The Starobinsky Model

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1 A little history...

Starobinsky's motivation was to find a cosmological model without a singularity in the past. Advances in the area of QFT on curved spaces [1–5] caused him to consider quantum corrections[1] to the Einstein equations in order to find non singular solutions.
In his famous model [7], Starobinsky did not a priori introduce a modification of the R-term in the Einstein-Hilbert action but simply considered the Einstein equations with then recently proposed quantum corrections to the right-hand-side, i.e.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \langle T_{\mu\nu} \rangle, \tag{1.1}$$

with $8\pi G = c = 1$ and

$$\begin{aligned} \langle T_{\mu\nu} \rangle =& K_1 \left(R^\sigma_\mu R_{\nu\sigma} - \frac{2}{3} R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_{\sigma\tau} R^{\sigma\tau} + \frac{1}{4} g_{\mu\nu} R^2 \right) \\ &+ K_2 \left(\nabla_\mu \nabla_\nu R - 2 g_{\mu\nu} \Box R - 2 R R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R^2 \right), \end{aligned} \tag{1.2}$$

where the constants K_1, K_2 depend on the number and spins of the quantum fields taken into account. The Einstein equations can then be solved to yield a universe where an early de Sitter phase extends infinitely to the past but comes to an end at a later point, thus the initial singularity is avoided and the understood evolution of the universe is not spoiled.
Only later, it was realised that Starobinsky's modification of (1.1) could be reproduced by modifying the Einstein-Hilbert action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} R \tag{1.3}$$

with [8]

$$R \to f(R) = R + \alpha R^2 + \beta R^2 \ln \frac{R}{C}, \tag{1.4}$$

where $\alpha \gg \beta$ and C is an arbitrary constant. So let's review $f(R)$ theory...

[1] He took into consideration the one-loop approximation of the interaction of quantum free matter fields with the gravitational field, see [6] for a comprehensive review.

2 f(R) Theory

The subsequent discussion follows the brilliant treatment of [9]. Consider the modified Einstein-Hilbert action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}f(R) + \int d^4x\mathcal{L}_M(g_{\mu\nu}, \Psi_M), \tag{2.1}$$

where the second term is the matter Lagrangian and $R = g^{\mu\nu}R_{\mu\nu}$, $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$. Varying the above action with respect to $g_{\mu\nu}$ yields

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) - \nabla_\mu\nabla\nu f'(R) + g_{\mu\nu}\Box f'(R) = T^{(M)}_{\mu\nu}, \tag{2.2}$$

where $f'(R) = \partial f/\partial R$, $\Box = \nabla^\mu\nabla\mu$ and $T^{(M)}_{\mu\nu}$ is the energy-momentum tensor of matter. Setting $f(R) = R - 2\Lambda$ reproduces the Einstein equations with a cosmological constant term. Taking the trace of (2.2) and considering $T^{(M)} = g^{\mu\nu}T^{(M)}_{\mu\nu}$, we find

$$3\Box \underbrace{f'(R)}_{scalaron\ \chi} + f'(R)R - 2f(R) = T^{(M)}. \tag{2.3}$$

The above determines the dynamics of the propagating scalar degree of freedom[2] $f'(R)$. Considering a vacuum, i.e. $T = 0$ and imposing a constant R, thus $\Box f'(R) = 0$, we find a de Sitter point at which $f'R - 2f = 0$. The model $f = \alpha R^2$ satisfies this condition and therefore gives rise to exact de Sitter space. A model of the type $f = R + \alpha R^2$ leads to a de Sitter phase for as long as the R^2-term dominates. When the linear term becomes significant, inflationary expansion ends and a phase of reheating follows in which oscillation of R leads to gravitational particle production.
Considering a spatially flat FLRW spacetime, i.e.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)d\mathbf{x}^2, \tag{2.4}$$

we have

$$R = 6\left(2H^2 + \dot{H}\right), \tag{2.5}$$

where the dot denotes differentiation with respect to coordinate time t and we find the field equations

$$3f'H^2 = \frac{1}{2}(f'R - f) - 3H\dot{f}' + \rho_M, \tag{2.6}$$

$$-2f'\dot{H} = \ddot{f}' - H\dot{f}' + (\rho_M + P_M), \tag{2.7}$$

$$\dot{\rho}_M = -3H(\rho_M + P_M), \tag{2.8}$$

with $T^{(M)}_{\mu\nu} = diag(\rho_M, -P_M, -P_M, -P_M)$.

[2]Inspecting (2.3) lets one recognise the term $f'(R)R = \chi R$ which indicates a non-minimal coupling of the Ricci scalar to the field χ. When the equation of motion is non-linear in R but shows a term where R is multiplied by some scalar function, one speaks of the equation being formulated in the Jordan frame. Strictly speaking, $f(R)$ is not a function of some scalar field but we will nevertheless refer to this representation as being in the Jordan frame.

3 Inflation in f(R) Theories

Consider models of the type

$$f(R) = R + \alpha R^n, \tag{3.1}$$

with $\alpha, n > 0$ in the absence of matter, i.e. $\rho_M = 0$. Equation(2.6) then yields

$$3(1 + n\alpha R^{n-1})H^2 = \frac{1}{2}(n-1)\alpha R^n - 3n(n-1)\alpha H R^{n-2}\dot{R}. \tag{3.2}$$

Inflation may be realised when the second term of (3.1) dominates, i.e. $f' = 1 + n\alpha R^{n-1} \gg 1$. We thus can approximate $f' \approx n\alpha R^{n-1}$ and recast (3.2) as

$$H^2 \approx \frac{n-1}{6n}\left(R - 6nH\frac{\dot{R}}{R}\right). \tag{3.3}$$

For inflation, we require the Hubble parameter to evolve slowly, i.e. $|\dot{H}/H^2|$, $|\ddot{H}/(H\dot{H})| \ll 1$. Then the above reduces to

$$\epsilon = -\frac{\dot{H}}{H^2} \approx \frac{-2+n}{(n-1)(2n-1)}. \tag{3.4}$$

For $n = 2$, the above equals zero and the exact de Sitter solution is recovered. Assuming $\epsilon > 0$, the above may be solved to yield[3]

$$H \approx \frac{1}{\epsilon t}, \tag{3.5}$$

$$a \propto t^{1/\epsilon}. \tag{3.6}$$

The standard inflationary scenario with decreasing H is realised within the regime $(1+\sqrt{3})/2 < n < 2$.

In the following, we will consider a reformulation [10] of Starobinsky's original work with

$$f(R) = R + \frac{R^2}{6M^2}, \tag{3.7}$$

where the constant M has dimensions of mass. Considering equations (2.6) and (2.7), we now have

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{1}{2}M^2 H = -3H\dot{H}, \tag{3.8}$$

$$\ddot{R} + 3H\dot{R} + M^2 R = 0. \tag{3.9}$$

We immediately see that R has an equation of motion of damped harmonic oscillator type where the damping is due to Hubble friction. This will eventually cause the linear R-term to dominate over the quadratic one and thus inflation

[3]Strictly speaking, solving (3.5) gives $a = a_0 C^{-n}(1 + x/n)^n$, where $n = \epsilon^{-1}$, $x = C \cdot t$ and C is an appearing constant of integration. Approaching the de Sitter limit gives $n \to \infty$ and the definition of the exponential function appears. However, the term $C^{-n} \to \infty$ or $\to 0$ depending on whether $C < 1$ or $C > 1$ respectively. Hence, choosing $C = H_0$ with H_0 being the Hubble parameter at the onset of inflation and working in units of H_0, i.e. $C = H_0 = 1$, one recovers $a \propto e^t$ which is exponential behaviour in the de Sitter limit as expected.

to end.
During inflation, the first two terms on the left-hand side of (3.10) may be omitted, hence we have $\dot{H} \approx -M^2/6$. We then find

$$H \approx H_0 - \frac{M^2}{6}(t - t_0) \tag{3.10}$$

$$a \approx a_0 e^{H_0(t-t_0)-(M^2/12)(t-t_0)^2} \tag{3.11}$$

$$R \approx 12H^2 - M^2, \tag{3.12}$$

where the zero denotes the quantities value at the onset of inflation. Inflation persists as long as

$$\epsilon = -\frac{\dot{H}}{H^2} \approx \frac{M^2}{6H^2} < 1. \tag{3.13}$$

Inflation ends when $H^2 = M^2/6$, thus M determines the energy scale of the end of inflation. From (3.10) we find inflation to end at $t_f \approx t_0 + 6H_0/M^2$. Thus the number of e-folds of the inflationary phase is[4]

$$N \equiv \int_{t_0}^{t_f} H dt \approx H_0(t_f - t_0) - \frac{M^2}{12}(t_f - t_i)^2 = \frac{3H_0^2}{M^2} \approx \frac{1}{2\epsilon(t_0)}. \tag{3.14}$$

So far it looks as if we have a very good candidate for an inflationary model. However, recalling the last talk, we might ask what mechanism might now be responsible for generating primordial perturbations as we don't have a quantum scalar field that undergoes fluctuations. So consider the following...

4 Conformal Transformations

We now seek an action that entails the physics of the above yet is only linear in R. But isn't that against the whole point of $f(R)$-theories? Consider the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{4.1}$$

where Ω^2 is the conformal factor and a tilde now denotes a quantity in the Einstein frame. When introducing

$$\omega \equiv \ln \Omega,\ \partial_\mu \omega \equiv \frac{\partial \omega}{\partial \tilde{x}^\mu},\ \tilde{\Box} \equiv \frac{1}{\sqrt{-\tilde{g}}}(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu \omega), \tag{4.2}$$

one may find

$$R = \Omega^2(\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega), \tag{4.3}$$

where $\sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}}$. The action may be recast as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}f'R - U\right) + \int d^4x\mathcal{L}_m, \tag{4.4}$$

with $U = 1/2(f'R - f)$. Combining all of the above, one may write

$$S = \int d^4x\sqrt{-\tilde{g}}\left(\frac{1}{2}f'\Omega^{-2}(\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega) - \Omega^{-4}U\right) + \int d^4x\mathcal{L}_M(\Omega^{-2}\tilde{g}_{\mu\nu}, \Psi_M). \tag{4.5}$$

[4] Again, note that for de Sitter we have $\epsilon = 0$, hence the expression diverges implying an ever lasting de Sitter phase.

By inspection, one may then choose $\Omega^2 = f'(R)$ for $f' > 0$. We furthermore introduce a canonical variable $\phi = \sqrt{3/2}\ln f'(R)$ and thus have[5] $\omega = \phi/\sqrt{6}$. Gauss' theorem will let a contribution of the type $\int d^4x\sqrt{-\tilde{g}}\tilde{\Box}\omega$ vanish and we can hence formulate the action in the Einstein frame as

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left(\frac{1}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) + \int d^4x\mathcal{L}_M\left(f'^{-1}(\phi)\tilde{g}_{\mu\nu}, \Psi_M\right), \tag{4.6}$$

where

$$V(\phi) = \frac{U}{f'^2} = \frac{f'R - f}{2f'^2}. \tag{4.7}$$

We have thus found an effective scalar field with the Lagrangian $\mathcal{L}_\phi = -\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$. Note that the integrand of the above action is identical to the action of Higgs inflation after canonical normalisation in the Einstein frame at large ϕ, i.e.

$$\frac{1}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - V(\tilde{\phi}), \tag{4.8}$$

where the tilde denotes the Higgs field in the Einstein frame and $f(\tilde{\phi}) = 1+\zeta\tilde{\phi}^2$. The Higgs potential[6] $V(\tilde{\phi}) = f(\tilde{\phi})^{-2}\frac{1}{4}\lambda_H(\phi(\tilde{\phi})^2 - v^2)$ in the Einstein frame is proportional to $\tilde{\phi}^2$ for small ϕ and flat at large ϕ.
This behaviour is similar to that of the effective scalar field's potential in R^2 inflation that will be subject of the next section.

5 Dynamics in the Einstein Frame

In the Einstein frame, the metric is given by

$$d\tilde{s}^2 = \Omega^2 ds^2 = f'(dt^2 - a^2 d\mathbf{x}^2) = d\tilde{t}^2 - \tilde{a}^2 d\mathbf{x}^2, \tag{5.1}$$

from which we deduce

$$\tilde{H} \equiv \frac{1}{\tilde{a}}\frac{d\tilde{a}}{d\tilde{t}} = \frac{1}{\sqrt{f'}}\left(H + \frac{\dot{f}'}{2f'}\right). \tag{5.2}$$

The effective scalar field is

$$\phi = \sqrt{\frac{3}{2}}\ln f' = \sqrt{\frac{3}{2}}\ln\left(1 + \frac{R}{3M^2}\right). \tag{5.3}$$

Considering the above, one has

$$V(\phi) = \frac{3M^2}{4}\left(1 - e^{-\sqrt{2/3}\phi}\right)^2, \tag{5.4}$$

which, as pointed out earlier, behaves similarly to the Higgs potential in the Einstein frame given as

$$V(\tilde{\phi}) = \frac{\lambda_H}{4\zeta^2}\left(1 - e^{-\sqrt{2/3}\phi}\right)^2. \tag{5.5}$$

[5]Obviously, we now may also express f' in terms of ϕ, namely $f'(\phi) = \exp(\sqrt{2/3}\phi)$.
[6]For a comprehensive discussion of Higgs inflation, see `desy.de/~westphal/workshop_seminar_fall_2010/Higgs_Inflation.pdf`.

The potential is flat for large field values thus leading to slow roll inflation and quadratic for low values of the field causing the field to oscillate around its minimum.
Now recalling $R = 6(2H^2 + \dot{H}) \approx 12H^2$, we may write

$$f' = 1 + \frac{R}{3M^2} \approx 4\frac{H^2}{M^2}. \tag{5.6}$$

Therefore, the time variable $\tilde{t}$ in the Einstein frame can be cast as

$$\tilde{t} = \int_{t_0}^{t} dt\sqrt{f'} \approx \frac{2}{M}\left(H_0(t-t_0) - \frac{M^2}{12}(t-t_0)^2\right). \tag{5.7}$$

Furthermore, we find

$$\tilde{a} \approx \left(1 - \frac{M^2}{12H_0^2}M\tilde{t}\right)\tilde{a}_0 e^{M\tilde{t}/2}, \tag{5.8}$$

where $\tilde{a}_0 = 2H_0a_0/M$. At last, the Hubble parameter in the Einstein frame is given as

$$\tilde{H} \approx \frac{M}{2}\left[1 - \frac{M^2}{6H_0^2}\left(1 - \frac{M^2}{12H_0^2}M\tilde{t}\right)^{-2}\right], \tag{5.9}$$

which for which $\dot{\tilde{H}} < 0$. The field equations for the action (4.6) are

$$3\tilde{H}^2 = \frac{1}{2}\left(\frac{d\phi}{d\tilde{t}}\right)^2 + V(\phi), \tag{5.10}$$

$$\frac{d^2\phi}{d\tilde{t}^2} + 3\tilde{H}\frac{d\phi}{d\tilde{t}} + \frac{dV}{d\phi} = 0, \tag{5.11}$$

which are of standard form. As usual, we can define the slow-roll parameters

$$\tilde{\epsilon} \equiv -\frac{d\tilde{H}/d\tilde{t}}{\tilde{H}^2} \approx \frac{1}{2}\left(\frac{dV/d\phi}{V}\right)^2 = \epsilon_v, \tag{5.12}$$

$$\tilde{\eta} \equiv \frac{d^2\phi/d\tilde{t}^2}{\tilde{H}(d\phi/d\tilde{t})} \approx \tilde{\epsilon} - \frac{\partial^2 V/\partial\phi^2}{3\tilde{H}^2} = \eta_v. \tag{5.13}$$

The number of e-folds in the Einstein frame is

$$\tilde{N} = \int_{\tilde{t}_0}^{\tilde{t}_f} \tilde{H}d\tilde{t}, \tag{5.14}$$

which we relate to the number of e-folds in the Jordan frame via

$$\begin{aligned} \tilde{H}d\tilde{t} &= \frac{1}{\sqrt{f'}}\left(H + \frac{\dot{f}'}{2f'}\right)\sqrt{f'}dt \\ &= Hdt\underbrace{\left(1 + \frac{\dot{H}}{H^2}\right)}_{\to 1} \\ &= dN, \end{aligned}$$

thus the number of e-folds during slow-roll is equal in both frames. Evaluating the slow-roll parameters with the potential of the effective scalar field in the Einstein frame and combining the results with the expression for the number of e-folds in terms of the potential

$$\tilde{N} = \int_{\phi_f}^{\phi_0} \frac{V}{V_{,\phi}} d\phi, \tag{5.15}$$

as well as the approximation $\tilde{H} \approx M/2$, we find

$$\tilde{\epsilon}_v \approx \frac{3}{4\tilde{N}^2}, \ \tilde{\eta}_v \approx \frac{1}{\tilde{N}}, \tag{5.16}$$

which will be of use later on.

6 Density Perturbations

In the Jordan frame, we cannot rely on the quantum fluctuations of a scalar field to provide a mechanism for the creation of primordial perturbations. The Einstein equations may hence only be perturbed with

$$F \to \bar{F} + \delta F, \tag{6.1}$$

where the bar denotes the unperturbed background value. Unlike the case with the scalar field, the physical origin of the perturbation remains unspecified. Nevertheless, one obtains a familiar result, namely a scale invariant spectrum of the curvature perturbation $\mathcal{R}$

$$P_{\mathcal{R}} \approx \frac{1}{Q_s} \left(\frac{H}{2\pi} \right)^2 \tag{6.2}$$

with $Q_s = \dot{\phi}^2/H^2$. For $f(R) = R + R^2/(6M^2)$, we furthermore obtain the results

$$N_k \approx \frac{1}{2\epsilon(t_*)}, \tag{6.3}$$

$$P_{\mathcal{R}} \approx \frac{N_*^2}{3\pi} \left(\frac{M}{m_{pl}} \right)^2, \tag{6.4}$$

$$n \approx 1 - \frac{2}{N_*}, \tag{6.5}$$

$$r \approx \frac{12}{N_*^2}, \tag{6.6}$$

where t_* is the time of (event) horizon exit of the perturbations we now observe in the CMB and N_* is the number of e-folds before the end of inflation for the horizon exit of those perturbations. WMAP five year data constraints $P_{\mathcal{R}} \approx (2.445 \pm 0.096) \cdot 10^{-9}$ and thus $M \approx 3 \cdot 10^{-6} m_{pl} \approx 10^{13}$GeV. For $N_k = 55$, the spectral index is $n = 1 - 2/55 = 0.964$ and the scalar-to-tensor ration $r \approx 0.004$ which are in perfect agreement with the results published by the Planck collaboration.
It may also be shown that the curvature perturbation $\mathcal{R}$ is conformally invariant, hence the observables remain unchanged under the change from Jordan to the Einstein frame.

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