Basics of Inflation

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1 Introduction

Inflation, postulated as a period of exponential expansion in the very early universe, at about $\gtrsim 10^{-34}\,$s $\left(\lesssim 10^{15}\,\text{GeV}\right)$ after the Big Bang, is assumed to be responsible both for the large-scale homogeneity of the universe and for the small fluctuations observed in the CMB temperature. However, the fundamental origin of inflation is still an open question. In particular, the physical nature of the inflaton field ϕ and the shape of the inflationary potential $V(\phi)$ are not known yet.

The aim of this talk is to review the basics of inflation focusing in particular on slow-roll inflation. If not marked otherwise, I will refer to the results presented in [1, 2].
We will start with a summary of the basic equations of the homogeneous Friedmann-Robertson-Walker universe needed for the description of the physics of inflation afterwards.

1.1 Friedmann-Robertson-Walker Spacetime

Friedmann-Robertson-Walker Metric. The spacetime of the homogeneous and isotropic universe is described by the Friedmann-Robertson-Walker (FRW) metric which in conformal time τ takes the form

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + d\Phi^2 \sin^2\theta\right)\right], \qquad d\tau \equiv \frac{1}{a(t)}\,dt\,. \tag{1}$$

Thereby, we have introduced the scale factor $a(\tau)$, normalized to $a_0 \equiv a\left(\tau_0\right) = 1$ at present time τ_0, and the curvature parameter $k = 0, \pm 1$ for a flat, closed or open universe, respectively.

Friedmann Equations. The dynamics of the universe are governed by the Einstein equations[1]

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{2}$$

wherein the Einstein tensor $G_{\mu\nu}$ is defined in terms of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,, \tag{3}$$

$$R_{\mu\nu} = \Gamma^{\,\alpha}_{\,\mu\nu,\alpha} - \Gamma^{\,\alpha}_{\,\mu\alpha,\nu} + \Gamma^{\,\alpha}_{\,\beta\alpha}\Gamma^{\,\beta}_{\,\mu\nu} - \Gamma^{\,\alpha}_{\,\beta\nu}\Gamma^{\,\beta}_{\,\mu\alpha} \tag{4}$$

$$\text{with } \Gamma^{\,\mu}_{\,\alpha\beta} \equiv \frac{g^{\mu\nu}}{2}\left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}\right), \tag{5}$$

$$R \equiv g^{\mu\nu} R_{\mu\nu}\,. \tag{6}$$

For a perfect fluid, the stress-energy tensor $T_{\mu\nu}$ reads

$$T^{\mu}_{\nu} = g^{\mu\alpha} T_{\alpha\nu} = (\rho + p)\, u^{\mu} u_{\nu} - p\, \delta^{\mu}_{\nu}\,, \tag{7}$$

where ρ and p constitute the energy density and the pressure in the fluid rest frame and $u^{\mu} \equiv \frac{dx^{\mu}}{d\tau}$ denotes the 4-velocity of the fluid. If we choose $u^{\mu} = (1,0,0,0)$ in a frame comoving with the fluid, (7) reduces to

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}. \tag{8}$$

In order to evaluate the Einstein equations in the FRW spacetime, we put the Einstein tensor $G_{\mu\nu}$, determined by inserting the metric tensor $g_{\mu\nu}$ of (1) in (3)-(6), and the stress-energy tensor $T_{\mu\nu}$ for a perfect fluid (8) in the Einstein equations (2). As a result, these transform into two coupled, non-linear differential equations, the so-called *Friedmann equations,*

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2} \qquad \text{with } H \equiv \frac{\dot{a}}{a}\,, \tag{9}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho + 3p\right), \tag{10}$$

which involve the Hubble parameter H corresponding to the expansion rate of the universe. Note that the overdots in the above equations denote derivatives with respect to physical time t.
By subsequently introducing the equation-of-state parameter as

$$w \equiv \frac{p}{\rho}\,, \tag{11}$$

where $w = 0, \frac{1}{3}, -1$ for a universe dominated by non-relativistic matter, radiation or a cosmological constant Λ, respectively, and by additionally defining the density parameter

$$\Omega \equiv \frac{\rho}{\rho_{cr}} = \frac{\rho}{3\,H^2} \tag{12}$$

as the ratio of the total energy density ρ to the critical energy density $\rho_{cr} \equiv 3\,H^2$, the Friedmann equations (9) and (10) can be rewritten as

$$\boxed{\frac{k}{(aH)^2} = \Omega - 1} \tag{13}$$

$$\boxed{\frac{\ddot{a}}{aH^2} = -\frac{\Omega}{2}\left(1 + 3w\right)}\,. \tag{14}$$

Via the first Friedmann equation (13) the density parameter Ω is related to the curvature k of the universe, as illustrated in Tab.1a. Furthermore, the Friedmann equations determine the time evolution of the scale factor $a(t)$ and hence the expansion history of the universe. The solutions of the Friedmann equations for a flat universe ($k = 0 \Leftrightarrow \Omega = 1$) are summarized in Tab.1b.

[1] In the following, we will use units where $8\pi G \equiv 1$ so that the reduced Planck mass $M_{Pl} = (8\pi G)^{-1/2} \equiv 1$ will neither appear in the Friedmann equations nor in the equations deduced therefrom, as for instance the slow-roll conditions for inflation (cf. Sec.2.2.1).

k	Ω	Geometry
1	>1	open
0	1	flat
-1	<1	closed

(a)

Epoch	w	$\rho(a)$	$a(t)$	$a(\tau)$	τ_i
RD	$\frac{1}{3}$	$\frac{1}{a^4}$	$t^{1/2}$	τ	0
MD	0	$\frac{1}{a^3}$	$t^{2/3}$	τ^2	0
Λ	-1	const.	e^{Ht}	$-\frac{1}{\tau}$	$-\infty$

(b)

Table 1: *Summary of characteristic physical properties of the FRW universe.* (a) Relation between the curvature k and the density parameter Ω. (b) Solutions of the Friedmann equations for a flat universe ($k = 0 \Leftrightarrow \Omega = 1$) dominated by matter (MD), radiation (RD) or a cosmological constant Λ.

1.2 Initial-Condition Problems of the Big Bang Theory

The standard Big Bang theory requires very special and fine-tuned initial conditions, namely exact initial homogeneity and flatness of the universe, to be in agreement with observations. The problems of initial homogeneity and flatness are referred to as the *horizon problem* and the *flatness problem,* respectively.

1.2.1 The Horizon Problem

The so-called *(comoving) particle horizon* χ_p constitutes the *causal horizon*, i.e. it is defined as the maximum distance light can travel between an initial time t_i and a later time t,

$$\chi_p \equiv \tau - \tau_i = \int_{t_i}^{t} \frac{dt'}{a(t')} = \int_{a_i}^{a} d\ln a \left(\frac{1}{aH}\right), \tag{15}$$

where $(aH)^{-1}$ denotes the *comoving Hubble radius.* Usually, the initial time is taken to be the "origin of the universe", $t_i = 0$, defined by the Big Bang singularity $a_i \equiv a(t_i = 0) = 0$.[2]
During the conventional Big Bang expansion ($\tau_i = 0$, $w \geq 0$) in the flat universe, the comoving Hubble radius, determined by the second Friedmann equation (14) as

$$(aH)^{-1} = \frac{1}{H_0}\, a^{\frac{1}{2}(1+3w)}, \tag{16}$$

grows monotonically so that the comoving horizon τ, i.e. the fraction of the universe in causal contact, increases with time (cf. (15)),

$$\tau \propto a^{\frac{1}{2}(1+3w)} = \begin{cases} a & \text{RD } \left(w = \frac{1}{3}\right) \\ \sqrt{a} & \text{MD } (w = 0) \end{cases}. \tag{17}$$

This means that the comoving scales which enter the horizon today must have been far outside the horizon at the time of CMB decoupling and therefore causally disconnected. However, observations of the CMB reveal that the universe was extremely homogeneous on these scales at the time of CMB decoupling.
The standard Big Bang theory does not provide a dynamical reason to explain the extremely homogeneous physical conditions on the causally independent regions of space in the early universe. This problem of initial homogeneity is called the horizon problem.

1.2.2 The Flatness Problem

The first Friedmann equation (13) for a non-flat universe ($k = \pm 1$),

$$(aH)^{-2} = |\Omega(a) - 1|, \tag{18}$$

implies that a monotonically growing comoving Hubble radius $(aH)^{-1}$ leads to an increase of the difference $|\Omega(a) - 1|$ and hence drives the universe away from flatness $\Omega(a) = 1$. Since observations

[2]Notice that $t_i = 0$ not necessarily implies $\tau_i = 0$, but that the initial value τ_i depends on the evolution of the scale factor $a(\tau)$ (cf. Tab.1b).

indicate a nearly flat universe at present time, $\Omega(a_o) \simeq 1$, this restricts the curvature of the early universe to have been extremely close to flatness. In detail, the deviations from flatness at the time of Big Bang Nucleosynthesis (BBN), during the GUT era and at the Planck scale are constrained by

$$|\Omega\left(a_{BBN}\right) - 1| \leq \mathcal{O}\big(10^{-16}\big)\,, \quad |\Omega\left(a_{GUT}\right) - 1| \leq \mathcal{O}\big(10^{-55}\big)\,, \quad |\Omega\left(a_{Pl}\right) - 1| \leq \mathcal{O}\big(10^{-61}\big)\,. \tag{19}$$

The problem of explaining the initial flatness of the universe is named the flatness problem.

2 Inflation

The main success of inflation is that it does not require extremely fine-tuned initial conditions as the standard Big Bang theory, but allows the universe to grow out of generic initial conditions and hence solves the horizon and flatness problem.

2.1 Basic Concept of Inflation

Since both the horizon and the flatness problem emerge due to the strictly increasing comoving Hubble radius, the concept of inflationary cosmology is based on the introduction of a *period in the very early universe, namely inflation, where the comoving Hubble radius decreases for a sufficiently long time.* This decrease of $(aH)^{-1}$ originates from an *exponential growth of the scale factor a* during inflation, while the Hubble parameter H remains nearly constant. Therefore, spacetime during inflation can be approximately considered as de Sitter ($\Lambda > 0$ and $w = -1$, cf. Tab.1b).

Solution to the horizon and flatness problem. Since the comoving Hubble radius and hence also the causal horizon (cf. (15)) decreases during inflation, the comoving scales entering the present universe were inside the horizon before inflation and therefore causally connected. Thus, the spatial homogeneity, as for example seen in the CMB, was established by causal physics before inflation. In this way, inflation provides a solution to the horizon problem.

As can be seen from (18), inflation, if introduced as a period in the very early universe where the comoving Hubble radius decreases, solves the flatness problem by driving the universe towards flatness ($\Omega(a) = 1$).

Conditions for Inflation. Via the Friedmann equations the decreasing comoving Hubble radius can be related to the acceleration and the pressure of the universe during inflation. Therefore, the three equivalent conditions for inflation are:

1. **Decreasing comoving Hubble radius:**
   The condition for a decreasing comoving Hubble radius
   $$\boxed{\frac{d}{dt}\left(aH\right)^{-1} < 0} \tag{20}$$
   can be used as fundamental definition of inflation as it is most closely connected to the flatness and horizon problem.

2. **Accelerated expansion:**
   Since a decreasing comoving Hubble radius directly implies accelerated expansion,
   $$\frac{d}{dt}\left(aH\right)^{-1} = \frac{d}{dt}\left(\dot{a}\right)^{-1} = -\frac{\ddot{a}}{\left(\dot{a}\right)^2} = -\frac{\ddot{a}}{\left(aH\right)^2} < 0 \quad \Rightarrow \quad \boxed{\ddot{a} > 0} \tag{21}$$
   $\big($with $H = \frac{\dot{a}}{a}\big)$, inflation is referred to as a *period of accelerated expansion.*

3. **Negative pressure:**
   Moreover, the second Friedmann equation (10) reveals that accelerated expansion $\ddot{a} > 0$ requires negative pressure,
   $$\frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho + 3p\right) > 0 \quad \Rightarrow \quad \boxed{p < -\frac{1}{3}\rho}\,. \tag{22}$$

2.2 Dynamics of Inflation

In the following calculations, we will assume a flat FRW universe ($k = 0$).
The basic theoretical model to describe the underlying dynamics of inflation assumes a single scalar field ϕ, the *inflaton,* which is minimally coupled to gravity[3] and moves within a potential $V(\phi)$. Hence, the action of the inflaton field,

$$\boxed{S = \int d^4x\,\sqrt{-g}\,\left(\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\upsilon\phi - V(\phi)\right) = S_{EH} + S_\phi} \tag{23}$$

with $g \equiv \det(g_{\mu\nu})$, comprises the sum of the gravitational Einstein-Hilbert action S_{EH} and the action of a scalar field with canonical kinetic term S_ϕ. The variation of the action S_ϕ with respect to ϕ,

$$0 \stackrel{!}{=} \frac{\delta S_\phi}{\delta\phi}\,, \tag{24}$$

determines the equation of motion of the inflaton. By using the metric tensor $g_{\mu\nu}$ of the FRW spacetime (1) and assuming a homogeneous inflaton field $\phi(t,\vec{x}) \equiv \phi(t)$, the resulting equation of motion is given by

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial\phi} = 0}\,, \tag{25}$$

wherein the term $\propto H\dot{\phi}$ constitutes the so-called Hubble friction.
Similarly, the stress-energy tensor of the inflaton field is obtained by varying the action S_ϕ in (23) with respect to the metric tensor $g_{\mu\nu}$,

$$T_{\mu\nu}(\phi) \equiv -\frac{2}{\sqrt{-g}}\,\frac{\delta S_\phi}{\delta g^{\mu\nu}} \tag{26}$$

In the FRW spacetime, $T_{\mu\nu}(\phi)$ takes the form as for a perfect fluid (cf. (8)) with

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{27}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \tag{28}$$

so that the Friedmann equations (9) and (10) transform into

$$\boxed{H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)}\,, \tag{29}$$

$$\boxed{\dot{H} + H^2 = -\frac{1}{3}\left(\dot{\phi}^2 - V(\phi)\right)} \tag{30}$$

and hence

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2\,. \tag{31}$$

From the equation of state,

$$\boxed{w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}}\,, \tag{32}$$

we conclude that acceleration driven by negative pressure occurs, when the potential energy $V(\phi)$ of the inflaton dominates over its kinetic energy $\frac{1}{2}\dot{\phi}^2$. This is depicted in Fig.1.

[3] *Minimally* coupled to gravity has to be understood in the sense that the inflaton field is not directly coupled to the metric.

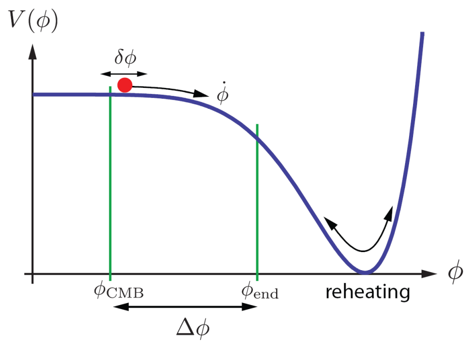


Figure 1: *Example of an inflationary potential* $V(\phi)$ [1]. Acceleration occurs when the potential energy of the inflaton dominates over its kinetic energy, $V(\phi) \gtrsim \frac{1}{2}\dot{\phi}^2$. As soon as the kinetic energy becomes comparable to the potential energy, $V(\phi) \simeq \frac{1}{2}\dot{\phi}^2$, inflation ends. The fluctuations in the CMB are created by quantum fluctuations $\delta\phi$ of the inflaton field about 60 e-folds before. After the end of inflation, the inflaton begins to oscillate around the minimum of the potential and its energy density is converted into radiation during reheating.

2.2.1 Slow-Roll Inflation

Hubble Slow-Roll Parameters. If we rewrite the second Friedmann equations (14) in the form

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{H^2}{2}\left(1 + 3w_\phi\right) = H^2\left(1 - \varepsilon\right) \tag{33}$$

and introduce the dimensionless so-called *Hubble slow-roll parameter* ε as

$$\epsilon \equiv \frac{3}{2}\left(w_\phi + 1\right) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}, \tag{34}$$

by using (32) and (29), the condition for accelerated expansion requires ε to be small,

$$\ddot{a} > 0 \quad \Rightarrow \quad \varepsilon < 1\,. \tag{35}$$

Since the slow-roll parameter ε is related to the evolution of the Hubble parameter H by (33),

$$\boxed{\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN}} \tag{36}$$

with $dN = Hdt = d\ln a$ denoting the number of e-folds, the above constraint (35) means that the fractional change of the Hubble parameter during inflation is small.
In order to guarantee that the accelerated expansion lasts for a sufficiently long time, we additionally require the friction term in the equation of motion (25) to be larger than the second time derivative of the inflaton field ϕ,

$$\left|\ddot{\phi}\right| \ll \left|H\dot{\phi}\right| \tag{37}$$

so that (25) reduces to

$$3H\dot{\phi} + \frac{\partial V(\phi)}{\partial\phi} \simeq 0\,. \tag{38}$$

The constraint (37) can be transformed into a condition for the second dimensionless *Hubble slow-roll parameter* η, defined as

$$\boxed{\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \varepsilon - \frac{1}{2\epsilon}\frac{d\varepsilon}{dN}}\,, \tag{39}$$

namely

$$\left|\ddot{\phi}\right| \ll \left|H\dot{\phi}\right| \quad \Rightarrow \quad \left|\eta\right| \ll 1. \tag{40}$$

It forces the fractional change of ε per e-fold to be small. For ensuring a successful period of inflation, we therefore have to require in total that the *slow-roll conditions*

$$\boxed{\varepsilon \ll 1, \quad \left|\eta\right| \ll 1} \tag{41}$$

for the Hubble slow-roll parameters ε and η are fulfilled.

Potential Slow-Roll Parameters. As explained in Sec.2.1, the spacetime during inflation is approximately de Sitter. In the de-Sitter limit ($w_\phi = -1$), corresponding to the case $\varepsilon = 0$ in (34), the potential energy dominates over the kinetic energy $V(\phi) \gg \frac{1}{2}\dot{\phi}^2$ (cf. (32)) and the first Friedmann equation (29) simplifies to

$$H^2 \simeq \tfrac{1}{3} V(\phi)\,. \tag{42}$$

Together with (38), this equation allows to express the slow-roll conditions as conditions on the shape of the inflationary potential $V(\phi)$. In detail, we use these equations to rewrite the Hubble slow-roll parameters

$$\begin{aligned}\varepsilon &= \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \simeq \frac{1}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \\ &\simeq \varepsilon_V\,,\end{aligned} \tag{43}$$

$$\begin{aligned}\eta &= -\frac{\ddot{\phi}}{H\dot{\phi}} \simeq \frac{V''(\phi)}{V(\phi)} - \frac{1}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \\ &\simeq \eta_V - \varepsilon_V\,,\end{aligned} \tag{44}$$

with $V'(\phi) \equiv \frac{\partial V(\phi)}{\partial \phi}$, $V''(\phi) \equiv \frac{\partial^2 V(\phi)}{\partial \phi^2}$, in terms of the corresponding *potential slow-roll parameters*

$$\varepsilon_V(\phi) \equiv \frac{1}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \tag{45}$$

$$\eta_V(\phi) \equiv \frac{V''(\phi)}{V(\phi)} \tag{46}$$

so that the *slow-roll conditions* transform into

$$\boxed{\varepsilon_V \ll 1, \quad \left|\eta_V\right| \ll 1}\,. \tag{47}$$

Inflation ends when $\varepsilon(\phi_{end}) \simeq \varepsilon_V(\phi_{end}) = 1$. To solve the horizon and the flatness problem, the number of e-folds $N(\phi)$ before the end of inflation,

$$\begin{aligned}N(\phi) &\equiv \ln\left(\frac{a_{end}}{a}\right) \\ &= \int_t^{t_{end}} H\, dt = \int_\phi^{\phi_{end}} \frac{H}{\dot{\phi}}\, d\phi \simeq \int_{\phi_{end}}^{\phi} \frac{V(\phi)}{\frac{dV(\phi)}{d\phi}}\, d\phi \\ &= \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\varepsilon}} \simeq \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\varepsilon_V}}\,,\end{aligned} \tag{48}$$

is required to exceed $N_{tot} \gtrsim 60$, whereas the CMB fluctuations are created at

$$N_{CMB} = \int_{\phi_{end}}^{\phi_{CMB}} \frac{d\phi}{\sqrt{2\varepsilon_V}} \simeq 40\ldots 60\,. \tag{49}$$

Note that the exact values of N_{tot} and N_{CMB} depend on the energy scale of inflation and on the details of reheating.

Example of $m^2\phi^2$ Inflation. As an example, we will perform the slow-roll analysis for the simplest model of inflation, namely a single inflaton field ϕ within a potential of the form

$$V(\phi) = \frac{1}{2} m^2 \phi^2\,. \tag{50}$$

In this model, the potential slow-roll parameters of (45) and (46) are given by

$$\varepsilon_V(\phi) \equiv \frac{M_{Pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = 2\,\frac{M_{Pl}^2}{\phi^2}\,, \qquad \eta_V(\phi) \equiv M_{Pl}^2\,\frac{V''(\phi)}{V(\phi)} = 2\,\frac{M_{Pl}^2}{\phi^2}\,, \tag{51}$$

where we reintroduced the Planck mass M_{Pl} to reveal that ε_V and η_V are dimensionless. The slow-roll conditions (47) are fulfilled, if the inflaton field acquires values significantly larger than the Planck mass,

$$\varepsilon_V \ll 1,\ |\eta_V| \ll 1 \quad \Rightarrow \quad \phi \gg \sqrt{2}\, M_{Pl} \equiv \phi_{end}\,. \tag{52}$$

Furthermore, we can deduce from the number of e-folds before the end of inflation,

$$N(\phi) \simeq \frac{1}{2M_{Pl}} \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\varepsilon_V}} = \frac{1}{4M_{Pl}^2}\left(\phi^2 - \phi_{end}^2\right) = \frac{\phi^2}{4M_{Pl}^2} - \frac{1}{2}\,, \tag{53}$$

that the CMB fluctuations are created at

$$N_{CMB} \simeq \frac{\phi_{CMB}^2}{4M_{Pl}^2} - \frac{1}{2} \simeq \frac{\phi_{CMB}^2}{4M_{Pl}^2} \quad \Rightarrow \quad \phi_{CMB} \simeq 2\, M_{Pl} \sqrt{N_{CMB}} \simeq 15\, M_{Pl}\,, \tag{54}$$

where we assumed $N_{CMB} \simeq 50$. Consequently, the difference $\Delta\phi \equiv \phi_{CMB} - \phi_{end} > M_{Pl}$ is super-Planckian. As we will discuss in the next section, this classifies the $m^2\phi^2$ model as so-called large-field inflationary model.

3 Classes of Inflationary Models

3.1 Single-Field Slow-Roll Inflation

Since the dynamics of a single inflaton field is determined by the shape of the inflationary potential $V(\phi)$, single-field models with an action of the form (23) can be classified by the characteristic distance $\Delta\phi \equiv \phi_{CMB} - \phi_{end}$ (measured in Planck units) of the potential. Depending on whether $\Delta\phi$ is sub- or super-Planckian, one distinguishes between small- and large-field inflationary models.

Small-Field Inflation. In small-field models of inflation, the inflaton field moves over a small sub-Planckian distance $\Delta\phi < M_{Pl}$. The inflationary potentials giving rise to a small-field evolution often arise in mechanisms of spontaneous symmetry breaking. Thus, typical examples of small-field inflation are:

- **Old Inflation [3]:**
  Models of old inflation incorporate Higgs-like potentials of the form

  $$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^2\right]^2 + \ldots\,, \tag{55}$$

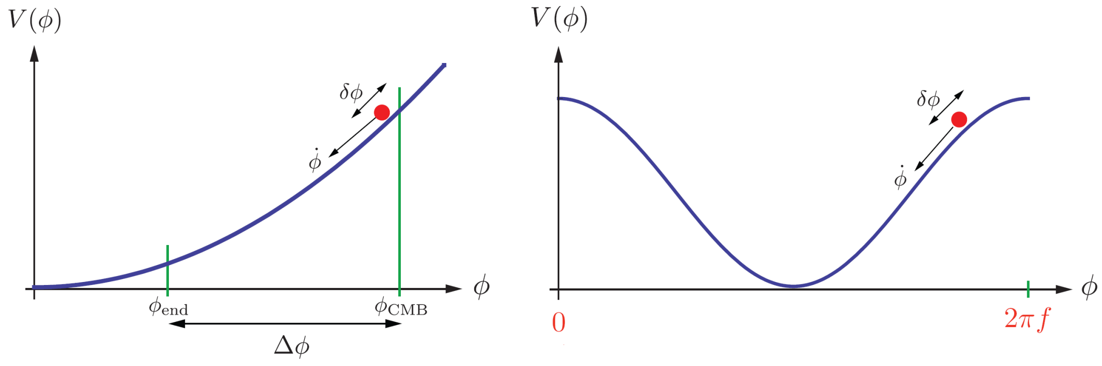
  where the dots refer to additionally added higher-order terms which become important near the end of inflation and during reheating. In this class of models, inflation proceeds via tunneling from the false to the true vacuum state.

- **New inflation [4, 5]:**
  In new inflation, the inflationary potential constitutes a Coleman-Weinberg potential,

  $$V(\phi) = V_0 \left\{ \left(\frac{\phi}{\mu}\right)^4 \left[\ln\left(\frac{\phi}{\mu}\right) - \frac{1}{4}\right] + \frac{1}{4} \right\}\,, \tag{56}$$

  which arises as the potential for radiatively-induced symmetry breaking in electroweak and grand-unified theories.

Large-Field Inflation. Large-field inflationary models are characterized by a super-Planckian evolution of the inflaton field, $\Delta\phi > M_{Pl}$. Important examples of large-field models include:



(a) *Chaotic Inflation.* (b) *Natural Inflation.*

Figure 2: *Examples of large-field inflationary potentials* $V(\phi)$ [1]

- **Chaotic inflation [6]:**
The simplest realizations of chaotic inflationary models consist of potentials with a single monomial term
$$V(\phi) = \lambda\phi^n \,, \tag{57}$$
as for instance discussed in the example at the end of Sec.2.2.1 and illustrated in Fig.2a. These models are referred to as "chaotic" because they allow arbitrary initial conditions.

- **Natural inflation [7]:**
One of the most elegant inflationary models is natural inflation (cf. Fig. 2b) where a potential of the form
$$V(\phi) = V_0 \left[\cos\left(\frac{\phi}{f}\right) + 1\right] \tag{58}$$
is assumed. This potential often arises if the inflaton field is taken to be an axion since in this case a shift symmetry can be employed to keep the potential flat even over large field ranges.

3.2 Extensions of Single-Field Slow-Roll Inflation

A large number of extensions of single-field slow-roll models with different theoretical motivations and observational predictions exist (for a review see e.g. [8]). In particular, single-field slow-roll models can be extended as follows:

- **Non-minimal coupling to gravity:**
By assuming a direct coupling of the inflaton field to the metric (the graviton), the action (23) is extended by an additional coupling term.

- **Modified gravity:**
Besides, the Einstein-Hilbert part of the action could be modified at high energies. The simplest realizations of this UV modification of gravity are the so-called $f(R)$ theories (with the Ricci scalar R), as for instance the Starobinsky model of R^2 inflation [9].

- **Non-canonical kinetic term:**
The Lagrangian $\mathcal{L}_\phi = X - V(\phi)$ of the action (23) possesses a canonical kinetic term $X \equiv \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$. If we assume that the high-energy theory includes non-canonical kinetic terms so that $\mathcal{L}_\phi = F(\phi, X) - V(\phi)$ with a function $F(\phi, X)$ of the inflaton field and its derivatives, it is possible that inflation is driven by this kinetic terms and occurs even in the presence of a steep potential.

- **Multiple fields:**
A large amount of possible inflationary models (see e.g. [10]) is gained by introducing additional fields to be dynamically relevant during inflation. Examples of these multi-field models are *hybrid inflation* [11, 12] with an additional "waterfall" scalar field, which triggers the end of inflation, or the *curvaton scenario* [13, 14], wherein the CMB fluctuations are generated by a second scalar field, called *curvaton*.

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