Constraints on $\boldsymbol{F}$-term Inflation

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1 Introduction

We want to study $\mathcal{N} = 1$ SUGRA theories in 4D. By demanding that the conditions for slow-roll inflation and for dS vacua are satisfied in scenarios where SUSY is broken by F-terms it is possible to derive constraints on the geometry of the scalar manifold in such theories [1]. As it turns out, the constraints become stronger as the ratio between the Hubble parameter H and the gravitino mass $m_{3/2}$ increases.

We will start with reviewing the slow-roll conditions for (multi-field) inflation scenarios and some properties of Kähler geometry. Since both has been covered in previous talks, we will be brief and just collect the necessary results. In the next section, we derive the aforementioned constraint on the holomorphic sectional curvature in the Goldstino direction and discuss its implications and its relation to the metastable dS condition. In the last section, we present some string-inspired SUGRA examples and discuss the consequences of the constraint and how it can be met in specific examples.

1.1 Review of slow-roll conditions

Throughout we will be working in Planck units ($M_P = 1$). In the single field inflation case, we start with the Lagrangian

$$\mathcal{L} = \frac{1}{2}R - \partial_\mu\phi\partial^\mu\phi - V(\phi,\overline{\phi})\,. \tag{1}$$

In order to realize inflation in such a setup, the slow-roll parameters ε, η defined by

$$\varepsilon := \frac{1}{2}\left(\frac{V'}{V}\right)^2\,, \qquad \eta := \frac{V''}{V} \tag{2}$$

have to be small, $\varepsilon, |\eta| \ll 1$. Here, the prime denotes differentiation with respect to the (canonically normalized) field ϕ.

In the case of multi-field inflation, (1) is changed to

$$\mathcal{L} = \frac{1}{2}R - g_{i\bar{\jmath}}\partial_\mu\phi^i\partial^\mu\overline{\phi}^{\bar{\jmath}} - V(\phi,\overline{\phi})\,, \tag{3}$$

where $g_{i\bar{\jmath}}$ is the (hermitian) Kähler metric, which is in general a function of the scalars. Roughly, the first and second derivatives occurring in the single field slow-roll parameters (2) turn into the calculation of the gradient and the Hessian, respectively:

$$\varepsilon := \frac{\nabla^i V\nabla_i V}{V^2} \tag{4a}$$

$$\eta := \text{min eigenvalue}(N)\,, \qquad N := \frac{1}{V}\begin{pmatrix}\nabla^i\nabla_j V & \nabla^i\nabla_{\bar{\jmath}}V\\ \nabla^{\bar{\imath}}\nabla_j V & \nabla^{\bar{\imath}}\nabla_{\bar{\jmath}}V\end{pmatrix}\,. \tag{4b}$$

The direction of inflation is $\nabla_i V/\sqrt{\nabla^j V \nabla_j V}$. Note that we use the field covariant derivatives ∇. Let us briefly argue how this comes about. Recall that in general relativity covariant derivatives occur because the metric is a non-constant function of the space-time coordinates. Hence upon partial integration, one obtains derivatives acting on $g_{\mu\nu}$, which is "repaired" by introducing a connection, which itself depends on derivatives of the metric. The same happens for the Kähler metric. In general $g_{i\bar{\jmath}}$ is a function of the scalar fields, $g_{i\bar{\jmath}} = g_{i\bar{\jmath}}(\phi, \overline{\phi})$. Hence in deriving the equations of motion for the inflaton from the variation of (3), we have to account for the field dependence of the Kähler metric. This is done by introducing the Levi-Civita connection which contains the Christoffel symbols Γ^a_{bc} in field space. As in GR, the connection is compatible with the Kähler metric, i.e. $\nabla g = 0$.

The mass matrix

$$M := \begin{pmatrix} \nabla_i \nabla_{\bar{\jmath}} V & \nabla_i \nabla_j V \\ \nabla_{\bar{\imath}} \nabla_{\bar{\jmath}} V & \nabla_{\bar{\imath}} \nabla_j V \end{pmatrix} \tag{5}$$

is connected to the matrix N via

$$N^I_J = \frac{L^{I\bar{J}} M_{\bar{J}J}}{V}\,, \qquad L_{I\bar{J}} := \begin{pmatrix} g_{i\bar{\jmath}} & 0 \\ 0 & g_{\bar{\imath}j} \end{pmatrix}\,. \tag{6}$$

1.2 Review of supergravity and Kähler geometry

The metric $g_{i\bar{\jmath}}$ of Kähler manifolds is given in terms of derivatives of a Kähler potential K via $g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K$. Due to this, geometric objects derived from the Kähler metric like the Christoffel symbols Γ^a_{bc} or the curvature R_{abcd} simplify. In particular, the only non-vanishing Christoffel symbols have only holomorphic or only anti-holomorphic indices. Consequently, the Riemann tensor is only non-vanishing if the index structure is (up to the usual symmetries in the indices) $R_{i\bar{\jmath}p\bar{q}}$. This leads to a simplification of the expressions in (4a) containing the covariant derivatives.

In supergravity, the scalar potential V can be conveniently written in terms of the expression $G = K + \ln|W|^2$ as

$$V = e^G (G^i G_i - 3)\,, \tag{7}$$

where we have introduced the quantity $G_i = \partial G/\partial \phi^i$. The Kähler metric $g_{i\bar{\jmath}}$ is then given by $g_{i\bar{\jmath}} = G_{i\bar{\jmath}} = \nabla_i \nabla_{\bar{\jmath}} G$, where the first equality follows since the W and $\overline{W}$ dependence of G is annihilated by $\partial_{\bar{\jmath}}$ and ∂_i, respectively. The second equality follows since G is a scalar and since there are no Christoffel symbols with mixed indices.

The algebraic equation of motion for the F-terms becomes $F^i = m_{3/2} G^i$ with the gravitino mass $m_{3/2} = e^{G/2}$. When $\langle F^i \rangle \neq 0$, SUSY is broken spontaneously. The direction G^i defines the Goldstino which is eaten by the gravitino upon SUSY breaking. The unit vector in this direction is given by

$$f_i = \frac{G_i}{\sqrt{G^i G_i}}. \tag{8}$$

This completes our review.

2 Constraints from slow-roll inflation

First we want to calculate the covariant derivatives occurring in (4a). For the first (covariant) derivative of V we find

$$\begin{aligned} \nabla_i V &= \nabla_i\, e^G (g^{j\bar{j}} G_j G_{\bar{j}} - 3) \\ &= V\, \nabla_i\, G + e^G \left[g^{j\bar{j}} (G_j \nabla_i G_{\bar{j}} + G_{\bar{j}} \nabla_i G_j) \right] \\ &= G_i V + e^G (G_j g^{j\bar{j}} g_{i\bar{j}} + G^j \nabla_i G_j) = e^G (G_i + G^j \nabla_i\, G_j) + G_i V \,. \end{aligned} \tag{9}$$

where we used in the second step that the metric can be pulled through the covariant derivative. The second derivatives can be evaluated in a similar manner. The calculation is straightforward but a bit tedious and yields

$$\nabla_i \nabla_{\bar{j}} V = e^G (g_{i\bar{j}} + \nabla_i G_k \nabla_{\bar{j}} G^k - R_{i\bar{j}p\bar{q}} G^p G^{\bar{q}}) + (G_i \nabla_{\bar{j}} V - G_i G_{\bar{j}} V) + (g_{i\bar{j}} V + G_{\bar{j}} \nabla_i V)\,, \tag{10}$$

$$\nabla_i \nabla_j V = e^G (\nabla_i G_j + \nabla_j G_i + G^k \nabla_i \nabla_j G_k) + (G_i \nabla_j V - G_i G_j V) + (V \nabla_i G_j + G_j \nabla_i V)\,. \tag{11}$$

Let us briefly explain the structure. The terms in the three brackets arise from differentiating in expression (9) the inner part of the bracket, the factor e^G in front, and the $G_i V$, respectively. In (10) the curvature term arises from the commutator $[\nabla_i, \nabla_{\bar{j}}]$ of the covariant derivatives and the metric expressions arise from the $\nabla_i G_{\bar{j}}$ terms. In (11), the commutator term is absent since the corresponding Riemann tensor vanishes (it has three holomorphic indices), and the metric expressions are replaced with expressions of the form $\nabla_i G_j$.

Notice that the expressions G_i, $\nabla_i G_j$, and $\nabla_i \nabla_j G_k$ are independent quantities that depend on (derivatives of) the superpotential. Thus by changing W, the slow-roll parameter ε can be made arbitrarily small by tuning $G^j \nabla_i G_j$ against G_i in (9). To tune the absolute value of the slow-roll parameter η, we have to investigate our freedom for adjusting N. We find that by tuning $\nabla_i \nabla_j G_k$ the value of $\nabla_i \nabla_j V$ can be set to any arbitrary value. Likewise, by tuning $\nabla_i G_j$, most of the eigenvalues of $\nabla_i \nabla_{\bar{j}} V$ can be adjusted. The only exception is that along the Goldstino direction f_i, we have used (part of) this freedom to make ε small. Hence we expect a constraint arising from the projection of (10) in the Goldstino direction f_i. Indeed, this gives the aforementioned constraint on the sectional curvature, as we shall see now.

To proceed further, we define the Goldstino vector in terms of the direction (8) as

$$f_{I\,(\alpha)} := \frac{1}{\sqrt{2}} \left(e^{-\mathrm{i}\alpha} f_i,\ e^{\mathrm{i}\alpha} f_{\bar{\imath}} \right)\,, \qquad f^J_{(\alpha)} := \frac{1}{\sqrt{2}} \left(e^{\mathrm{i}\alpha} f^j,\ e^{-\mathrm{i}\alpha} f^{\bar{\jmath}} \right)\,, \tag{12}$$

where $\alpha \in \mathbb{R}$ is a phase. Using that for any unit vector $\eta \leq u_I N^I_J u^J$ and choosing two orthogonal Goldstino directions, say $\alpha = 0, \pi/2$, one obtains

$$\eta \leq \frac{\nabla_i \nabla_{\bar{j}} V}{V} f^i f^{\bar{j}}\,, \tag{13}$$

which can be expressed after some algebra as

$$\frac{\nabla_i \nabla_{\bar{j}} V}{V} f^i f^{\bar{j}} = -\frac{2}{3} + \frac{4}{\sqrt{3}} \frac{1}{\sqrt{1+\gamma}} \mathrm{Re} \left[\frac{\nabla_i V}{V} f^i \right] + \frac{\gamma}{\gamma+1} \frac{\nabla^i V \nabla_i V}{V^2} + \frac{\gamma+1}{\gamma} \widehat{\sigma}(f^i)\,, \tag{14}$$

where we introduced

$$\gamma := \frac{1}{3} \frac{V}{m_{3/2}^2} \simeq \frac{H^2}{m_{3/2}^2} = H^2 e^{-G}\,, \qquad \widehat{\sigma}(f^i) := \frac{2}{3} - R(f^i) := \frac{2}{3} - R_{i\bar{j}p\bar{q}} f^i f^{\bar{j}} f^p f^{\bar{q}}\,, \tag{15}$$

to parameterize the ratio of the Hubble scale to the gravitino mass and the sectional curvature $R(f)$, respectively. From the definition (4a) of ε it is clear that for the unit vector f^i we have $|f^i\nabla_i V/V| \leq \sqrt{\varepsilon}$. Using this, we can rewrite (14) as

$$\eta \leq \eta_{\max} := -\frac{2}{3} + \frac{4}{\sqrt{3}}\frac{1}{\sqrt{1+\gamma}}\sqrt{\varepsilon} + \frac{\gamma}{\gamma+1}\varepsilon + \frac{\gamma+1}{\gamma}\widehat{\sigma}(f^i)\,. \tag{16}$$

Thus, to meet the slow-roll condition $|\eta| \ll 1$, we need $\eta_{\max}$ to be either negative and small or positive. Inserting $\varepsilon \ll 1$ the bound becomes

$$\boxed{\widehat{\sigma}(f^i) \gtrsim \tfrac{2}{3}\tfrac{\gamma}{\gamma+1}} \qquad \text{or} \qquad \boxed{R(f^i) \lesssim \tfrac{2}{3}\tfrac{1}{\gamma+1}}\,. \tag{17}$$

It is worthwhile to compare this with the result we obtain for the mass matrix (5). Going through the same steps in the computation, one obtains [2]

$$m^2 = 3m_{3/2}^2\left(\frac{2}{3} - (\gamma+1)R(f^i)\right)\,. \tag{18}$$

Thus for $m^2 > 0$ we need $R(f^i) < 2/[3(\gamma+1)]$. It is useful to consider the limiting cases:

- In the limit $\gamma \ll 1$, i.e. $m_{3/2} \gg H$, the constraint (17) becomes $R(f^i) \lesssim 2/3$, which coincides with the mass bound (18).
- In the limit $\gamma \gg 1$, i.e. $m_{3/2} \gg H$, the constraints becomes much stronger, $R(f^i) \lesssim 0$.

Let us apply our results to some example models to see why slow-roll inflation works or cannot work in these setups.

3 Examples

In this section we discuss some examples and apply the bound we just derived. Most of the examples are string-inspired, with the inflaton given by the moduli sector.

3.1 Canonical Kähler potential

In the case of a canonical Kähler potential,

$$K = \sum_i \overline{X}^i X^i\,, \tag{19}$$

the Kähler metric is the unit matrix and the resulting curvature is flat. Hence (17) is satisfied for any value of γ.

3.2 Kähler potential from simple string compactifications

In the case of simple string compactifications, one finds for the moduli Kähler potentials of the form

$$K = -n\ln(T+\overline{T})\,. \tag{20}$$

The scalar manifold is one-dimensional and its sectional curvature is constant and given by $R = 2/n$. Hence for $n < 3$ we find that γ has to be negative and for $n = 3$ we are at the boundary where $\gamma = 0$. By including subleading effects, $0 < \gamma \ll 1$ is possible, but $\gamma \gg 1$ is out of range since typically $n = \mathcal{O}(1)$ in these models.

The situation can be improved by adding an uplifting sector to the Kähler potential,

$$K = -n \ln(T + \overline{T}) + \overline{X}X\,. \tag{21}$$

In this case the curvature of the submanifold in X direction vanishes. Hence by aligning the Goldstino along this direction, the situation reduces to the one discussed in section 3.1 and the bound (17) can be satisfied.

Lastly, we want to discuss a Kähler potential that occurs for example when compactifying string theory on orbifolds. It is of the form

$$K = -n \ln(T + \overline{T} - \overline{X}X)\,. \tag{22}$$

We find again a constant curvature $R = 2/n$ and thus the additional term $\overline{X}X$ does not improve the situation in this case.

3.3 SUGRA from Heterotic string compactifications

The moduli sector of string theory on a Calabi-Yau threefold in the large volume limit exhibits the no-scale property [3] $K^i K_i = 3$, which means that the curvature along the direction $k^i = K^i/\sqrt{3}$ takes the critical value $R(k^i) = 2/3$. Thus the question arises whether there is another direction $f^i \neq k^i$ along which the sectional curvature reduces.

For the special cases where the CY is an orbifold or a K3 fibration over a large $\mathbb{P}^1$, it can actually be shown that the direction $f^i = k^i$ is a minimum [4] and thus the situation is as the one discussed in section 3.2.

Let us consider more general CY compactifications of the heterotic string. Neglecting the complex structure and the bundle moduli, the Kähler potential is given by

$$\begin{aligned} K &= -\ln V\,, \\ V &= \frac{2^3}{3!}\int_X J \wedge J \wedge J = \frac{2^3}{3!}\int_X t^i D_i \wedge t^j D_j \wedge t^k D_k = \frac{1}{6} d_{ijk}(T^i + \overline{T}^i)(T^j + \overline{T}^j)(T^k + \overline{T}^k)\,, \end{aligned} \tag{23}$$

where the Kähler parameters t_i are the real part of the lowest (scalar) component of the chiral moduli superfields T^i. Whether or not the direction k^i is a minimum of the curvature or a saddle point can be studied by looking at the discriminant of the cubic polynomial V in the t_i, which gives a condition on the intersection numbers d_{ijk}.

3.4 SUGRA from Type II string compactifications

As a final example we want to present a recent analysis that has been carried out in [5]. In classical type IIB string theory the Kähler potential is found to be [6]

$$K = -2\ln(V) - \ln(S + \overline{S})\,. \tag{24}$$

The volume V is again given in terms of the triple intersection numbers d_{ijk} and the Kähler parameters t^i. Instead of being the real part of the lowest component of a chiral multiplet, the t^i's are connected to the $T^i = \rho^i + \mathrm{i}b^i$ via

$$\rho^i = \frac{1}{16} d^{ijk} t_j t_k \,. \tag{25}$$

For simplicity we assume that $h^{1,1} = h^{1,1}_+ = 1$. In this case

$$V = \frac{\sqrt{2}}{3\sqrt{d_{111}}} (T + \overline{T})^{\frac{3}{2}} \,. \tag{26}$$

Furthermore, it is assumed that the complex structure moduli have been stabilized by fluxes. Next the non-perturbative contributions superpotential and the quantum corrections to the Kähler potential are included. The former correct the Gukov-Vafa-Witten superpotential $W = \int \Omega \wedge F$ by non-perturbative terms coming from gaugino condensation on $D7$-branes wrapping the four-cycle T,

$$W = W_0 + \sum_{i=1}^{n} A_i e^{-a_i T} \,. \tag{27}$$

The Kähler corrections arise from α'^2 [7] and α'^3 [6] terms which can be derived from M-theory and connected to Type IIB via the Sen limit of the dual F-Theory. In the end, the corrected Kähler potential reads

$$\begin{aligned} K &= -2 \ln \left[\sqrt{T + \overline{T}} \left((T + \overline{T} - \frac{15}{8} k^2) \right) + \frac{\widehat{\xi}}{2} \right] \,, \\ \widehat{\xi} &= -\frac{\zeta(3)}{2} \chi(X) (S - \overline{S})^{\frac{3}{2}} \,, \qquad k = \frac{1}{\sqrt{3}} \left[\int_{B_3} c_1(B_3) \wedge c_1(B_3) \wedge c_1(B_3) \right]^{\frac{1}{3}} \,, \end{aligned} \tag{28}$$

where $\chi(X)$ is the Euler number of the CY threefold and $c_1(B_3)$ is the first Chern class of the CY fourfold base. Using this Kähler potential the (scalar) sectional curvature becomes in the limit $\widehat{\xi} \ll V$, $k \ll t$

$$R = \frac{2}{3} + \frac{5}{32} \left(5 \frac{k^4}{t^2} - \frac{7}{3\sqrt{2}} \frac{\widehat{\xi}}{\gamma\, t^{\frac{3}{2}}} \right) \overset{!}{<} \frac{2}{3} \,. \tag{29}$$

For the expression in the brackets to become negative we need the α'^3 to come with the correct sign and to counter-balance the contribution from the α'^2 corrections which are always positive, hence forcing t to be very large and/or γ being very small.

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