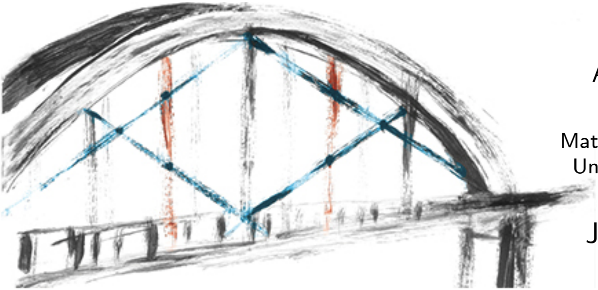
Non perturbative results for $\mathcal{N} = 4$ SCFT



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Goal of this talk

- Use superconformal symmetry and the associativity of the operator product expansion at the level of four point function $\rightarrow$ bootstrap equations
- Put bounds on the dimension of operators transforming in different representation of the R-symmetry group

Conformal algebra

The conformal group is defined as the set of transformations that preserve angles.

- Translations: P_μ
- Lorentz transformations: $M_{\mu\nu}$
- Scale transformations: D
- Special conformal transformations: K_μ

The conformal algebra is

$$
\begin{aligned}
[D, K_\mu] &= iK_\mu \\
[D, P_\mu] &= -iP_\mu \\
[P_\mu, K_\nu] &= 2i(\delta_{\mu\nu} D - M_{\mu\nu})
\end{aligned}
$$

How it acts on fields

Primary fields are local operators $\phi(x)$ characterised by the fact that they are annihilated by the special conformal transformations generator at $x = 0$. The behaviour of $\phi(0)$ is

$$[M_{\mu\nu}, \phi(0)] = \Sigma_{\mu\nu}\phi(0) \to \text{SPIN}$$
$$[D, \phi(0)] = -i\Delta\phi(0) \to \text{DIMENSION}$$
$$[K_\mu, \phi(0)] = 0 \to \text{PRIMARY FIELD}$$

Primary fields

- P_μ raises the scaling dimension while K_μ lowers it. In unitary CFT there is a lower bound on the dimensions of the fields.
- Each representation of the conformal algebra must have some operator of lowest dimension, which must then be annihilated by $K_\mu \rightarrow$ PRIMARY OPERATOR
- By acting with P_μ on a primary $\rightarrow$ DESCENDANTS

2 and 3 pt functions

- All the information of a CFT is encoded in the set of dimensions and structure constants of local operators
- Conformal symmerty fixes the space-time dependence of 2 and 3 point functions. If we consider scalar operators:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{\delta_{12}}{x_{12}^{2\Delta}}$$

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{c_{123}}{|x_{12}|^{\Delta_{123}}|x_{23}|^{\Delta_{231}}|x_{13}|^{\Delta_{132}}}$$

where $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$

- For 2 and 3 point functions of different classes of operators there are tensorial structures to be taken into account, but still fully fixed by conformal symmetry.

OPE

- At least in principle, using the OPE all the higher point correlation functions can be constructed

$$\phi_A(x)\phi_B(y) = \sum_D c_{ABD}(x-y)^{\Delta_D-\Delta_A-\Delta_B} \sum_n \beta^{(n)}_{ABD}(x-y)^{|n|}\phi^{(n)}_D(y)$$

where n is the descendant level, $\phi^{(0)}_D$ are the primary operators and $\beta^{(0)}_{ABD} = 1$.

Four point function

- For the case of four point function, conformal symmetry does not fix the full coordinate dependence.
  The four point function of identical scalar primaries with dimension d takes this form

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{g(u,v)}{|x_{12}|^{2d}|x_{34}|^{2d}}$$

where $g(u,v)$ is a function of the conformal invariant cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

Conformal blocks I

By considering the OPE $\phi(x_1) \times \phi(x_2)$ we can write

$$g(u, v) = 1 + \sum_{\ell,\Delta} a_{\Delta,\ell} g_{\Delta,\ell}(u, v)$$

- the first term is the contribution of the identity operator, which is present in the OPE
- the sum runs over the the tower of primaries present in the OPE
- ℓ and Δ denote the spin and the dimension of the intermediate primary
- $a_{\Delta,\ell} = c^2_{\Delta,\ell}$ is the square of the structure constants and is non-negative due to unitarity
- $g_{\Delta,\ell}(u, v)$ are the conformal blocks...

Conformal blocks II

...conformal blocks

- repack the contributions of all descendants of a given primary
- transform under the conformal group in the same way as the four point function
- depend on the spin and the dimension of the intermediate state and on the dimension of the primary operator
- are known in a closed form in 4 dimensions:

$$g_{\Delta,\ell}(u,v) = \frac{(z\bar{z})^{\frac{\Delta-\ell}{2}}}{z-\bar{z}} \left((-\frac{z}{2})^{\ell} z k_{\Delta+\ell}(z) k_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z}) \right)$$

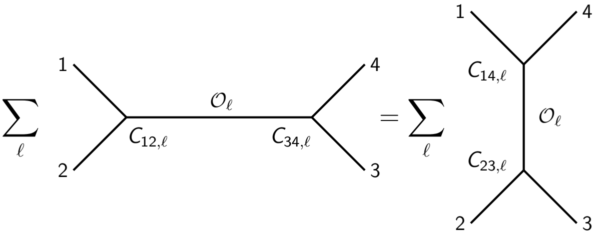
with $k_{\beta}(z) = {}_2F_1(\beta/2, \beta/2, \beta; z)$ and $u = z\bar{z}, \quad v = (1-z)(1-\bar{z})$.

[Dolan, Osborn, 2005]

4 point and OPE

Associativity of the conformal algebra implies that

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$



$$\frac{g(u,v)}{x_{12}^{2d}x_{34}^{2d}} = \frac{g(v,u)}{x_{23}^{2d}x_{14}^{2d}} \;\rightarrow\; v^d g(u,v) = u^d g(v,u)$$

- d is the dimension of ϕ

Sum rule

$$\sum_{\ell,\Delta} a_{\Delta,\ell} F_{\Delta,\ell}(u,v) = 1, \qquad a_{\Delta,\ell} \geq 0$$

$$F_{\Delta,\ell}(u,v) \equiv \frac{v^d g_{\Delta,\ell}(u,v) - u^d g_{\Delta,\ell}(v,u)}{u^d - v^d}$$

[Rattazzi, Rychkov, Tonni, Vichi, 2008]

- We apply a linear operator Φ
- If $\Phi\left(F_{\Delta,\ell}(u,v)\right) \geq 0$ and $\Phi\left(1\right) \leq 1$ then the sum rule has no solution for $a_{\Delta,\ell}$ non negative!
- By considering trial families of spectra it is possible to put bounds on the dimension of the leading twist operator for a given spin.

Superconformal symmetry

- $\mathcal{N}=4$ SYM has superconformal symmetry, combination of conformal symmetry and supersymmety
- conformal group $SO(2,4)$ +
  - supersymmetry generators (superpartners of translations) Q^a_α and $\bar{Q}_{\dot{\alpha}a}$ with $a = 1, \ldots, 4$
  - special superconformal generators(superpartners of special conformal transformations) $S_{\alpha a}$ and $\bar{S}^{\bar{a}}_{\dot{\alpha}}$
  - SO(6) R-symmetry generators T^A with $A = 1, \ldots, 15$
- (a little bit of) superconformal algebra

$$[L, S] \sim \bar{S} \qquad [D, Q] = \frac{i}{2} Q \qquad [D, S] = -\frac{i}{2} S$$

Operators

- Among primary operators there is a subclass annihilated by conformal supercharges $S \rightarrow$ SUPERCONFORMAL PRIMARIES.
- By acting with S (Q) the dimension is lowered (raised) by $\frac{1}{2}$;
- Superconformal primaries that commute with at least one of the supercharges $\rightarrow$ CHIRAL PRIMARIES.
- They are also called BPS operators because they belong to shortened representations and their dimension is protected.

Four point function I

- The lowest component of a $\frac{1}{2}$-BPS multiplet in $\mathcal{N} = 4$ SYM is a real scalar field of dimension p transforming in the irrep $[0, p, 0]$ of the $SO(6)$ R-symmetry group
- This operator can be written as

$$\mathcal{O}^{[p]}(x, t) = t_{r_1} \dots t_{r_p} \, \mathrm{Tr} \left(\Phi^{r_1} \cdots \Phi^{r_p} \right)$$

where t is a complex six-dimensional null vector ($t \cdot t = 0$) and $r_i = 1, ..., 6$.

Four point function II

- The four point function of four such identical operator has the form

$$\langle \mathcal{O}^{[p]}(x_1, t_1)\mathcal{O}^{[p]}(x_2, t_2)\mathcal{O}^{[p]}(x_3, t_3)\mathcal{O}^{[p]}(x_4, t_4)\rangle = \left(\frac{t_1 \cdot t_2 \; t_3 \cdot t_4}{x_{12}^2 x_{34}^2}\right)^p \mathcal{G}^{(p)}(u, v, \sigma, \tau)$$

with

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

CROSS RATIOS

$$\sigma = \frac{t_1 \cdot t_3 \; t_2 \cdot t_4}{t_1 \cdot t_2 \; t_3 \cdot t_4} \qquad \tau = \frac{t_1 \cdot t_4 \; t_2 \cdot t_3}{t_1 \cdot t_2 \; t_3 \cdot t_4}$$

HARMONIC CROSS RATIOS

OPE decomposition

- The function $\mathcal{G}^{(p)}(u, v, \sigma, \tau)$ can be decomposed in the $SO(6)$ R-symmetry representations appearing in the OPE of $\mathcal{O}^{[p]}(x_1, t_1) \times \mathcal{O}^{[p]}(x_2, t_2)$, determined by

$$[0, p, 0] \times [0, p, 0]$$

  and containing $\frac{1}{2}(p+1)(p+2)$ terms
- Each of these contributions can be expanded in conformal partial waves, corresponding to CONFORMAL PRIMARY OPERATORS with dimensions Δ and spin ℓ transforming in the appropriate representation

$$\mathcal{G}^{(p)}(u, v, \sigma, \tau) = \sum_{0 \leq m \leq n \leq p} a_{nm}(u, v) Y_{nm}(\sigma, \tau)$$

  where n and m specify the representation $[n-m, 2m, n-m]$ and $a_{nm} = \sum_{\ell,\Delta} a^{[nm]}_{\Delta,\ell} g_{\Delta,\ell}(u, v)$

Superconformal decomposition

- Superconformal symmetry requires that each conformal primary belongs to a given supermultiplet, with a corresponding superconformal primary
- Superconformal Ward identities dictate the decomposition of $\mathcal{G}(u, v, \sigma, \tau)$ in terms of:
  - long multiplets, containing all the dynamical non-trivial information $\rightarrow \mathcal{H}(u, v, \sigma, \tau)$
  - short and semi-short multiplets, which are fully determined by symmetries and the free field theory results
- Consider the decomposition in conformal partial wave of $\mathcal{H}(u, v, \sigma, \tau)$, it receives contributions only from $p(p-1)/2$ representations.

Superconformal decomposition II

- It can be written as

$$\mathcal{H}(u,v,\sigma,\tau) = \sum_{0\leq m\leq n\leq p-2} \mathcal{H}^{[nm]}(u,v) Y_{nm}(\sigma,\tau)$$

$$\mathcal{H}^{[nm]}(u,v) = \sum_{\Delta,\ell} A^{[nm]}_{\Delta,\ell}\, g^{(\ell)}_{\Delta+4}(u,v)$$

- The sum runs over SUPERCONFORMAL PRIMARY OPERATORS with dimensions Δ and spin ℓ, where the spin is even/odd if $n+m$ is even/odd.
- F.i. for $p=2$ superconformal primaries transform only in the singlet representation $[0,0,0]$ of $SU(4)$ R-symmetry, for $p=3$ they transform under $[0,0,0]$, $[0,2,0]$ and $[1,0,1]$.

Superconformal decomposition III

Actually not all $A^{[nm]}_{\Delta,\ell}$ are non negative:

- unitarity requires that only contributions for $\Delta \geq 2n + \ell + 2$
- long multiplet decomposes into semi-short multiplets at the unitary threshold

$$\mathcal{H}(u,v,\sigma,\tau) = \sum_{0\leq m\leq n\leq p-2} \hat{\mathcal{H}}^{[nm]}(u,v) Y_{nm}(\sigma,\tau)$$

$$\hat{\mathcal{H}}^{[nm]}(u,v) = \sum_{\Delta,\ell} a^{[nm]}_{\Delta,\ell}\, g^{(\ell)}_{\Delta+4}(u,v) + F^{[nm]}_{(p)}(u,v)$$

- All $a^{[nm]}_{\Delta,\ell}$ are non negative and $F^{[nm]}_{(p)}(u,v)$ contain only contributions from short and semi-short multiplets for each specific $SU(4)$ representation and do not depend on the coupling constant.

Crossing symmetry

- Crossing symmetry requires invariance of the four-point function under exchanging (x_1, t_1) with (x_3, t_3)
- At the level of cross ratios this is equivalent to $u \to v$, $v \to u$, $\sigma \to \frac{\sigma}{\tau}$ and $\tau \to \frac{1}{\tau}$ and implies

$$\mathcal{G}\left(u, v, \sigma, \tau\right) = (\tau)^p \left(\frac{u}{v}\right)^p \mathcal{G}\left(v, u, \frac{\sigma}{\tau}, \frac{1}{\tau}\right)$$

- Plugging back the expansion in conformal partial waves of the four point function, it is possible to obtain an equation for $\mathcal{H}(u, v, \sigma, \tau)$.

Comparison

Conformal

$$\sum_{\ell,\Delta} a_{\Delta,\ell} F_{\Delta,\ell}(u,v) = 1$$

- The rhs denotes the contribution of the identity operator
- The sum on the lhs runs over the dimension and the spin of the conformal primaries appearing in the OPE

Super-conformal (e.g. p=2)

$$\sum_{\ell,\Delta} a_{\Delta,\ell} F_{\Delta+4,\ell}(u,v) = F^{short}(u,v)$$

[Beem, Rastelli, van Rees]

- The rhs denotes the contribution of short and semishort operators (protected part)
- The sum on the lhs runs over the dimension and the spin of the superconformal primaries appearing in the OPE

p=3

- For $p = 3$ the representations that contribute to the conformal partial wave decomposition of $\hat{\mathcal{H}}^{[nm]}(u, v)$ are $[0, 0, 0]$, $[1, 0, 1]$ and $[0, 2, 0]$.
- We have 3 equations involving different combinations of $\hat{\mathcal{H}}^{[nm]}(u, v)$ (remember that there is a factor in front of $\mathcal{H}^{[nm]}(u, v)$ in the crossing relation depending on the different R-symmetry representations!)
- It is possible to write these equations in a vectorial form.

Final equations

$$\sum_{\substack{\Delta \geq \ell+2 \\ \ell=0,2,\dots}} a^{[00]}_{\Delta,\ell} \begin{pmatrix} F^{(3)}_{\Delta,\ell} \\ 0 \\ H^{(3)}_{\Delta,\ell} \end{pmatrix} + \sum_{\substack{\Delta \geq \ell+4 \\ \ell=1,3,\dots}} a^{[10]}_{\Delta,\ell} \begin{pmatrix} 0 \\ F^{(3)}_{\Delta,\ell} \\ 3H^{(3)}_{\Delta,\ell} \end{pmatrix} + \sum_{\substack{\Delta \geq \ell+4 \\ \ell=0,2,\dots}} a^{[11]}_{\Delta,\ell} \begin{pmatrix} 5F^{(3)}_{\Delta,\ell} \\ F^{(3)}_{\Delta,\ell} \\ -4H^{(3)}_{\Delta,\ell} \end{pmatrix} = \begin{pmatrix} F^1_{short}(u,v) \\ F^2_{short}(u,v) \\ F^3_{short}(u,v) \end{pmatrix}$$

where

$$H^{(p)}_{\Delta,\ell}(u,v) = v^p g^{(\ell)}_{\Delta+4}(u,v) + u^p g^{(\ell)}_{\Delta+4}(v,u)$$

- $F^1_{short}(u,v)$, $F^2_{short}(u,v)$ and $F^3_{short}(u,v)$ are simple combinations of $F^{[00]}_3(u,v)$, $F^{[10]}_3(u,v)$ and $F^{[11]}_3(u,v)$.

Linear operator

$$\sum_{\Delta,\ell} a^{[00]}_{\Delta,\ell} \vec{V}^{[00]}_{\Delta,\ell} + \sum_{\Delta,\ell} a^{[10]}_{\Delta,\ell} \vec{V}^{[10]}_{\Delta,\ell} + \sum_{\Delta,\ell} a^{[11]}_{\Delta,\ell} \vec{V}^{[11]}_{\Delta,\ell} = \vec{F}_{short}$$

- $a^{\mathcal{R}}_{\Delta,\ell}$ are non-negative coefficients
- Unitarity requires that

$$\Delta \geq \ell + 2 \ \text{ for } \ [00], \qquad \Delta \geq \ell + 4 \ \text{ for } \ [10] \text{ and } [11]$$

- A given spectrum can be ruled out if we can find a linear functional $\Phi : \vec{V} \to R$ such that

$$\Phi \ \vec{V}^{[00]}_{\Delta,\ell} \geq 0, \qquad \text{for } a^{[00]}_{\Delta,\ell} \neq 0, \ \ell = 0, 2, \ldots$$

$$\Phi \ \vec{V}^{[10]}_{\Delta,\ell} \geq 0, \qquad \text{for } a^{[10]}_{\Delta,\ell} \neq 0, \ \ell = 1, 3, \ldots$$

$$\Phi \ \vec{V}^{[11]}_{\Delta,\ell} \geq 0, \qquad \text{for } a^{[11]}_{\Delta,\ell} \neq 0, \ \ell = 0, 2, \ldots$$

$$\Phi \ \vec{F}_{short} < 0$$

Dependence on N

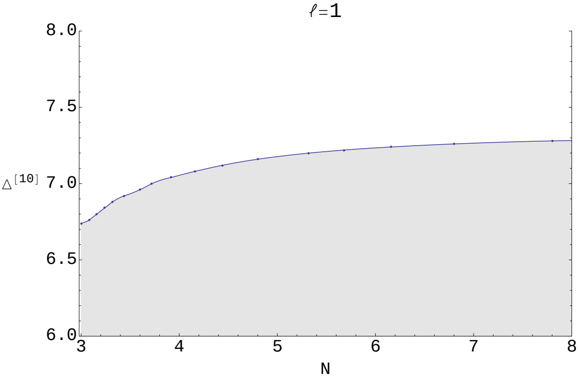
- $\vec{F}_{short}$ depends on 3 factors a_1, a_2 and a_3 which are related to the topologies of the free field theory graphs
- For the case of $\mathcal{N}=4$ SYM with gauge group $SU(N)$ they are

$$a_1 = 9(N^2-1)^2(N-\frac{4}{N})^2, \quad a_2 = \frac{9}{N^2-1}a_1, \quad a_3 = 162(N^2-1)\frac{48-16N^2+N^4}{N^2}$$

- For different gauge groups they are different, however a_1 can always be set to 1 and a_2 is related to the central charge.
- Notice that for $p=2$, there are only a_1 and a_2, then the only input needed is the central charge of the theory.

Bounds on $[1, 0, 1]$

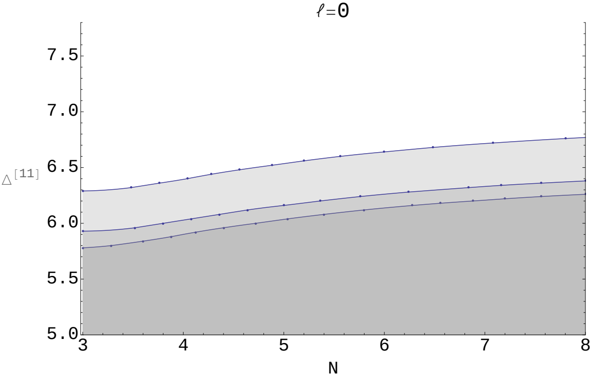
$$\mathrm{Tr}\Phi^I D^\ell \Phi^J \Phi^K \Phi^L + ..., \quad \ell = 1, 3, ...$$



$$\Delta_\infty \leq 7.24$$

Bounds on $[0, 2, 0]$

$$\mathrm{Tr}\Phi^I D^\ell \Phi^I \Phi^{(J}\Phi^{K)} + \ldots, \quad \ell = 0, 2, \ldots$$



$$\Delta_\infty \leq 6.48$$

Comments

- The bounds for the dimension of these operators represent rigorous, non-perturbative, information about non-planar $\mathcal{N} = 4$ SYM
- They can be improved by using more sophisticated numerical techniques
- We expect the leading twist operators to be given by double trace operators and the dimension to behave as $\Delta \approx \Delta_0 + 2 - \kappa/N^2$. It is possible to extrapolate with our method the values of κ, which has not been computed with any other method yet.
- For the singlet case, it has been computed in the context of AdS/CFT and it is -16. It has been extracted via bootstrap techniques in [Beem, Rastelli, van Rees] and it is consistent with the value computed.

Conclusions

- Crossing symmetry + superconformal symmetry → coupled bootstrap equations
- Upper bounds to the scaling dimension of unprotected superconformal primary operators transforming non-trivially under the $SU(4)$ R-symmetry group
- These bounds depend not only on the central charge but also on additional parameters that appear in the OPE of two symmetric traceless tensor fields
- Bounds for operators in the $[1,0,1]$ and $[0,2,0]$ representations for $\mathcal{N}=4$ SYM with gauge group $SU(N)$. These bounds represent rigorous, non-perturbative, information about non-planar $\mathcal{N}=4$ SYM.

Extra: linear operator

- The linear operator takes the form

$$\Phi^{(\Lambda)}\begin{pmatrix} f_1(a,b) \\ f_2(a,b) \\ f_3(a,b) \end{pmatrix} = \sum_{i,j=0}^{i+j=\Lambda}\left(\frac{\xi_{ij}^{(1)}}{i!j!}\partial_a^i\partial_b^j f_1(0,0) + \frac{\xi_{ij}^{(2)}}{i!j!}\partial_a^i\partial_b^j f_2(0,0) + \frac{\xi_{ij}^{(3)}}{i!j!}\partial_a^i\partial_b^j f_3(0,0)\right)$$

where

$$z = 1/2 + a + b, \quad \bar{z} = 1/2 + a - b$$