Observational consequences of Inflation

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1 Quantum Fluctuations in de Sitter Space

1.1 Scalar Perturbations

Consider action for single-field slow-roll model of inflation

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,\left[R - (\nabla\phi)^2 - 2V(\phi)\right]\,, \tag{1}$$

with the following gauge for the dynamical fields g_{ij} and ϕ

$$\delta\phi = 0\,, \qquad g_{ij} = a^2[(1-2\mathcal{R})\delta_{ij} + h_{ij}]\,, \qquad \partial_i h_{ij} = h^i_i = 0\,. \tag{2}$$

(Aided by Scalar-Vector-Tensor decomposition. We ignore the vector perturbations S_i and F_i aren't created by inflation and decay with the expansion of the universe.)

$\mathcal{R}$ remains constant outside the horizon, and we compute its correlation functions at horizon crossing.

1.1.1 Free Field Action

Expand the action (1) up to $\mathcal{O}(\mathcal{R}^3)$

$$\boxed{S_{(2)} = \frac{1}{2}\int \mathrm{d}^4x\, a^3\frac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right]}\,. \tag{3}$$

$$\boxed{S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x\,\left[(v')^2 + (\partial_i v)^2 + \frac{z''}{z}v^2\right]}\,, \qquad (...)' \equiv \partial_\tau(...)\,. \tag{4}$$

with

$$v \equiv z\mathcal{R}\,, \qquad \text{where} \qquad z^2 \equiv a^2\frac{\dot\phi^2}{H^2} = 2a^2\varepsilon\,, \tag{5}$$

and transitioning to conformal time τ

Fourier expand the field v

$$v(\tau,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, v_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{6}$$

where

$$\boxed{v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0}\,. \tag{7}$$

1.1.2 Quantization

Promote the field v to quantum operator via

$$v_{\mathbf{k}} \quad \to \quad \hat{v}_{\mathbf{k}} = v_k(\tau)\hat{a}_{\mathbf{k}} + v^*_{-k}(\tau)\hat{a}^\dagger_{-\mathbf{k}}\,, \tag{8}$$

where the creation and annihilation operators $\hat{a}^\dagger_{-\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ satisfy

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\,, \tag{9}$$

$$\langle v_k, v_k\rangle \equiv \frac{i}{\hbar}(v_k^* v_k' - v_k^{*\prime} v_k) = 1\,. \tag{10}$$

1.1.3 Boundary Conditions and Bunch-Davies Vacuum

A vacuum state for the fluctuations

$$\hat{a}_{\mathbf{k}}|0\rangle = 0 \tag{11}$$

can be chosen in a standard way, as the Minkowski vacuum of a comoving observer in the far past (when all comoving scales were far inside the Hubble horizon), $\tau \to -\infty$ or $|k\tau| \gg 1$ or $k \gg aH$.

$$\boxed{\lim_{\tau\to-\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}}\,. \tag{12}$$

1.1.4 Solution in de Sitter Space

Consider the de Sitter limit $\varepsilon \equiv -\frac{\dot{H}}{H} \to 0$ and

$$\frac{z''}{z} = \frac{a''}{a} = \frac{2}{\tau^2}\,. \tag{13}$$

The mode equation

$$\boxed{v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0} \tag{14}$$

has an exact solution

$$v_k = \alpha\,\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta\,\frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right). \tag{15}$$

After considering the boundary conditions, the unique Bunch-Davies mode function is

$$\boxed{v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)}\,. \tag{16}$$

1.1.5 Power Spectrum in Quasi-de Sitter

Compute the power spectrum of the field $\hat{\psi}_{\mathbf{k}} \equiv a^{-1}\hat{v}_{\mathbf{k}}$,

$$\langle \hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{|v_k(\tau)|^2}{a^2} = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H^2}{2k^3}(1+k^2\tau^2)\,. \tag{17}$$

On superhorizon scales, $|k\tau| \ll 1$, this approaches a constant

$$\langle \hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}_{\mathbf{k}'}(\tau)\rangle \to (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H^2}{2k^3} = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\Delta_\psi^2\,. \tag{18}$$

or

$$\Delta_\psi^2 = \left(\frac{H}{2\pi}\right)^2\,. \tag{19}$$

Compute the power spectrum of $\mathcal{R} = \frac{H}{\dot\phi}\psi$ at horizon crossing, $a(t_\star)H(t_\star) = k$, with dimensionless power spectrum $\Delta^2_{\mathcal{R}}(k)$ by

$$\langle \mathcal{R}_{\mathbf{k}}\mathcal{R}_{\mathbf{k}'}\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')P_{\mathcal{R}}(k)\,, \qquad \Delta^2_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2}P_{\mathcal{R}}(k)\,, \tag{20}$$

such that the real space variance of $\mathcal{R}$ is $\langle \mathcal{R}\mathcal{R}\rangle = \int_0^\infty \Delta^2_{\mathcal{R}}(k)\,\mathrm{d}\ln k$. This gives

$$\boxed{\Delta^2_{\mathcal{R}}(k) = \frac{H_\star^2}{(2\pi)^2}\frac{H_\star^2}{\dot\phi_\star^2}}\,. \tag{21}$$

1.2 Tensor Perturbations

1.2.1 Action

By expansion of the Einstein-Hilbert action one may obtain the second-order action for tensor fluctuations is

$$S_{(2)} = \frac{M_{\rm pl}^2}{8}\int \mathrm{d}\tau \mathrm{d}x^3 a^2\left[(h'_{ij})^2 - (\partial_l h_{ij})^2\right]\,. \tag{22}$$

We define the following Fourier expansion

$$h_{ij} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{s=+,\times}\epsilon^s_{ij}(k)h^s_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{23}$$

where $\epsilon_{ii} = k^i\epsilon_{ij} = 0$ and $\epsilon^s_{ij}(k)\epsilon^{s'}_{ij}(k) = 2\delta_{ss'}$. The tensor action (22) becomes

$$S_{(2)} = \sum_s \int \mathrm{d}\tau\mathrm{d}\mathbf{k}\frac{a^2}{4}M_{\rm pl}^2\left[h^s_{\mathbf{k}}{}'h^s_{\mathbf{k}}{}' - k^2h^s_{\mathbf{k}}h^s_{\mathbf{k}}\right]\,. \tag{24}$$

We define the canonically normalized field

$$v^s_{\mathbf{k}} \equiv \frac{a}{2}M_{\rm pl}h^s_{\mathbf{k}}\,, \tag{25}$$

to get

$$\boxed{S_{(2)} = \sum_s \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}^3\mathbf{k} \left[(v_{\mathbf{k}}^{s\prime})^2 - \left(k^2 - \frac{a''}{a}\right) (v_{\mathbf{k}}^s)^2 \right]}, \tag{26}$$

where

$$\frac{a''}{a} = \frac{2}{\tau^2} \tag{27}$$

holds in de Sitter space.

1.2.2 Quantization

Each polarization of the gravitational wave is just a renormalized massless field in de Sitter space

$$h_{\mathbf{k}}^s = \frac{2}{M_{\mathrm{pl}}} \psi_{\mathbf{k}}^s , \qquad \psi_{\mathbf{k}}^s \equiv \frac{v_{\mathbf{k}}}{a} . \tag{28}$$

The power spectrum for a single polarization of tensor perturbations is

$$\Delta_h^2(k) = \frac{4}{M_{\mathrm{pl}}^2} \left(\frac{H_\star}{2\pi}\right)^2 . \tag{29}$$

1.2.3 Power Spectrum

The dimensionless power spectrum of tensor fluctuations therefore is

$$\boxed{\Delta_{\mathrm{t}}^2 = 2\Delta_h^2(k) = \frac{2}{\pi^2} \frac{H_\star^2}{M_{\mathrm{pl}}^2}} . \tag{30}$$

1.3 The Energy Scale of Inflation

The *tensor-to-scalar ratio* is

$$r \equiv \frac{\Delta_{\mathrm{t}}^2(k)}{\Delta_{\mathrm{s}}^2(k)} . \tag{31}$$

Since Δ_{s}^2 is fixed and $\Delta_{\mathrm{t}}^2 \propto H^2 \approx V$, r is a direct measure of the energy scale of inflation

$$\boxed{V^{1/4} \sim \left(\frac{r}{0.01}\right)^{1/4} 10^{16}\,\mathrm{GeV}} . \tag{32}$$

2 Primordial Spectra

The results for the power spectra of the scalar and tensor fluctuations created by inflation are

$$\Delta_{\mathrm{s}}^2(k) \equiv \Delta_{\mathcal{R}}^2(k) = \left. \frac{1}{8\pi^2} \frac{H^2}{M_{\mathrm{pl}}^2} \frac{1}{\varepsilon} \right|_{k=aH} , \tag{33}$$

$$\Delta_{\mathrm{t}}^2(k) \equiv 2\Delta_h^2(k) = \left. \frac{2}{\pi^2} \frac{H^2}{M_{\mathrm{pl}}^2} \right|_{k=aH} , \tag{34}$$

where

$$\varepsilon = -\frac{d\ln H}{dN}\,. \tag{35}$$

The tensor-to-scalar ratio is

$$r \equiv \frac{\Delta_{\rm t}^2}{\Delta_{\rm s}^2} = 16\,\varepsilon_\star\,. \tag{36}$$

2.1 Scale-Dependence

The spectral indices are

$$n_{\rm s}-1 \equiv \frac{d\ln\Delta_{\rm s}^2}{d\ln k} = \frac{d\ln\Delta_{\rm s}^2}{dN}\times\frac{dN}{d\ln k}\,, \qquad n_{\rm t} \equiv \frac{d\ln\Delta_{\rm t}^2}{d\ln k}\,. \tag{37}$$

The derivative with respect to e-folds is

$$\frac{d\ln\Delta_{\rm s}^2}{dN} = \frac{d\,(\ln H^2/\varepsilon + \text{const})}{dN} = 2\frac{d\ln H}{dN} - \frac{d\ln\varepsilon}{dN}\,. \tag{38}$$

$$\frac{d\ln\varepsilon}{dN} = 2(\varepsilon-\eta)\,, \qquad \text{where} \quad \eta = -\frac{d\ln H_{,\phi}}{dN}\,. \tag{39}$$

$$\ln k = N + \ln H\,. \tag{40}$$

Hence

$$\frac{dN}{d\ln k} = \left[\frac{d\ln k}{dN}\right]^{-1} = \left[1+\frac{d\ln H}{dN}\right]^{-1} = (1-\varepsilon)^{-1} \approx 1+\varepsilon\,. \tag{41}$$

To first order in the Hubble slow-roll parameters

$$n_{\rm s}-1 = \left[-2\varepsilon - 2(\varepsilon-\eta)\right](1+\varepsilon) = 2\eta_\star - 4\varepsilon_\star, \quad n_{\rm t} = -2\varepsilon_\star. \tag{42}$$

2.2 Slow-Roll Results

In the slow-roll approximation the Hubble and potential slow-roll parameters are related as follows

$$\varepsilon \equiv -\frac{\dot H}{H^2} \approx \epsilon_{\rm v}\,, \qquad \eta \equiv -\frac{\ddot\phi}{H\dot\phi} \approx \eta_{\rm v} - \epsilon_{\rm v}\,. \tag{43}$$

The scalar and tensor spectra are then expressed purely in terms of $V(\phi)$ and $\epsilon_{\rm v}$ (or $V_{,\phi}$)

$$\Delta_{\rm s}^2(k) \approx \frac{1}{24\pi^2}\frac{V}{M_{\rm pl}^4}\frac{1}{\epsilon_{\rm v}}\bigg|_{k=aH}\,, \qquad \Delta_{\rm t}^2(k) \approx \frac{2}{3\pi^2}\frac{V}{M_{\rm pl}^4}\bigg|_{k=aH}\,. \tag{44}$$

The scalar spectral index is

$$\boxed{n_{\rm s}-1 = 2\eta_{\rm v}^\star - 6\epsilon_{\rm v}^\star}\,. \tag{45}$$

The tensor spectral index is

$$\boxed{n_{\rm t} = -2\epsilon_{\rm v}^\star}\,, \tag{46}$$

and the tensor-to-scalar ratio is

$$\boxed{r = 16\epsilon_{\rm v}^\star}\,. \tag{47}$$