Conformal SUGRA

Weyl trans. or Local conformal trans.

$$g_{\mu\nu} \longrightarrow e^{-2\sigma(x)} g_{\mu\nu} \qquad \varphi \rightarrow \varphi e^{k\sigma(x)}$$

φ has conformal weight k.

[(SM) fields: scalars $k=1$; fermions $k=\frac{3}{2}$; gauge bosons $k=0$]

Very powerful tool in physics { We'll just scratch a tip of the iceberg in considering its role in classical cosmology, and specifically the SUGRA implementation. As time permits we could duelve into a current debate of geodesical completeness. }

Literature: (1) Wess & Bagger

(2) D. Z. Freedman & A. Van Proeyen "Supergravity", Cambridge, UK: Cambridge Univ. Pr. (2012), 607 p.

(3) gr-qc/0110012 A. Borde, A. H. Guth, A. Vilenkin

(4)* 1311.3326, 1307.1848 Kallosh & Linde | Bars, Steinhardt & Turok

(5) General Relativity, Robert Wald, (section 9.1)

The basic idea, if we don't have any dim-ful params. in our theory, we'll have classical scale inv. $\Rightarrow$ conformal inv. $\Rightarrow$ local conformal inv. ?

The Newton constant/Planck mass obviously spoils this invariance.

Solution: "Lift" the theory to a Weyl inv. one

How?

~~Make~~ Have a scalar field. The VEV of the scalar field will give the Newton constant. Thus, having an action which is classically Weyl inv. with spontaneous sym. breaking.

Name of the scalar: Weyl scalar or conformon

The idea was especially useful in deriving the SUGRA Lagrangian, dubbed "the conformal compensator"

Toy Model of a cosmological constant

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}\partial_\mu \chi \partial_\nu \chi g^{\mu\nu} + \frac{\chi^2}{12} R(g) - \frac{\lambda}{4}\chi^4\right]$$

conformally inv. $g_{\mu\nu} \to e^{-2\sigma(x)} g_{\mu\nu}$; $\chi \to e^{\sigma(x)}\chi$ (*)

χ has the wrong sign in the kinetic term. This is not a real problem because by fixing the symmetry (*) we remove it.

Example I: $\chi = \sqrt{6}\, M_{pl}$ (From now on $M_{pl} \equiv 1$)

$$\Rightarrow \quad \mathcal{L} = \sqrt{-g}\Big[\frac{1}{2}R(g) - \underbrace{9\lambda}_{\text{C.C.}}\Big]$$

$\Rightarrow$ dS solution $a = e^{\sqrt{3\lambda}\, t}$; $H = \sqrt{3\lambda}$

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\vec{x}^2\right]$$

η - conformal time $\quad dt = a\, d\eta$

$$t \in (-\infty, \infty) \quad ; \quad \eta \in (-\infty, 0]$$

Example II: $a = 1$ (!) $\Rightarrow ds^2 = -d\eta^2 + d\bar{x}^2$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \chi \partial_\nu \chi \eta^{\mu\nu} - \frac{\lambda}{4}\chi^4$$

$$\chi'' = \lambda \chi^3$$

χ is unstable. Equivalent to the correct sign but with a negative potential $V = -\frac{\lambda}{4}\chi^4$

solution for $\chi \longrightarrow +\infty$ when η growing

$$\chi = -\sqrt{\frac{2}{\lambda}}\frac{1}{\eta} \qquad \eta \in (-\infty, 0)$$

Relation between the gauges $e^{\sigma(x)} = -\sqrt{3\lambda}\,\eta$

So inflation can be viewed as a fixed Planck mass and exponentially expanding space, or as a Minkowski space with exponentially growing conformon.
(Not a very big deal)
Better idea: Keep the conformal setting as much as possible and fix a gauge only at the end.

Before discussing the Conformal SUGRA case, let us note that the following discussion is valid in general for Weyl inv. theories.
SUGRA will be a particular case where
a. The Kähler metric has to be derived, so field space is really of Kähler type.
b. The potential $V = V_F + V_D$ will be derived from a holomorphic Superpotential and the gauge kinetic function.

$$V_E = e^K\{ G^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - 3|W|^2 \} + \frac{1}{2}(\mathrm{Re} f)^{-1}_{ab} D^a D^b$$

SUGRA and Superconformal inv.

By not fixing a frame we have

$$\mathcal{L}_{conf.} = \sqrt{-g}\left(-\tfrac{1}{6}N(X,\bar{X})R - G_{I\bar{J}}\,\partial^{\mu}X^{I}\partial_{\mu}\bar{X}^{\bar{J}} - V(X,\bar{X})\right)$$

$I, \bar{I} = 0, \ldots, n$, $n+1$ scalars (physical fields + conformon)

$$G_{I\bar{J}} = \frac{\partial^2 N}{\partial X^I \partial \bar{X}^{\bar{J}}} \quad \text{not positive definite!}$$

$N(X,\bar{X})$ homogeneous of first degree in $X, \bar{X}$

$V(X,\bar{X})$ " of degree 2 in $X, \bar{X}$

Enjoys the ~~standard~~ Kähler geometry structure with N playing the role of the Kähler potential

Throughout the analysis $Y_I \equiv \frac{\partial Y}{\partial X^I}$; $Y_{\bar{J}} \equiv \frac{\partial Y}{\partial \bar{X}^{\bar{J}}}$; $Y^I \equiv G^{I\bar{J}} Y_{\bar{J}}$

<u>Einstein frame</u>

(1) $N(X,\bar{X}) = -3$

$$\Rightarrow \mathcal{L}_{conf.} = \sqrt{-g}\left(\tfrac{1}{2}R - G_{I\bar{J}}(\partial X^{I})(\partial \bar{X}^{\bar{J}}) - V(X,\bar{X})\right)$$

A natural choice is to use eq (1) to fix the conformon $\Rightarrow X, \bar{X}$ will correspond only to n physical scalars $z_i, \bar{z}_{\bar{i}}$ (not necessary!)

Example; consider a conformon and a scalar ϕ with $X^2 - \phi^2 = -3 = N$

Solution $X = \sqrt{3}\sinh\frac{z}{\sqrt{3}}$; $\phi = \sqrt{3}\cosh\frac{z}{\sqrt{3}}$

gives a canonical complex scalar field in the Einstein Frame.

(5)

Jordan Frame

$$\mathcal{N}(X,\bar{X}) = -3\Phi(z,\bar{z}) \quad (2)$$

$$\Rightarrow \mathcal{L} = \sqrt{-g_J}\left(+\frac{1}{2}\Phi(z,\bar{z})R_J - G_{I\bar{J}}\partial^\mu X^I \partial_\mu X^{\bar{J}} - V(X,\bar{X})\right)$$

Similarly we can use (2) to fix the conformon. The Higgs inflation model basically uses the <u>non-susy</u> version of the Jordan frame.

For successful SUSY + Higgs inflation, the simplest framework above doesn't work because it fails to meet the necessary condition of $R(f_i) \leq \frac{2}{3}$ (Fabien's talk)

$\Rightarrow$ One must add additional <u>dimensionful</u> parameters.

Elaborating on the relation between the Jordan & Einstein Frame

$$g_J^{\mu\nu} = e^{2\sigma} g_E^{\mu\nu}, \quad e^{2\sigma} = -\frac{1}{3}\Phi(z,\bar{z}) > 0$$

We got rid of the conformon and there's a factor of -3 different in terms of normalization

$$\Rightarrow \mathcal{L}_J = \sqrt{-g_J}\left(-\frac{1}{6}\Phi R(g_J) + \frac{1}{3}\Phi G_{I\bar{J}} D_\mu z^I D^\mu \bar{z}^{\bar{J}} - \frac{1}{4\Phi}\partial_\mu \Phi \partial^\mu \Phi - V_J + \text{vectors} + \text{fermions}\right)$$

$$V_J = \frac{\Phi^2}{9} V_E$$

The frame function $\Phi(z^I \bar{z}^I)$ is inv. under gauge trans. that's why $D_\mu \Phi = \partial_\mu \Phi$

$D_\mu \Phi = \Phi_I D_\mu z^I + \Phi_{\bar{I}} D_\mu \bar{z}^{\bar{I}}$

$$\Rightarrow \quad -\frac{1}{4\Phi}(D_\mu \Phi)(D^\mu \Phi) = -\frac{1}{4\Phi}\left(\Phi_I D_\mu z^I - \Phi_{\bar{I}} D_\mu \bar{z}^{\bar{I}}\right)^2 - \frac{\Phi_I \Phi_{\bar{J}}}{\Phi}(D_\mu z^I)(D^\mu \bar{z}^{\bar{J}})$$

$$\Rightarrow \mathcal{L}_J = \sqrt{-g_J}\Big\{ -\frac{\Phi}{6}R(g_J) + \left(\frac{1}{3}\Phi G_{I\bar{J}} - \frac{\Phi_I \Phi_{\bar{J}}}{\Phi}\right) D_\mu z^I D^\mu \bar{z}^{\bar{J}} + \Phi A_\mu A^\mu - V_J$$

+ vectors + fermions $\Big\}$

where $A_\mu = -\frac{i}{2\Phi}\left(\Phi_I D_\mu z^I - \Phi_{\bar{J}} D_\mu \bar{z}^{\bar{J}}\right)$ is the bosonic part of the auxilary field of SUGRA

① The metric here is not Hermitian because of this $\frac{1}{4\Phi}(D_\mu \Phi)(D^\mu \Phi)$ term.
Kähler geometry is not apparent.

② The Einstein frame is the specific case of $\Phi = -3$

Canonical kinetic terms in the Jordan Frame

$\Phi = -3e^{-\frac{1}{3}K}$, where K is the Kähler potential
(I don't think there's a real freedom here, ~~Φ~~
According to Wess & Bagger this constraint is necessary)

$$\Rightarrow \quad G_{I\bar{J}} = -3\frac{\Phi_{I\bar{J}}}{\Phi} + 3\frac{\Phi_I \Phi_{\bar{J}}}{\Phi^2}$$

$$\Rightarrow \mathcal{L}_J = \sqrt{-g_J}\left(-\frac{\Phi}{6}R(g_J) - \Phi_{I\bar{J}} D_\mu z^I D^\mu \bar{z}^{\bar{J}} + \Phi A_\mu A^\mu - \frac{\Phi^2}{9}V_E + \ldots\right)$$

Hence, if we choose $\Phi = -3 + |z^I|^2 + f(z^I) + \bar{f}(\bar{z}^{\bar{I}})$

and $A_\mu = 0$

we'll get canonical kinetic terms.

This is ~~rather~~ not generic.

From the pheno. point of view $K = -3\ln\left(1 - \frac{|z|^2 + f + \bar{f}}{3}\right)$

which has a tachyonic instability

if one wants inflation/dS $\Rightarrow$ needs modification

(with dim-full parameters)

~~Geodesic Completeness~~

Metric dependent gauge fixing

$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(\beta^i dt + dx^i)(\beta^j dt + dx^j)$

$\alpha, \beta^i, \gamma_{ij}$ ≡ 10 arbitrary functions of the coordinates.

No gauge fixing at all.

This time we will get rid of $-\det g_{\mu\nu} = \alpha^2 \det \gamma = 1$

$\mathcal{L}_{conf} \longrightarrow \mathcal{L}\left(X^I(x), \bar{X}^{\bar{J}}(x), \alpha(x), \beta^i(x), \gamma_{ij}(x)\right)$
and first derivatives

The conformon is still here.

In the FLRW case this corresponds to $a(\eta) = 1$.

$\Rightarrow ds^2 = -d\eta^2 + d\vec{x}^2$

the general eqs. for scalars and the metric ~~will come from~~ are related due to the Weyl sym.:

$$\frac{\delta S}{\delta X^I} X^I + \frac{\delta S}{\delta \bar{X}^{\bar{J}}} \bar{X}^{\bar{J}} - 2 \frac{\delta S}{\delta g_{\mu\nu}} g_{\mu\nu} = 0$$

For the FLRW $a=1$ case we'll have

$$X^{I\,\prime\prime} + \Gamma^I_{JK} X^{J\,\prime} X^{K\,\prime} + G^{I\bar{J}} V_{\bar{J}} = 0$$

where $'$ denotes derivative w.r.t. η.
So instead of having the Friedmann eqs. ~~+ n KG eqs.,~~ ~~...~~ + n scalar field eqs in a gravitational field, we have $n+1$ eqs. for scalars in Minkowski spacetime.

Final comment for the model builders:
The Jordan frame, or non-minimal coupling to gravity opens the way to a plethora of inflation models.
They don't have to be SUSY, or Higgs, and for the sake of honesty has been suggested already in the 80s–90s

Geodesical Completeness

The notion of geodesical completeness is crucial in analyzing singularities in GR (and from that the potential appearence of new physics).

def: If there exist geodesics that are inextendible in at least one direction, but have only finite ~~affine~~ affine length, then such geodesics are incomplete.

(For time-like / space like geodesics one requires finite proper time / proper length)

This is useful because it "detects" singularities even when R, R_{abcd} or similar quantities remain finite or even vanish!

(Borde, Guth, Vilenking)

In gr-qc/0110012 AAA^r have proven that if the average expansion rate $H_{av} > 0$ for any spacetime then it is past-incomplete. This includes inflationary spacetimes, so even with inflation, there has to be "new physics" beyond that.

We are still stuck with the "Big Bang singularity"

Solutions(?) ① A bounce, a period where $H_{av} < 0$? ② Quantum cosmology?
Both?

In Benedict's talk on the Starobinski model an eternal de-Sitter space was possible if the gravity Lagrangian was $\mathcal{L}_{grav.} = \sqrt{-g}\, R^2$

However, even the eternal dS is past incomplete according to AAA.

The simple case: Null geodesic in FLRW

$$ds^2 = dt^2 - a^2(t)\, d\bar{x}^2$$

the affine param. $d\lambda \propto a(t)\, dt$ for a null geodesic eq.

Let us normalize $d\lambda = \frac{a(t)}{a(t_f)}\, dt$

such that $\left.\frac{d\lambda}{dt}\right|_{t=t_f} = 1$ at $t = t_f$ (some reference time)

$$\Rightarrow \int_{\lambda(t_i)}^{\lambda(t_f)} H(\lambda)\, d\lambda = \int_{t_i}^{t_f} \frac{\dot{a}}{a} \frac{d\lambda}{dt}\, dt = \int_{t_i}^{t_f} \frac{\dot{a}}{a(t_f)}\, dt = \int_{a(t_i)}^{a(t_f)} \frac{da}{a(t_f)} = 1 - \frac{a(t_i)}{a(t_f)} \leq 1$$

Defining $0 < H_{av} \equiv \frac{1}{\lambda(t_f) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_f)} H(\lambda)\, d\lambda \leq \frac{1}{\lambda(t_f) - \lambda(t_i)}$

so we have a finite affine length for $H_{av} > 0$

$\Rightarrow$ past incomplete!

(Maximum affine length is bounded)

How is the geodesical completeness relevant to conformal SUGRA?

There's no doubt that even in the "lifted" Weyl theory, many curvature scalars diverge.

For example
$$I = \left[\frac{3}{N(x,\bar{x})}\right]^2 C_{\mu\nu\lambda\delta} C^{\mu\nu\lambda\delta}$$

$C_{\mu\nu\lambda\delta}$ is the Weyl tensor ~~(traceless~~

~~$C_{\mu\nu\lambda\delta} = R_{\mu\nu\lambda\delta} - \frac{1}{2}$~~ (Linear combination of $R_{\mu\nu\lambda\delta}$, $R_{\mu\lambda}$, R)

and $C^{\mu}{}_{\nu\mu\sigma} = 0$

$C_{\mu\nu\lambda\delta} = 0 \iff \exists$ coordinate system with ~~$g_{\mu\nu} \propto$ const~~

$$g_{\mu\nu} = \underbrace{\alpha}_{\text{constant}} \eta_{\mu\nu}$$

Specifically in the Einstein frame
$$I_E = \left(\frac{3}{-3}\right)^2 C_{\mu\nu\lambda\delta} C^{\mu\nu\lambda\delta} \to \infty$$
towards the Big Bang singularity.

The Weyl "lifting" does not change that.

Does it change geodesical completeness?

Bars, Steinhart & Turok claim it does(!) in the following sense:

(1) One can smoothly continue the geodesics ~~along~~ through the big bang and big crunch singularity

(2) The action of geodesics that reach arbitrarily far into the past should diverge

The way they do it. compute geodesics $x^\mu(\lambda)$ by extremizing the particle action

$$S = -\int d\lambda\, m \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

$$m = g h(x(\lambda))$$
↳ Higgs field   still no dim-full params.

for $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$

$$\bar{x}(\eta) = \bar{q} + \bar{p}\int_{\eta_0}^{\eta} \frac{d\eta'}{\sqrt{\bar{p}^2 + m^2(\eta')a^2(\eta')}}$$

so if $m(\eta)$, $a(\eta)$ is given in all patches and are smoothly continued through singularities, then the geodesics will be matched and continued as well.
The Weyl invariance with $\mathcal{L} = \sqrt{-g}\left[\frac{1}{12}(\phi^2 - h^2)R + ..\right]$ is crucial to ensure continuation.

vs. $|S_A| = \int_{\eta_i}^{\eta_f} d\eta \frac{m^2(\eta)a^2(\eta)}{\sqrt{\bar{p}^2 + m^2(\eta)a^2(\eta)}} \longrightarrow \infty$