Evolution equations beyond one loop from conformal symmetry

Alexander Manashov

University of Regensburg & Sankt-Petersburg University

based on

V. Braun, A. Manashov: Eur.Phys.J. C73 (2013) 2544 [arXiv:1306.5644]

DESY, 09.12.2013

- DIS: Structure functions : scale dependence is governed by anomalous dimensions of twist-2 operators (known at three loop order, **Larin, Vermaseren, Moch, Vogt, et al**)
- DVCS: Generalized Parton Distributions: scale dependence $\longleftrightarrow$ *full* anomalous dimension matrix (nonforward kernel)
- At one loop level: anomalous dimensions+conformal symmetry$\rightarrow$ full anomalous dimension matrix.
  Conformal symmetry is extremely useful at one loop.

  "The use of conformal symmetry in QCD", Braun, Korchemsky, Müller, 2003
- In any realistic $d = 4$ QFT the conformal symmetry is broken, $\beta(g) \neq 0$.

  Is any use of conformal symmetry beyond one loop ?
- **D. Müller,** Constraints for anomalous dimensions of local light cone operators in ϕ^3 in six-dimensions theory, Z. Phys. C **49** (1991) 293.
  (Conformal Ward Identities, Conformal anomaly, Conformal scheme, etc)

  Belitsky, Müller, (2000) two loop kernels in QCD.

Motivation

- **What does conformal symmetry tell us about evolution equations in conformal field theories?**

$O(n)$ **symmetric** φ^4 **model in** $d = 4 - 2\epsilon$

$$S(\varphi) = \int d^d x \left[\frac{1}{2}(\partial\varphi)^2 + \frac{gM^{2\epsilon}}{24}(\varphi^2)^2 \right] ,$$

$$\beta(\alpha) = -\,2\epsilon\alpha + \frac{\alpha^2(n+8)}{3} - \frac{\alpha^3(3n+14)}{3} + \mathcal{O}(\alpha^4)\,, \qquad \left[\alpha = \frac{g^2}{(4\pi)^2} \right]$$

Critical point α_***,** $\beta(\alpha_*) = 0 \Longrightarrow$ **Scale and Conformal invariance**

(describes phase transition in Ising model)

[Large N_f QCD has a critical point in $d = 4 - 2\epsilon$, $\quad \beta(\alpha_*) = 0$.]

We consider twist-2 symmetric and traceless operators

$$\mathcal{O}_{\mu_1\ldots\mu_N}(x) = \sum_k C_k\,\mathbf{S}\,\partial_{\mu_1}\ldots\partial_{\mu_k}\varphi(0)\partial_{\mu_{k+1}}\ldots\partial_{\mu_N}\varphi(0) - \text{traces}$$

The standard trick is to contract all indices with null vector n, $n^2 = 0$.

$$\mathcal{O}_N(x) = n^{\mu_1}\ldots n^{\mu_N}\mathcal{O}_{\mu_1\ldots\mu_N}(x) = \sum_{\substack{k,m \\ k+m=N}} C_{km}\,O_{mk}$$

$\mathcal{O}_N$ is a linear combination of the basis twist-2 operators

$$O_{mk} = \frac{1}{m!k!}(n\partial)^m\varphi(0)(n\partial)^k\varphi(0)$$

Renormalization: MS scheme

$$O_{mk} \to [O_{mk}] = Z_{mk}^{m'k'}(\alpha,\epsilon)\, O_{m'k'} \qquad Z = 1 + \sum_{j=1}^{\infty} \frac{Z_j(\alpha)}{\epsilon^j}$$

$[O_{mk}]$ are finite operators (i.e. their correlation functions with basic field are finite)

$$\langle [O_{mk}](x)\varphi(x_1)\ldots\varphi(x_k)\rangle = N^{-1}\int D\varphi e^{-S_R(\varphi)}[O_{mk}](x)\varphi(x_1)\ldots\varphi(x_k)$$

We keep $\epsilon-$finite (do not send $\epsilon \to 0$).
The correlators depend explicitly on ϵ, **but** in the MS scheme $Z-$factors contain only pole terms in ϵ

(In MOM scheme Z factors contain both singular and regular terms in ϵ)

Renormalization group equation:

$$\left(\left[M\partial_M + \beta(\alpha)\partial_\alpha\right]\delta_m^{m'}\delta_k^{k'} + \gamma_{mk}^{m'k'}(\alpha)\right)[O_{m'k'}] = 0$$

γ is an anomalous dimension matrix (in the MS-like schemes)

$$\gamma(\alpha) = -M\frac{dZ}{dM}Z^{-1} = \alpha\gamma^{(1)} + \alpha^2\gamma^{(2)} + \dots$$

Important: $\longrightarrow$ **$\gamma^{(k)}$ do not depend on ϵ!!**

Critical dimensions:

$$\gamma(\alpha_*) = \alpha_*\gamma^{(1)} + \alpha_*^2\gamma^{(2)} + \dots, \qquad [\gamma(\alpha) = \alpha\gamma^{(1)} + \alpha^2\gamma^{(2)} + \dots]$$

$$\alpha_* = 6\epsilon/(n+8) + O(\epsilon^2) \qquad \beta(\alpha_*) = 0$$

MS scheme: if one knows $\gamma(\alpha_*)$ then he knows $\gamma(\alpha)$, $\quad \gamma(\alpha_*) \Longleftrightarrow \gamma(\alpha)$

At $\alpha = \alpha_*$ the theory enjoys an *exact* conformal symmetry. One can expect that this symmetry reveals itself in properties of $\gamma(\alpha_*)$.

Nonlocal operator is a generating function for local ones

$$[\mathcal{O}(x; z_1, z_2)] = [\varphi(x + z_1 n)\varphi(x + z_2 n)] \equiv \sum_{mk} z_1^m z_2^k [O_{mk}(x)].$$

It can be represented in the following form

$$[\mathcal{O}(x; z_1, z_2)] = [Z\,\mathcal{O}](x; z_1, z_2) = \int du dv\, \mathcal{Z}(u, v)\, \mathcal{O}(x; z_{12}^u, z_{21}^v)$$

$\underline{z_{12}^u = z_1(1-u) + z_2 u}$ and $\mathcal{Z}(u, v) = 1 + \sum_{j=1}^{\infty} Z_j(\alpha, u, v)/\epsilon^j$.

The RG equation takes the form

$$\Big(\Big[M\partial_M + \beta(\alpha)\partial_\alpha\Big] + \mathbf{H}(\alpha)\Big)[\mathcal{O}(x; z_1, z_2)] = 0$$

Here $H(u)$ is the evolution kernel (Hamiltonian)

$$\mathbf{H}(\alpha) = \alpha\, \mathbf{H}^{(1)} + \alpha^2\, \mathbf{H}^{(2)} + \ldots$$

and

$$[\mathbf{H}^{(k)} f](z_1, z_2) = \int du dv\, h^{(k)}(u, v)\, f(z_{12}^u, z_{21}^v)\,.$$

One-loop examples

Notations: $\bar{u} = 1 - u$, $\tau = uv/\bar{u}\bar{v}$.

- φ^4 theory

$$[H^{(1)}_{\varphi^4} f](z_1, z_2) = \int_0^1 du\, f(z_{12}^u, z_{12}^u), \qquad [h(u,v) = \frac{1}{\bar{u}\bar{v}}\delta\left(1-\tau\right)] \quad (j = 1/2)$$

- φ^3 theory $(d = 6 - 2\epsilon)$

$$[H^{(1)}_{\varphi^3} f](z_1, z_2) = \int_0^1 du \int_0^{\bar{u}} dv\, f(z_{12}^u, z_{21}^v), \qquad [h(u,v) = 1] \quad (j = 1)$$

- QCD, nonsinglet operators: $H^{(1)}_{QCD} = \widehat{H} - H_{\varphi^3}$ $(j = 1)$

$$[\widehat{H} f](z_1, z_2) = \int_0^1 du \frac{\bar{u}}{u}\left[2f(z_1, z_2) - f(z_{12}^u, z_2) - f(z_1, z_{21}^u)\right], \qquad [h(u,v) = \delta\left(\tau\right)]$$

The Hamiltonians commute with the generators of the $SL(2,R)$ group: $[H, S_{\pm,0}] = 0$

$$S_- = -\partial_{z_1} - \partial_{z_2}, \qquad S_0 = z_1\partial_{z_1} + z_2\partial_{z_2} + 2j, \qquad S_+ = z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2j(z_1 + z_2)$$

$$[S_+, S_-] = 2S_0, \qquad [S_0, S_\pm] = \pm S_\pm$$

S_-—translations, S_0—dilatations, S_+—inversions, j—conformal spin

Finite symmetry transformations:

$$f(z_1,z_2)\to f'(z_1,z_2)=\frac{1}{(cz_1+d)^{2j}(cz_2+d)^{2j}}f\left(\frac{az_1+b}{cz_1+d},\frac{az_2+b}{cz_2+d}\right)$$

General form of $SL(2,R)$ invariant operator:

$$[\mathbf{K}f](z_1,z_2)=\int dudv\bar{u}^{2j-2}\bar{v}^{2j-2}\,\omega\left(\frac{uv}{\bar{u}\bar{v}}\right)f(z_{12}^{u},z_{21}^{v})$$

Eigenfunctions: $\Psi_{Nk}=S_+^k(z_1-z_2)^N,\quad \mathbf{K}\Psi_{Nk}=\kappa_N\Psi_{Nk}$

$$\kappa_N=\int dudv\bar{u}^{2j-2}\bar{v}^{2j-2}\,\omega\left(\frac{uv}{\bar{u}\bar{v}}\right)(1-v-u)^N\,.$$

An invariant $\mathbf{K}$ can be restored by its spectrum, κ_N.

One-loop examples

Momentum representation:

$$f(z_1, z_2) = \int du_1\, du_2\, e^{-iu_1 z_1 - iu_2 z_2}\, \tilde{f}(u_1, u_2)$$

$$[\mathcal{H}\tilde{f}](u_1, u_2) = \int_{-\infty}^{\infty} dv_1\, dv_2 \delta(u_1 + u_2 - v_1 - v_2)\, \mathcal{H}(u_1, u_2 | v_1, v_2)\, \tilde{f}(v_1, v_2)\,,$$

φ^3 Hamiltonian:

$$\mathcal{H}(u_1, u_2 | v_1, v_2) = \Theta(-u_1, u_2, u_1 - v_1)\,\frac{v_1 - u_1}{v_1 v_2} + \Theta(u_1, u_2, v_2 - u_2)\,\frac{u_2}{v_2(u_1 + u_2)}$$
$$+ \Theta(u_1, u_2, v_1 - u_1)\,\frac{u_1}{v_1(u_1 + u_2)}\,,$$

where

$$\Theta(a_1, \ldots, a_n) = \prod_{k=1}^{n} \theta(a_i) - \prod_{k=1}^{n} \theta(-a_i)\,.$$

Eigenfuctions are Gegenbauer polynomials $(u_1 + u_2)^N C_N^{(3/2)}\left(\frac{u_1 - u_2}{u_1 + u_2}\right)$

- Light-ray operators technique provides a convenient framework for study of evolution equations.
- Evolution kernels have a simple form in this representation
- Symmetry generators have the standard form
- The eigenfucntions are simple powers, $(z_1 - z_2)^N$

At the critical point $\alpha = \alpha_*$ theory enjoys an exact conformal symmetry

Operators can be classified according the representations of the conformal group:

$$
\begin{aligned}
i\Big[\mathbf{P}^\mu, \mathcal{O}_N(x)\Big] &= \partial^\mu \mathcal{O}_N(x) \\
i\Big[\mathbf{D}, \mathcal{O}_N(x)\Big] &= (x\partial + \Delta_N)\mathcal{O}_N(x) \\
i\Big[\mathbf{K}^\mu, \mathcal{O}_N(x)\Big] &= \Big[2x^\mu(x\partial) - x^2\partial^\mu + 2\Delta_N x^\mu + 2x_\nu \Sigma^{\mu\nu}\Big]\mathcal{O}_N(x)\,,
\end{aligned}
$$

$\mathcal{O}_N-$ **conformal operator**. Its transformation properties are completely determined by the scaling dimension Δ_N and spin

$$
\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta'}(y)\rangle = \delta_{\Delta\Delta'}\frac{C(\Delta)}{|x-y|^{2\Delta}}
$$

Scaling dimensions Δ_N are physical observables.

A tower of operators:

$$\{\mathcal{O}_{Nk} = \partial_+^k \mathcal{O}_N(0),\ \ k = 0, 1, \ldots \infty\}$$

form the representation of the collinear subgroup of the conformal group:

$$\mathbf{L}_- = \mathbf{K}_- = \bar{n}^\mu K_\mu, \qquad \mathbf{L}_+ = \mathbf{P}_+ = n^\mu P_\mu, \qquad \mathbf{L}_0 = \mathbf{D} - \mathbf{M}_{n\bar{n}}$$

$$[\mathbf{L}_+, \mathcal{O}_{Nk}] = \mathcal{O}_{Nk+1}, \quad [\mathbf{L}_0, \mathcal{O}_{Nk}] = (j_N + k)\mathcal{O}_{Nk+1}, \quad [\mathbf{L}_-, \mathcal{O}_{Nk}] = k(j_N + k)\mathcal{O}_{Nk-1},$$

where

$$j_N = (\Delta_N + N)/2 \qquad\qquad \Delta_N = \Delta_N^{can} + \gamma_N(\alpha_*)$$

In perturbation theory:

$$\mathcal{O}_N = \sum_{k+m=N} c_{mk}(\epsilon)[O_{mk}]\,.$$

$\mathcal{O}_N$ is the solution of the RG equation

$$(M\partial_M + \gamma_N^*)\mathcal{O}_N = 0.$$

How the symmetry generators act on the nonlocal operator ?

Re-expansion

$$[\varphi(z_1 n)\varphi(z_2 n)] \equiv \sum_{mk} z_1^m z_2^k [O_{mk}] = \sum_{Nk} \Psi_{Nk}(z_1, z_2)\,\mathcal{O}_{Nk}.$$

$[\varphi(z_1 n)\varphi(z_2 n)]-$vector in "operator space", $\mathcal{O}_{Nk}-$"basis vectors",
$\Psi_{Nk}(z_1, z_2)-$ coefficient functions ("coordinates") [depend on ϵ]

$$(M\partial_M + \mathbf{H}(\alpha_*))[\mathcal{O}(z_1, z_2)] = 0 \Longrightarrow \mathbf{H}(\alpha_*)\Psi_{Nk}(z_1, z_2) = \gamma_N(\alpha_*)\Psi_{Nk}(z_1, z_2)$$

Symmetry of nonlocal operators

The leading order symmetry generators

$$S_-^{(0)} = -\partial_{z_1} - \partial_{z_2}, \quad S_0^{(0)} = z_1\partial_{z_1} + z_2\partial_{z_2} + 1, \quad S_+^{(0)} = z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + z_1 + z_2.$$

In interacting theory the generators become modified by quantum corrections:

$$S_\alpha^{(0)} \to S_\alpha = S_\alpha^{(0)} + \Delta S_\alpha$$

$$[\mathbf{L}_\alpha, [\mathcal{O}(z_1, z_2)]] = \sum_{Nk} \Psi_{Nk}(z_1, z_2)[\mathbf{L}_\alpha, \mathcal{O}_{Nk}] =$$

$$= \sum_{Nk} S_\alpha \Psi_{Nk}(z_1, z_2)\mathcal{O}_{Nk} = S_\alpha[\mathcal{O}(z_1, z_2)]$$

Two generators, S_-, S_0, can be fixed in all orders without calculations

$\mathbf{L}_+$ is shift along $n-$direction $\Longrightarrow$ $\boxed{S_- = -\partial_{z_1} - \partial_{z_2} = S_-^{(0)}.}$

$\mathbf{L}_0$ counts "dimension":

$$\mathbf{L_0}\mathcal{O}_{Nk} = (k + j_N)\mathcal{O}_{Nk} \Longrightarrow \boxed{S_0 = S_0^{(0)} - \epsilon + \frac{1}{2}\mathbf{H}(\alpha_*)}$$

The S_+ generators can be calculated order by order in perturbation theory:

Explicit two loop calculation gives:

$$S_+ = S_+^{(0)} + (z_1 + z_2)\left(-\epsilon + \frac{1}{2}\alpha_* \mathbf{H}^{(1)}\right) + \frac{1}{4}\alpha_*^2 \left\{z_1 + z_2, \mathbf{H}^{(2)}\right\} + O(\epsilon^3)$$

$$= S_+^{(0)} + (z_1 + z_2)\left(-\epsilon + \frac{1}{2}\mathbf{H}(\alpha_*)\right) + \frac{1}{4}\alpha_*^2 \left[\mathbf{H}^{(2)}, z_1 + z_2\right] + \mathcal{O}(\epsilon^3)$$

$$[S_+, S_-] = 2S_0 \qquad \left(S_0 = S_0^{(0)} - \epsilon + \frac{1}{2}\mathbf{H}(\alpha_*)\right)$$

Constraints for the evolution kernels

The generators have to satisfy $sl(2)$ algebra commutation relations

$$[S_0, S_-] = -S_-, \qquad [S_+, S_-] = 2S_0, \qquad \underline{[S_0, S_+] = S_+}.$$

The last one is equivalent to $\longrightarrow$ $\boxed{[\mathbf{S}_+, \mathbf{H}(\alpha_*)] = \mathbf{0}}$

By construction $[S_-^{(0)}, \mathbf{H}(\alpha_*)] = [S_0^{(0)}, \mathbf{H}(\alpha_*)] = 0 \to [S_-, \mathbf{H}(\alpha_*)] = [S_0, \mathbf{H}(\alpha_*)] = 0$

In the perturbative expansion

$$\mathbf{H}(\alpha_*) = \alpha_* \mathbf{H}^{(1)} + \alpha_*^2 \mathbf{H}^{(2)} + \ldots$$

$$S_+(\alpha_*) = S_+^{(0)} + \alpha_* \Delta S_+^{(1)} + \alpha_*^2 \Delta S_+^{(2)} + \ldots$$

$[S_-^{(0)}, \mathbf{H}^{(k)}] = [S_0^{(0)}, \mathbf{H}^{(k)}] = 0$, for all k.

α_* $\qquad [S_+^{(0)}, \mathbf{H}^{(1)}] = 0$ $\longrightarrow$ $\boxed{\mathbf{H}^{(1)} \text{ is } sl(2) \text{ invariant operator}}$

α_*^2 $\qquad [S_+^{(0)}, \mathbf{H}^{(2)}] = [\mathbf{H}^{(1)}, \Delta S_+^{(1)}],$

α_*^3 $\qquad [S_+^{(0)}, \mathbf{H}^{(3)}] = [\mathbf{H}^{(1)}, \Delta S_+^{(2)}] + [\mathbf{H}^{(2)}, \Delta S_+^{(1)}],$

and so on.

General form of the kernel

$$[\mathbf{H}^{(k)}f](z_1, z_2) = \int dudv\, h^{(k)}(u,v) f(z_{12}^u, z_{21}^v)$$

The kernels are fixed up to invariant terms $[S_\alpha, \mathbf{H}_{inv}] = 0$, that is

$$\mathbf{H}^{(k)} = \mathbf{H}^{(k)}_{inv} + \Delta\mathbf{H}^{(k)}$$

where $\Delta\mathbf{H}^{(k)}$ (i.e. $\Delta h^{(k)}(u,v)$) is determined by the consistency equations.

Then if the anomalous dimensions $\gamma_N^{(k)}$ are known

$$\mathbf{H}^{(k)}(z_1 - z_2)^N = \gamma_N^{(k)}(z_1 - z_2)^N$$

one can find the spectrum of the operator $\mathbf{H}^{(k)}_{inv}$ and restore the operator itself.

(The kernel of an invariant invariant operator

$$h_{inv}(u,v) = (\bar{u}\bar{v})^{2j-2} h\left(\frac{uv}{\bar{u}\bar{v}}\right)$$

and it is uniquely determined by its spectrum.)

Results

- 3-loop evolution kernels for the twist-2 operators in φ^4 theory
- 2-loop kernels in φ^3 theory
- 2-loop kernels for flavor nonsinglet operators in QCD

One loop evolution kernel

$$\mathbf{H} = \frac{\alpha_s}{\pi} C_F \left[\widehat{\mathcal{H}} - \mathcal{H} + \frac{1}{2}\right]$$

$$[\widehat{\mathcal{H}}\varphi](z_1, z_2) = \int_0^1 du \frac{\bar{u}}{u} \Big[2\varphi(z_1, z_2) - \varphi(z_{12}^u, z_2) - \varphi(z_1, z_{21}^u)\Big]\,,$$

$$[\mathcal{H}\varphi](z_1, z_2) = \int_0^1 du \int_0^{\bar{u}} dv\, \varphi(z_{12}^u, z_{21}^v)$$

$$\boxed{S_+ = S_+^{(0)} + (z_1 + z_2) a_* \Big(- b_0 + \frac{1}{2}\mathbf{H}^{(1)}\Big) - 2a_* C_F z_{12} \widetilde{\mathcal{H}}}$$

$a = \alpha_s/4\pi$ and $\epsilon = b_0 a_* + O(a_*^2)$.

$$[\widetilde{\mathcal{H}}\varphi](z_1, z_2) = \int_0^1 du \left[\frac{\bar{u}}{u} + \ln u\right] \Big(\varphi(z_{12}^u, z_2) - \varphi(z_1, z_{21}^u)\Big)$$

$$\boxed{\mathbf{H}^{(2)} = b_0\, C_F\, \mathbf{H}_{b_0} + C_F^2\, \mathbf{H}_F + C_F\, C_A\, \mathbf{H}_A + \left(C_F^2 - \frac{1}{2} C_F\, C_A \right) P_{z_1 z_2} \mathbf{H}_P}$$

$\mathbf{H}_A$ and $\mathbf{H}_P$ invariant operators

$$[\mathbf{H}_P \varphi](z_1, z_2) = 8 \int_0^1 du \int_0^{\bar{u}} dv\, \varphi(z_{12}^u, z_{21}^v) \Big\{ \ln^2 \bar{\tau} - 2\tau \ln \bar{\tau} \Big\}$$

$$\tau = \frac{uv}{\bar{u}\bar{v}} \quad \text{and} \quad \bar{\tau} = 1 - \tau$$

$$\mathbf{H}_A = 4 \left(-\frac{8}{3} + \frac{2}{3}\pi^2 - 6\zeta(3) - \frac{4}{3}\widehat{H} + \Delta\mathcal{H} \right)$$

$$[\Delta\mathcal{H}\varphi](z_1, z_2) = \int_0^1 du \int_0^{\bar{u}} dv\, \varphi(z_{12}^u, z_{21}^v) \Big\{ 2\Big(\mathrm{Li}_2(\tau) - \mathrm{Li}_2(1) \Big) + \ln^2 \bar{\tau} - \frac{2}{\tau} \ln \bar{\tau} + \frac{10}{3} \Big\}$$

Two loop evolution kernel

$$\mathbf{H}_F = 4\left(6 + \frac{2}{3}\pi^2 - 12\zeta(3) + \mathcal{H}_{inv} + \mathcal{H}_{ninv}\right)$$

$$[\mathcal{H}_{inv}\varphi](z_1, z_2) = -\int_0^1 du \int_0^{\bar{u}} dv\, \varphi(z_{12}^u, z_{21}^v)\left\{4\,\mathrm{Li}_2(\tau) + 2\ln^2\bar{\tau} + \ln\tau - 2\left(\frac{2-\tau}{\tau}\right)\ln\bar{\tau}\right\}$$

$$\begin{aligned}[\mathcal{H}_{ninv}\varphi](z_1, z_2) = &\int_0^1 du \int_0^{\bar{u}} dv\, \varphi(z_{12}^u, z_{21}^v)\Big\{\ln^2\bar{u} + \ln^2\bar{v} - 2\ln u \ln\bar{u} - 2\ln v\ln\bar{v} \\ &+ \ln^2(1-u-v) - \ln u - \ln v + 2\left(\frac{\bar{u}}{u}\ln\bar{u} + \frac{\bar{v}}{v}\ln\bar{v}\right)\Big\} \\ &- 2\int_0^1 du\frac{\bar{u}}{u}\ln\bar{u}\left[\frac{3}{2} - \ln\bar{u} + \frac{1+\bar{u}}{\bar{u}}\ln u\right]\Big(2\varphi(z_1, z_2) - \varphi(z_{12}^u, z_2) - \varphi(z_1, z_{21}^u)\Big)\end{aligned}$$

Two loop evolution kernel

$$\mathbf{H}_{b_0} = 4\left[-\frac{5}{3}\widehat{\mathcal{H}} + \frac{11}{3}\mathcal{H} - \frac{13}{12} - \widehat{\mathcal{H}}' + \mathcal{H}'\right]$$

$$[\widehat{\mathcal{H}}'\varphi](z_1, z_2) = \int_0^1 du \frac{\bar{u}}{u} \ln \bar{u} \Big[2\varphi(z_1, z_2) - \varphi(z_{12}^u, z_2) - \varphi(z_1, z_{21}^u)\Big] ,$$

$$[\mathcal{H}'\varphi](z_1, z_2) = \int_0^1 du \int_0^{\bar{u}} dv\, \varphi(z_{12}^u, z_{21}^v) \ln(1 - u - v)$$