varepsilon}.$$

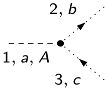
- Vertices



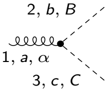
$$= -g\, f^{abc} \left[(\ell_1 - \ell_2)^\gamma g^{\alpha\beta} + (\ell_2 - \ell_3)^\alpha g^{\beta\gamma} + (\ell_3 - \ell_1)^\beta g^{\gamma\alpha} \right] ,$$



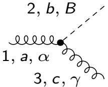
$$= -g\, f^{abc}\, \ell_2^\alpha ,$$



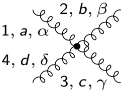
$$= -ig\, f^{abc}\, \mu Q^A ,$$



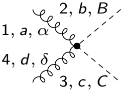
$$= -g\, f^{abc}\, (\ell_2 - \ell_3)^\alpha\, G^{BC} ,$$



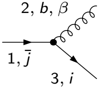
$$= \mp g\, f^{abc}\, (i\mu)\, g^{\gamma\alpha}\, Q^B \quad (\tilde{\ell}_1 = 0, \quad \tilde{\ell}_3^\gamma = \pm\mu^\gamma)$$



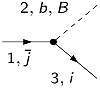
$$= -ig^2 \Big[+ f^{xad} f^{xbc} \left(g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\gamma} g^{\beta\delta} \right) + f^{xac} f^{xbd} \left(g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\delta} g^{\beta\gamma} \right) + f^{xab} f^{xdc} \left(g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta} \right) \Big] ,$$



$$= 2ig^2 g^{\alpha\delta} \left(f^{xab} f^{xcd} + f^{xac} f^{xbd} \right) G^{BC} ,$$



$$= -ig \left(t^b \right)^i_{\bar{\jmath}} \gamma^\beta ,$$



$$= -ig \left(t^b \right)^i_{\bar{\jmath}} \gamma^5 \Gamma^B$$

- Selection rules (-2ϵ SRs)
  In the -2ϵ-dimensional vector space the following rules

$$
\begin{aligned}
G^{AB}G^{BC} &= G^{AC}, & G^{AA} &= 0, & G^{AB} &= G^{BA}, \\
\Gamma^{A}G^{AB} &= \Gamma^{B}, & \Gamma^{A}\Gamma^{A} &= 0, & Q^{A}\Gamma^{A} &= 1, \\
Q^{A}G^{AB} &= Q^{B}, & Q^{A}Q^{A} &= 1
\end{aligned}
$$

  completely define our four dimensional formulation and agree with the integrands in **Four Dimensional Helicity scheme** up to spurious terms (which are nul after integration) as explicitly checked for the following QCD amplitudes

$$
\begin{aligned}
q\,\bar{q} &\to t\,\bar{t}\,, & g\,g &\to t\,\bar{t}\,, & t\,\bar{t} &\to t\,\bar{t}\,, \\
g\,g &\to g\,g\,, & q\,\bar{q} &\to t\,\bar{t}\,g\,, & g\,g &\to t\,\bar{t}\,g\,, \\
q\,\bar{q} &\to t\,\bar{t}\,q'\,\bar{q}'\,.
\end{aligned}
$$

Generalized Internal lines

- Generalized subluminal Dirac equation

$$\left(\not{\ell} - i\mu\gamma^5 - m\right)\ u\left(\ell, \pm\frac{1}{2}\right) = 0\,,$$

$$\left(\not{\ell} - i\mu\gamma^5 + m\right)\ v\left(\ell, \pm\frac{1}{2}\right) = 0\,,$$

$$\ell^\mu = \ell^{\flat\mu} + \frac{m^2 + \mu^2}{2l \cdot q_\ell} q^\mu{}_\ell; \qquad (\ell^\flat)^2 = 0 = q_\ell^2\,.$$

- Generalized spinors in four dimensional helicity formalism

$$u_+\left(\ell\right) = \left|\ell^\flat\right\rangle - \frac{(m - i\mu)}{\left[\ell^\flat\, q_\ell\right]}\left|q_\ell\right]\,, \quad u_-\left(\ell\right) \quad = \left|\ell^\flat\right] - \frac{(m + i\mu)}{\left\langle\ell^\flat\, q_\ell\right\rangle}\left|q_\ell\right\rangle\,,$$

$$v_-\left(\ell\right) = \left|\ell^\flat\right\rangle + \frac{(m - i\mu)}{\left[\ell^\flat\, q_\ell\right]}\left|q_\ell\right]\,, \quad v_+\left(\ell\right) \quad = \left|\ell^\flat\right] + \frac{(m + i\mu)}{\left\langle\ell^\flat\, q_\ell\right\rangle}\left|q_\ell\right\rangle\,.$$

- Polarization sum of the generalized fermions

$$\sum_{\lambda=\pm} u_\lambda(\ell)\,\bar{u}_\lambda(\ell) = \not{\ell} - i\mu\gamma^5 + m\,,$$
$$\sum_{\lambda=\pm} v_\lambda(\ell)\,\bar{v}_\lambda(\ell) = \not{\ell} - i\mu\gamma^5 - m\,.$$

- D dimensional Polarization Vectors
  In Arnowitt-Fickler gauge the helicity sum of the transverse polarization vectors is

$$\sum_{i=1}^{D-2} \varepsilon^{\alpha}_{i\,(D)}\left(\bar{\ell},\bar{\eta}\right)\varepsilon^{*\beta}_{i\,(D)}\left(\bar{\ell},\bar{\eta}\right) = -\bar{g}^{\alpha\beta} + \frac{\bar{\ell}^{\alpha}\,\bar{\eta}^{\beta} + \bar{\ell}^{\beta}\,\bar{\eta}^{\alpha}}{\bar{\ell}\cdot\bar{\eta}} - \frac{\bar{\eta}^{2}\,\bar{\ell}^{\alpha}\,\bar{\ell}^{\beta}}{(\bar{\eta}\cdot\bar{\ell})^{2}}$$

$$\bar{\ell}\cdot\bar{\eta}\neq 0$$

  From the gauge invariance in D dimensions the choice of the fixed D-dimensional gauge vector

$$\bar{\eta}^{\alpha} = \mu^{\alpha}$$

  allows for the disentanglement

$$\begin{aligned}\sum_{i=1}^{D-2} \varepsilon^{\alpha}_{i\,(D)}\left(\bar{\ell},\bar{\eta}\right)\varepsilon^{*\beta}_{i\,(D)}\left(\bar{\ell},\bar{\eta}\right) &= \left(-g^{\alpha\beta} + \frac{\ell^{\alpha}\ell^{\beta}}{\mu^{2}}\right) \\ &- \left(\tilde{g}^{\alpha\beta} + \frac{\mu^{\alpha}\mu^{\beta}}{\mu^{2}}\right).\end{aligned}$$

- Generalized Polarization Vectors
  By the light cone decomposition

$$\ell^\alpha = \ell^{\flat\,\alpha} + \hat{q}_\ell^\alpha$$

the μ-massive polarizations vectors are

$$\varepsilon_+^\alpha(\ell) = -\frac{\left[\ell^\flat \left|\gamma^\alpha\right| \hat{q}_\ell\right\rangle}{\sqrt{2}\mu}\,, \qquad \varepsilon_-^\alpha(\ell) = -\frac{\left\langle\ell^\flat \left|\gamma^\alpha\right| \hat{q}_\ell\right]}{\sqrt{2}\mu}\,,$$

$$\varepsilon_0^\alpha(\ell) = \frac{\ell^{\flat\alpha} - \hat{q}_\ell^\alpha}{\mu}$$

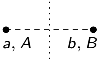
$$\sum_{\lambda=\pm,0} \varepsilon_\lambda^\alpha(\ell)\,\varepsilon_\lambda^{*\beta}(\ell) = -g^{\alpha\beta} + \frac{\ell^\alpha \ell^\beta}{\mu^2}$$

$$\begin{aligned} \varepsilon_\pm^2(\ell) &= 0\,, & \varepsilon_\pm(\ell)\cdot\varepsilon_\mp(\ell) &= -1\,, \\ \varepsilon_0^2(\ell) &= -1\,, & \varepsilon_\pm(\ell)\cdot\varepsilon_0(\ell) &= 0\,, \\ \varepsilon_\lambda(\ell)\cdot\ell &= 0. \end{aligned}$$

The numerator of cut propagator of the scalar can be expressed in terms of the (-2ϵ)-SRs:

$$\tilde{g}^{\alpha\beta} + \frac{\mu^\alpha \mu^\beta}{\mu^2} \quad \rightarrow \quad \hat{G}^{AB} \equiv G^{AB} - Q^A Q^B \, .$$

The factor $\hat{G}^{AB}$ can be easily accounted for by the cut propagator



$$= \hat{G}^{AB} \, \delta^{ab} \, .$$

Four point massless one-loop color ordered amplitudes A_4

From the reduction theorem a dimensionally regularized A_4 is decomposed in a cut-constructible part and in a rational part ($\mathcal{R}$) expressed in terms of scalar integrals in $D = 4 - 2\epsilon$ dimensions. The coefficients c_i are rational functions of the external momenta and polarizations.

$$
\begin{aligned}
A_4 = \frac{1}{(4\pi)^{2-\epsilon}} \Big[& c_{1|2|3|4;\,0}\, I_{1|2|3|4} + \big(c_{12|3|4;\,0}\, I_{12|3|4} \\
& + c_{1|2|34;\,0}\, I_{1|2|34} + c_{1|23|4;\,0}\, I_{1|23|4} + c_{2|3|41;\,0}\, I_{2|3|41} \big) \\
& + \big(c_{12|34;\,0}\, I_{12|34} + c_{23|41;\,0}\, I_{23|41} \big) \Big] + \mathcal{R} + O(\epsilon)\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R} = \frac{1}{(4\pi)^{2-\epsilon}} \Big[& c_{1|2|3|4;\,4}\, I_{1|2|3|4}[\mu^4] + \big(c_{12|3|4;\,2}\, I_{12|34}[\mu^2] \\
& + c_{1|2|34;\,2}\, I_{1|2|34}[\mu^2] + c_{1|23|4;\,2}\, I_{1|23|4}[\mu^2] \\
& + c_{2|3|41;\,2}\, I_{2|3|41}[\mu^2] \big) \\
& + \big(c_{12|34;\,2}\, I_{12|34}[\mu^2] + c_{23|41;\,2}\, I_{23|41}[\mu^2] \big) \Big]\,.
\end{aligned}
$$

The amplitude in four dimensions (fewer than $D = 4 - 2\epsilon$) can be obtained by superposition in the superflous dimensions to elminate the dependence on such variables. The integrals over μ^2 can be performed by separating the integration into 4 and $D - 4$ dimensional parts

$$\int \frac{d^D \bar{\ell}}{(2\pi)^D} = \int \frac{d^{-\epsilon}(\mu^2)}{(2\pi)^{-2\epsilon}} \int \frac{d^4 \ell}{(2\pi)^4}.$$

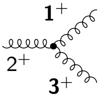
By using polar coordinates in the -2ϵ dimensional Euclidean vector space, all the integrals in $\mathcal{R}$ can be computed. In particular

$$\lim_{\epsilon \to 0} I^{4-2\epsilon}_{1|2|3|4}[\mu^4] = \lim_{\epsilon \to 0} \left(\epsilon(\epsilon - 1) I^{8-2\epsilon}_{1|2|3|4} \right) = -\frac{1}{6}$$

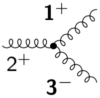
The rational terms cannot be obtained from unitarity cuts with loop momenta in $D = 4$.

The all helicity-plus four gluons planar amplitude with a gluonic loop

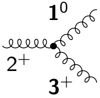
In order to reconstruct the full μ dependence and to obtain by cut construction the rational coefficients of the master integral decomposition, the following color-ordered trees are needed. With all outgoing **on shell** complex momenta



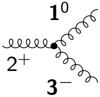
$$= 0\,,$$



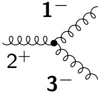
$$= ig\left(\frac{[\mathbf{1}^\flat|2][\hat{q}_{\mathbf{1}}|2]}{\mu} + \frac{\langle r_2|\mathbf{1}|2]}{\langle r_2|2\rangle}\right)\; r_2 - \text{independent},$$



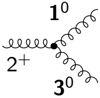
$$= 0\,,$$



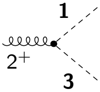
$$= \frac{\sqrt{2} ig\, [\hat{q}_{\mathbf{1}}|2]^2}{\mu}\ ,$$



$$= ig \frac{[\hat{q}_{\mathbf{1}}|2]\, [\hat{q}_{\mathbf{3}}|2]\, \langle \mathbf{1}^\flat|\mathbf{3}^\flat\rangle}{\mu^2}\ ,$$



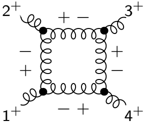
$$\begin{aligned} = & - ig\, \frac{\langle r_2|\mathbf{1}|2]}{\langle r_2|2\rangle} \Big\{ 1 - \frac{(1+\xi)}{\xi\, \mu^2} \Big[(1+\xi)\, \mu^2 \\ & + \xi\, \langle \hat{q}_{\mathbf{1}}|2|\hat{q}_{\mathbf{1}}] \Big] \Big\}\, , \end{aligned}$$

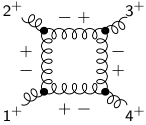


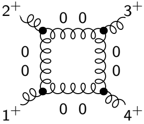
$$\begin{aligned} &= \frac{ig}{\sqrt{2}}\, (\mathbf{3} - \mathbf{1})^\mu\, \varepsilon^+_\mu\, (2, r_2)\, G^{AB} \\ &= -ig\, \frac{\langle r_2|\mathbf{1}|2]}{\langle r_2|2\rangle} G^{AB} \end{aligned}$$

where $\hat{q}_3 = \xi \hat{q}_1$.

The box coefficients are obtained by the following merging procedure, with the external legs of the trees on the generalized mass-shell

$C^{[0]}_{1|2|3|4} =$ 

$+$ 

$+$ 

,

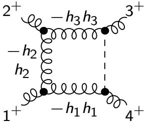
$$c^{[0]}_{1|2|3|4;\,4} = 3g^4 i \frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}$$

in which the relations

$$\langle \mathbf{j}^\flat | \hat{q}_\mathbf{j} \rangle = [\hat{q}_\mathbf{j} | \mathbf{j}^\flat] = \mu$$

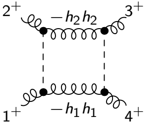
allow to obtain a polinomial numerator in μ.

$$C^{[1]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1$$

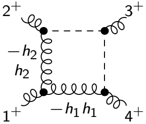


$$+ \text{ c.p. },$$

$$C^{[2]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1^2$$

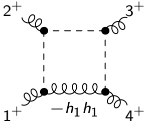


$$+ \mathcal{T}_2$$



$$+ \text{ c.p. },$$

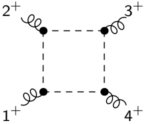
$$C^{[3]}_{1|2|3|4} = \sum_{h_1=\pm,0} \mathcal{T}_3$$



$$+ \text{ c.p. },$$

,

$$C^{[4]}_{1|2|3|4} = \mathcal{T}_4$$



$$\begin{aligned}
\mathcal{T}_1 &= Q^A \hat{G}^{AB} Q^B &&= 0\,, \\
\mathcal{T}_2 &= Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} Q^D &&= 0\,, \\
\mathcal{T}_3 &= Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} G^{DE} \hat{G}^{EF} Q^F &&= 0\,, \\
\mathcal{T}_4 &= \mathrm{tr}\left(G\, \hat{G}\, G\, \hat{G}\, G\, \hat{G}\, G\, \hat{G} \right) &&= -1\,.
\end{aligned}$$

$$c_{1|2|3|4;\,4} = c^{[0]}_{1|2|3|4;\,4} + c^{[4]}_{1|2|3|4;\,4} = 3g^4 i \frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle} - ig^4 \frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}\,,$$

$$A_4^{1-\mathrm{loop}}\left(1_g^+, 2_g^+, 3_g^+, 4_g^+\right) = \frac{2ig^4}{16\pi^2} \times \left(-\frac{1}{6}\right) \times \frac{[12]\,[34]}{\langle 12\rangle \langle 34\rangle}\,.$$

The effective coupling of Higgs to gluons in NLO amplitudes

- For 2 gluons $\to$ Higgs we use an effective operator with $m_{\mathrm{top}} \to \infty$.

$$L_{\mathrm{int}} = \frac{C}{2} H F^{a}_{\mu\nu} F^{a\mu\nu}$$

- The leading tree-level color ordered amplitude $0 \to gggH$

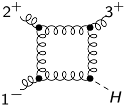
$$A^{\text{tree}}_{4,H}(1^- 2^+ 3^+ H) = i \frac{[23]^4}{[12]\,[23]\,[31]} \,.$$

- As an example of application of our regularization scheme to an effective field theory consider at **two loops** in the large m_{top} limit the color ordered primitive amplitude

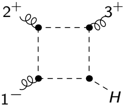
$$A^{\text{1-loop}}_{4,H}(1^- 2^+ 3^+ H)$$

To see just how the procedure works consider firstly some quadruple cuts:

$$C_{1|2|3|H} = $$



$$+$$

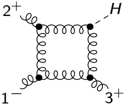


$$,$$

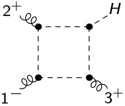
$$c_{1|2|3|H;\,0} = -\frac{1}{2} A^{\text{tree}}_{4,H} s_{12} s_{23}\,,$$

$$c_{1|2|3|H;\,4} = 0\,;$$

$$C_{1|2|H|3} = $$



$$+$$

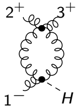


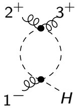
$$,$$

$$c_{1|2|H|3;\,0} = -\frac{1}{2} A^{\text{tree}}_{4,H} s_{13} s_{12}\,,$$

$$c_{1|2|H|3;\,4} = 0\,.$$

......and by omitting for reasons of time many other contributions.....and just considering some among the double cuts

$C_{23|H1} =$ 

$+$ 

$$c_{23|H1;\,0} = 0$$

$$c_{23|H1;\,2} = 4A^{\text{tree}}_{4,H}\frac{s_{12}s_{13}}{s_{23}^3}\,.$$

By collecting all master integrals coefficients we recognize a full agreement with the the full Feynman diagrams calculations in the FDH scheme performed by Schmidt in 1997, the result is

$$
\begin{aligned}
A_4^{1-loop}\left(1^-,2^+,3^+,H\right) = r_\Gamma A_4^{tree}\times \\
\Bigg\{ \frac{1}{\epsilon^2}\left[(-s_{12})^{-\epsilon}+(-s_{13})^{-\epsilon}+(-s_{23})^{-\epsilon}\right]-\frac{\pi^2}{2} \\
+\left[2\mathrm{Li}_2\left(1-\frac{s_{12}}{m_H^2}\right)+2\mathrm{Li}_2\left(1-\frac{s_{13}}{m_H^2}\right)+2\mathrm{Li}_2\left(1-\frac{s_{23}}{m_H^2}\right)\right] \\
+\left[\log\left(\frac{s_{12}}{m_H^2}\right)\log\left(\frac{s_{23}}{m_H^2}\right)+\log\left(\frac{s_{12}}{m_H^2}\right)\log\left(\frac{s_{13}}{m_H^2}\right)\right. \\
\left.+\log\left(\frac{s_{13}}{m_H^2}\right)\log\left(\frac{s_{23}}{m_H^2}\right)\right] \\
-\frac{1}{3}\frac{s_{12}s_{13}+s_{12}s_{23}+s_{13}s_{23}}{m_H^4}+1\Bigg\}
\end{aligned}
$$

Generalized Open Loop

- The FDF of d-dimensional one-loop amplitudes is compatible with methods generating recursively the integrands of one-loop amplitudes and leads to the complete reconstruction of the numerator of the Feynman integrands as a polynomial in the loop variables ℓ and μ.
- We focus on the **OPEN-LOOP** technique by **Cascioli-Maierhöfer-Pozzorini (PRL 2012)**.

- $$w^{\beta}(i) = \frac{X^{\beta}_{\gamma\delta}(i,j,k) w^{\gamma}(j) w^{\delta}(k)}{p_i^2 - m_i^2 + i\varepsilon}$$

- $$\mathcal{N}^{\beta}{}_{\alpha}\left(\mathcal{I}_n, \ell, \mu\right) = X^{\beta}_{\gamma\delta}\left(\mathcal{I}_n, i_n, \mathcal{I}_{n-1}\right) \mathcal{N}^{\gamma}{}_{\alpha}\left(\mathcal{I}_{n-1}, \ell, \mu\right) w^{\delta}\left(i_n\right)$$

- $$X^{\beta}_{\gamma\delta} = Y^{\beta}_{\gamma\delta} + \ell^{\nu}\, Z^{\beta}_{\nu;\,\gamma\delta} + \mu\, W^{\beta}_{\gamma\delta}\,.$$

- $$\mathcal{N}^{\beta}{}_{\alpha}\left(\mathcal{I}_n, \ell, \mu\right) = \sum_{j=0}^{R} \sum_{a=0}^{R-j} \mathcal{N}^{[a]\,\beta}_{\nu_1\cdots\nu_j;\,\alpha}\left(\mathcal{I}_n\right)\, \ell^{\nu_1} \ldots \ell^{\nu_j}\, \mu^{a}$$

- $$\mathcal{N}^{[a]\,\beta}_{\nu_1\cdots\nu_j;\,\alpha}\left(\mathcal{I}_n\right) = \Big[\, Y^{\beta}_{\gamma\delta}\, \mathcal{N}^{[a]\,\gamma}_{\nu_1\cdots\nu_j;\,\alpha}\left(\mathcal{I}_{n-1}\right) + Z^{\beta}_{\nu_1;\,\gamma\delta}\, \mathcal{N}^{[a]\,\gamma}_{\nu_2\cdots\nu_j;\,\alpha}\left(\mathcal{I}_{n-1}\right)$$
$$+ W^{\beta}_{\gamma\delta}\, \mathcal{N}^{[a-1]\,\gamma}_{\nu_1\cdots\nu_j;\,\alpha}\left(\mathcal{I}_{n-1}\right) \Big]\, w^{\delta}(i_n)$$

Conclusions and perspectives

- A four-dimensional formulation (*FDF*) of dimensional regularization of one-loop scattering amplitudes has been applied to generalized unitarity techniques. At one loop the cut-constructible part and the rational part of scattering amplitudes have been computed by the same on-shell methods.
- The *FDF* Feynman rules have been extended to the recursive methods for generating the integrand of one-loop amplitudes.
- The inclusion of the fermion mass for a one loop amplitude like $0 \rightarrow ggt\bar{t}$ at one loop in *FDF* will be analysed.
- More loops and more jets in *FDF* is another goal to achieve.
- An important issue is to apply FDF for real corrections and corresponding subtraction terms of infrared divergencies.