The Exact Effective Couplings of 4D $\mathcal{N} = 2$ gauge theories

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Based on [V.M., Elli Pomoni, arXiv:1406.3629] and work in progress

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What can we say about less supersymmetric theories?

Techniques

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- Dual holographic description
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What is the reason for integrability?

- Planarity?
- Conformality?
- Supersymmetry?
- Transformation properties of the fields?

The claim

The purely gluonic sector

Pick your favorite $\mathcal{N} = 2$ SCFT
and take one vector multiplet

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is closed to all loops

[Pomoni, 2013]

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The purely gluonic sector in every $\mathcal{N} = 2$ SCFT is integrable

[Pomoni, 2013]
see also [Pomoni, Sieg, 2011], [Gadde, Liendo, Rastelli, Yan, 2012]

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$$g^2 = \frac{g_{YM}^2 N_c}{(4\pi)^2} = \frac{\lambda}{(4\pi)^2}$$

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Example: anomalous dimensions

$$\gamma^{\mathcal{N}=2}(g^2, \ldots) = \gamma^{\mathcal{N}=4}(f(g^2, \ldots))$$

Effective tension

Orbifolds of $\mathcal{N} = 4$ SYM are well known [Kachru, Silverstein, 1998], [Lawrence, Nekrasov, Vafa, 1998]

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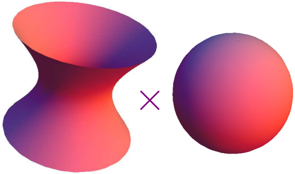

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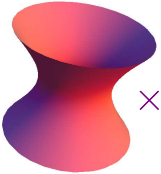
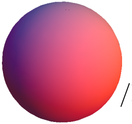


$\times$



$/\Gamma \subset \mathrm{SU}(2) \subset \mathrm{SO}(6)$

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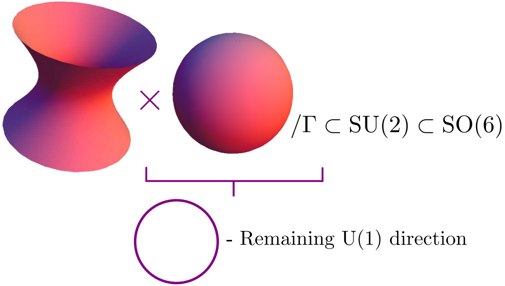

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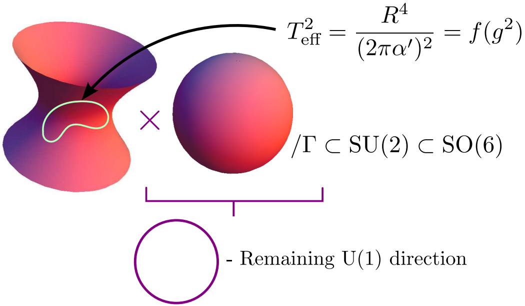
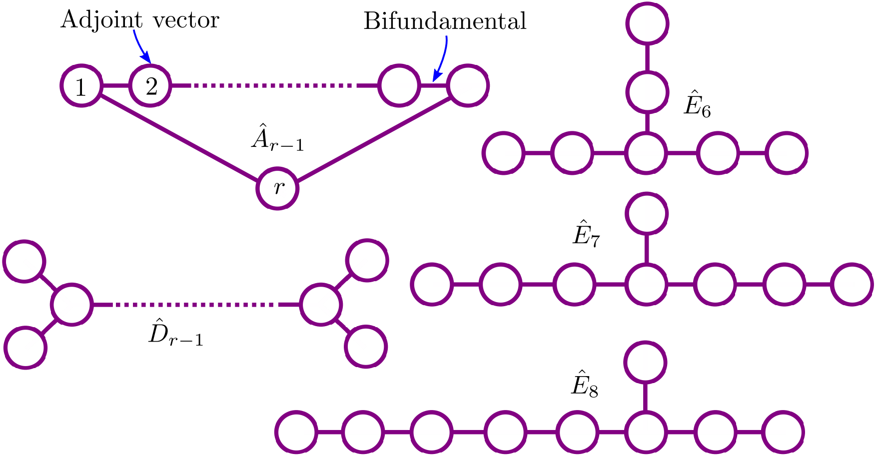
$$\text{Effective couplings} = \text{effective string tension}$$

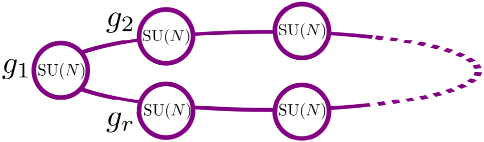


$\hat{A}_{r-1}$ quiver gauge theories are dual to $AdS_5 \times S^5/\mathbb{Z}_r$

ADE classification of superconformal gauge theories



$\hat{A}_{r-1}$: the cyclic quiver



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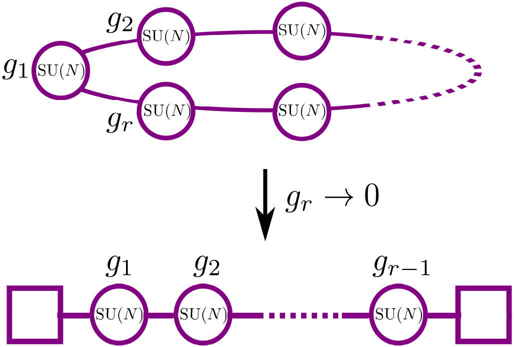

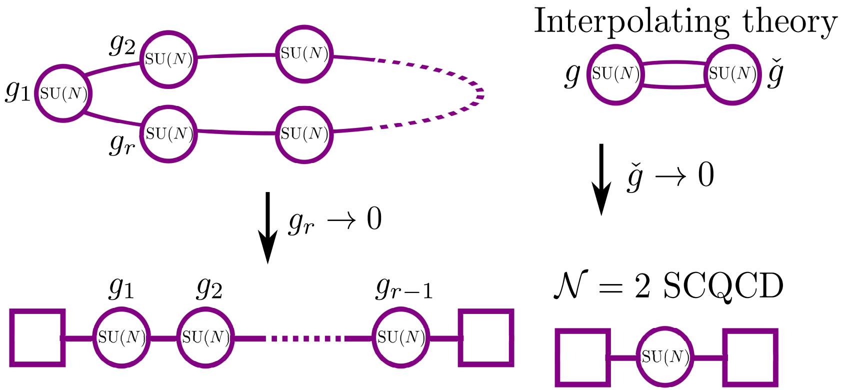
g_2
g_1
g_r
SU(N)
$g_r \to 0$
g_1
g_2
g_{r-1}
Interpolating theory
g
$\check{g}$
$\check{g} \to 0$
$\mathcal{N} = 2$ SCQCD

The diagrammatic argument

Background field formalism

Want to keep as much of the manifest local gauge invariance as possible

Background field

$$A_\mu = \underbrace{\mathcal{A}_\mu}_{\text{classical background}} + \underbrace{\mathbb{A}_\mu}_{\text{quantum fluctuations}}$$

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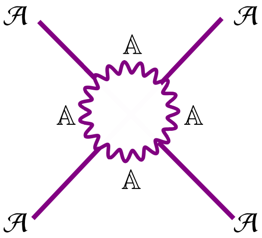

- $\mathcal{A}$ outside
- $\mathbb{A}$ inside

Simplifications

Renormalization factors

$$\mathcal{A}^{\mu}_{\text{bare}} = \sqrt{\mathcal{Z}_{\mathcal{A}}}\mathcal{A}^{\mu}_{\text{ren}} \qquad \mathbb{A}^{\mu}_{\text{bare}} = \sqrt{\mathcal{Z}_{\mathbb{A}}}\mathbb{A}^{\mu}_{\text{ren}}$$
$$g_{\text{bare}} = \mathcal{Z}_g g_{\text{ren}} \qquad \xi_{\text{bare}} = \mathcal{Z}_{\xi}\xi_{\text{ren}}$$

- We have the relations

$$\mathcal{Z}_g\sqrt{\mathcal{Z}_{\mathcal{A}}} = 1 \qquad \text{and} \qquad \mathcal{Z}_{\mathbb{A}} = \mathcal{Z}_{\xi}$$

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- The renormalization factors $\mathcal{Z}_{\mathbb{A}}$ cancel for each individual diagram

Cancellations

Compute $\langle \mathcal{O}(y)\mathcal{A}(x_1)\cdots\mathcal{A}(x_L)\rangle$ for $\mathcal{O} \sim \mathrm{tr}\left(\phi^L\right)$.

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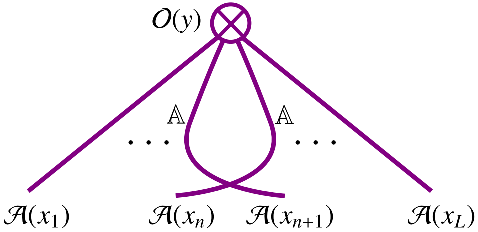

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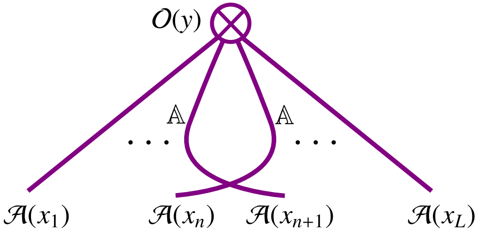

$$\mathcal{O}_i^{\mathrm{ren}}\left(\mathbb{A}_{\mathrm{ren}}, \mathcal{A}_{\mathrm{ren}}\right) = \sum_j \mathcal{Z}_{ij}\mathcal{O}_j^{\mathrm{bare}}\left(\mathcal{Z}_{\mathbb{A}}^{1/2}\mathbb{A}, \mathcal{Z}_{\mathcal{A}}^{1/2}\mathcal{A}\right)$$

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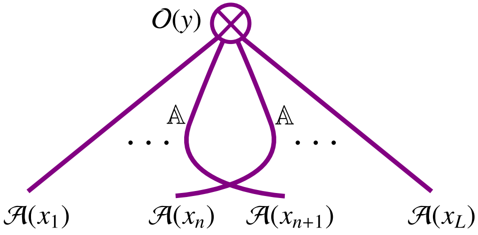

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- $\langle \mathbb{A}\mathbb{A}\mathcal{A}\mathcal{A}\rangle$ renormalizes as $\mathcal{Z}_{\mathbb{A}}\mathcal{Z}_{\mathcal{A}}\langle \mathbb{A}\mathbb{A}\mathcal{A}\mathcal{A}\rangle$
- The $\mathbb{A}$ propagators as $\mathcal{Z}_{\mathbb{A}}^{-1}$
- The $\mathcal{O}^{ren}$ has two more $\mathcal{Z}_{\mathbb{A}}^{1/2}$

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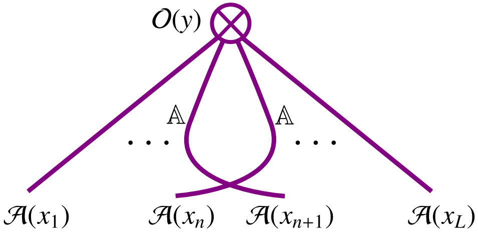

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- The $\mathcal{O}^{ren}$ has two more $\mathcal{Z}_{\mathbb{A}}^{1/2}$
- All the $\mathcal{Z}_{\mathbb{A}}$ cancel
- The final result depends only on $\mathcal{Z}_{\mathcal{A}} = \mathcal{Z}_g^{-2}$

Novel regularization

Regularization prescription for $\mathcal{N} = 2$ theories

Subtract from a given $\mathcal{N} = 2$ diagram the $\mathcal{N} = 4$ diagram with the same external states

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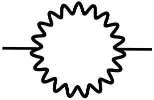
Example: the divergent bubble



The difference is zero since the gluonic tree level terms in both the $\mathcal{N} = 2$ and the $\mathcal{N} = 4$ Lagrangians are identical

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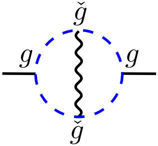
Example: First different diagram



The diagram is finite, the difference non-zero

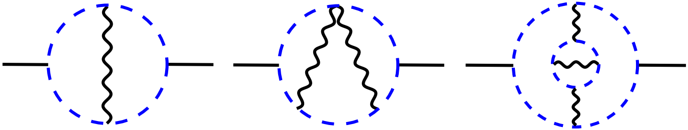
Novel regularization

Regularization prescription for $\mathcal{N} = 2$ theories

Subtract from a given $\mathcal{N} = 2$ diagram the $\mathcal{N} = 4$ diagram with the same external states

Diagrams different from the $\mathcal{N} = 4$ ones $\Rightarrow$ make hypermultiplet loops and then let fields from the other vector multiplets propagate inside

[Pomoni, Sieg, 2011], [Pomoni, 2013]



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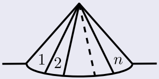
Basic building block: Fan Integrals



$$= 2 \binom{2n-1}{n} \zeta(2n-1) \frac{1}{p^2}$$

New vertices appear in the $\mathcal{N} = 2$ effective action

New vertices

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However: for operators in the SU(2,1|2) sector, the non-renormalization theorem of [Fiamberti, Santambrogio, Sieg, Zanon, 2008], [Sieg, 2010]

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However: for operators in the SU(2,1|2) sector, the non-renormalization theorem of [Fiamberti, Santambrogio, Sieg, Zanon, 2008], [Sieg, 2010]

None of these new vertices can contribute to the anomalous dimensions
Only renormalized tree level vertices can contribute

Effective couplings

For each gauge group

$$f_k(g_1^2, \ldots, g_r^2) = g_k^2 + g_k^2 \Big[\underbrace{\left(\mathcal{Z}_{g_k}^{\mathcal{N}=2}\right)^2}_{\text{propagator of } k^{\text{th}}-\text{group}} - \underbrace{\left(\mathcal{Z}_{g_1=\cdots=g_r}^{\mathcal{N}=4}\right)^2}_{\text{orbifold point}} \Big]$$

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Wilson Loops

From Localization

Circular Wilson Loop in $\mathcal{N} = 4$ SYM

Circular Wilson loop in $\mathcal{N} = 4$

$$W^{\mathcal{N}=4}(g) = \frac{I_1(4\pi g)}{2\pi g} = \begin{cases} 1 + 2\pi^2 g^2 + \frac{4\pi^4 g^4}{3} + \frac{4\pi^6 g^6}{9} + \mathcal{O}(g^8) \\ \frac{e^{4\pi g}}{\sqrt{32\pi^4 g^3}}\Big(1 + \mathcal{O}(g^{-1})\Big) \end{cases}$$

[Erickson, Semenoff, Zarembo, 2000]

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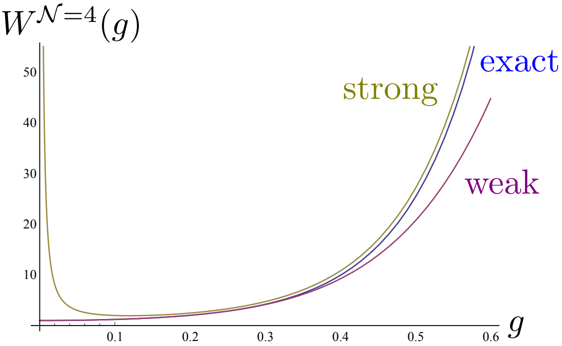

Wilson loops in $\mathcal{N} = 2$ gauge theories

Purely gluonic observable

$$W_k^{\mathcal{N}=2} = \left\langle \frac{1}{N_c} \mathrm{tr}_{\square} \mathrm{Pexp} \oint_{\mathcal{C}} ds \left(i A_\mu^{(k)}(x) \dot{x}^\mu + \phi^{(k)}(x) |\dot{x}| \right) \right\rangle$$

- $\square$ is fundamental representation of $SU(N_c)$
- $\mathcal{C}$ is the circular loop located at the equator of S^4
- The adjoint scalar $\phi^{(k)}$ and the gauge field $A_\mu^{(k)}$ are in the vector multiplet of the k-th gauge group.

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$$W_k^{\mathcal{N}=2}(g_1, \ldots, g_r) = W^{\mathcal{N}=4}(f_k(g_1, \ldots, g_r))$$

$$f_k(g_1, \ldots, g_r) = g_k^2 + \cdots$$

the effective coupling constant of the k-th gauge group

Partition function from localization

Expectation value

$$\left\langle \phi^{(k)} \right\rangle = \mathrm{diag}\left(a_1^{(k)}, \ldots, a_{N_c}^{(k)}\right)$$

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Partition function on S^4

$$Z = \int \prod_{k=1}^{r} da^{(k)} \prod_{i<j=1}^{N_c} \left(a_i^{(k)} - a_j^{(k)}\right)^2 e^{-\frac{N_c}{2g_k^2} \sum_{i=1}^{N_c} \left(a_i^{(k)}\right)^2} Z_{\text{1-loop}} \cancel{|Z_{\text{inst}}|^2} \text{ planar limit}$$

Perturbative Part

Vector multiplet: $$Z^{\text{vect}}_{\text{1-loop}} = \prod_{i<j=1}^{N_c} H^2(a_i - a_j)$$

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$$H(x) = G(1 + ix)G(1 - ix)e^{-(1+\gamma)x^2} = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}}$$

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The full one loop part

$$Z_{\text{1-loop}} = \prod_{k,l=1}^{r} \prod_{i,j=1}^{N_c} H^{\frac{\mathbf{c}_{kl}}{2}}\left(a_i^{(k)} - a_j^{(l)}\right)$$

Saddle point approximation

Effective action

$$Z = \int \prod_{k=1}^{r} d^{N_c-1} a^{(k)} e^{-N_c \mathcal{S}_{\text{eff}}} \Longrightarrow \frac{\partial \mathcal{S}_{\text{eff}}}{\partial a_i^{(k)}} = 0$$

Densities

$$\rho_k(x) = \frac{1}{N_c} \sum_{i=1}^{N_c} \delta\left(x - a_i^{(k)}\right) \Rightarrow \int_{-\mu_k}^{\mu_k} \rho_k(x) dx = 1$$

Integral equations

$$\frac{x}{2g_k^2} = \fint_{-\mu_k}^{\mu_k} \frac{\rho_k(y)}{x-y} - \frac{1}{2}\sum_{l=1}^{r} \mathbf{c}_{kl} \int_{-\mu_l}^{\mu_l} \rho_l(y) K(x-y) dy$$

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Wilson loop expectation values

$$W_k^{\mathcal{N}=2} = \left\langle \frac{1}{N_c}\sum_{i=1}^{N_c} e^{2\pi a_i^{(k)}} \right\rangle = \int_{-\mu_k}^{\mu_k} \rho_k(x) e^{2\pi x} dx.$$

Weak coupling result

For the simplest quiver $\hat{A}_1$

$$W^{\mathcal{N}=2}(g,\check{g}) = 1 + 2\pi^2 g^2 + \frac{4}{3}\pi^4 g^4 + \pi^6\left[\frac{4}{9}g^6 - 24g^4(\check{g}^2 - g^2)\frac{\zeta(3)}{\pi^4}\right]$$
$$+ \pi^8\left[\frac{4}{45}g^8 + (\check{g}^2 - g^2)\left(32g^6\frac{\zeta(3)}{\pi^4} - 80g^4(3\check{g}^2 + g^2)\frac{\zeta(5)}{\pi^6}\right)\right]$$
$$+ \pi^{10}\left[\frac{8}{675}g^{10} + (\check{g}^2 - g^2)\left(16g^8\frac{\zeta(3)}{\pi^4} - 80g^6\left(13g^2 + 4\check{g}^2\right)\frac{\zeta(5)}{3\pi^6}\right.\right.$$
$$\left.\left. - 288g^4\left(2g^4 - g^2\check{g}^2 + \check{g}^4\right)\frac{\zeta(3)^2}{\pi^8} + 280g^4\left(8g^4 + 5g^2\check{g}^2 + \check{g}^4\right)\frac{\zeta(7)}{\pi^8}\right)\right] + \cdots$$

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Effective couplings

$$W^{\mathcal{N}=2}(g,\check{g}) = W^{\mathcal{N}=4}(f(g,\check{g}))$$

$$\begin{aligned}\Rightarrow f(g,\check{g}) \;=\; & g^2 + 2\left(\check{g}^2 - g^2\right)\Big[6\zeta(3)g^4 - 20\zeta(5)g^4\left(\check{g}^2 + 3g^2\right) \\ & + g^4\Big(70\zeta(7)\Big(\check{g}^4 + 5\check{g}^2g^2 + 8g^4\Big) - 2\zeta(2)(20\zeta(5))g^4 \\ & - 2(6\zeta(3))^2\Big(\check{g}^4 - \check{g}^2g^2 + 2g^4\Big)\Big)\Big] + \cdots\end{aligned}$$

$\mathbb{Z}_2$ symmetry: $\check{f}(g,\check{g}) = f(\check{g},g)$

Feynman diagram interpretation

First correction

First $\zeta(3)$ correction computed in [Pomoni, Sieg, 2011]

$$f(g,\check{g}) = g^2 + \underline{2(\check{g}^2 - g^2)}\Big[\underline{6\zeta(3)g^4} - 20\zeta(5)g^4\left(\check{g}^2 + 3g^2\right) + g^4\Big(70\zeta(7)\Big(\check{g}^4 + 5\check{g}^2g^2 + 8g^4\Big) - 2\zeta(2)(20\zeta(5))g^4 - 2(6\zeta(3))^2\Big(\check{g}^4 - \check{g}^2g^2 + 2g^4\Big)\Big)\Big] + \cdots$$

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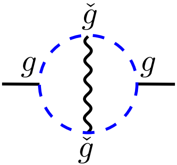
\end{aligned}
$$



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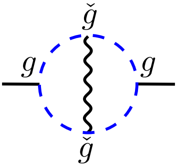

$$
\Rightarrow \underbrace{12g^4\check{g}^2\zeta(3)}_{\mathcal{N}=2} - \underbrace{12g^6\zeta(3)}_{\mathcal{N}=4} = 2(6\zeta(3))g^4(\check{g}^2 - g^2).
$$

Second correction

$$f(g,\check{g}) = g^2 + \underline{2(\check{g}^2 - g^2)}\Big[6\zeta(3)g^4 \underline{-20\zeta(5)g^4(\check{g}^2 + 3g^2)}$$
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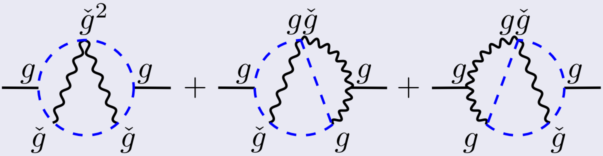

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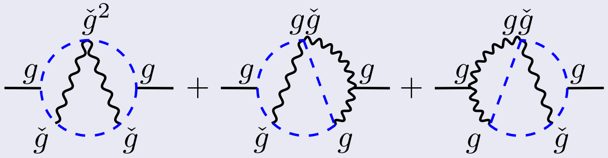
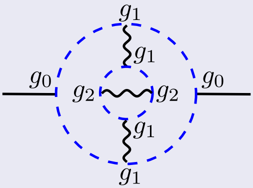


$$
\underbrace{-20\zeta(5)\Big(\check{g}^4g^4 + 2\check{g}^2g^6\Big)}_{\mathcal{N}=2} + \underbrace{20\zeta(5)\Big(3g^8\Big)}_{\mathcal{N}=4} = -20\zeta(5)g^4(\check{g}^2 - g^2)(\check{g}^2+3g^2)
$$

Less than maximum transcendentality corrections

Quiver with $r > 2 \Longrightarrow$ next to nearest neighbor gauge groups

Nested diagram



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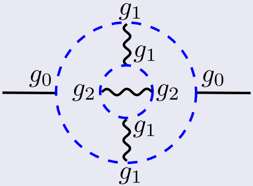

$$
\begin{aligned}
f_0 \quad = \quad & \cdots + (6\zeta(3))^2 g_0^4 \Big[8g_0^6 - 2g_{-1}^6 - 2g_1^6 + \underline{g_1^4 g_2^2 + g_{-1}^4 g_{-2}^2} \\
& -6g_0^4 \left(g_{-1}^2 + g_1^2 \right) + 2g_0^2 \left(g_{-1}^4 + g_{-1}^2 g_1^2 + g_1^4 \right) \Big] + \cdots
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Strong coupling

$$\text{Large couplings } g_k \to \infty \implies W_k^{\mathcal{N}=2} \sim e^{2\pi\mu_k}$$

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Effective coulings

$$\frac{1}{f_k} = \frac{1}{r}\left(\frac{1}{g_1^2} + \cdots + \frac{1}{g_r^2}\right)$$

Agrees with an AdS/CFT computation

[Lawrence, Nekrasov, Vafa, 1998]
[Gadde, Pomoni, Rastelli, 2009]
[Gadde, Liendo, Rastelli, Yan, 2012]

Outlook

Twist-two descendent of Konishi

$$\Delta(g,\check{g}) = 4 + 12g^2 - 48g^4 + 48g^4\left[7g^2 - 3\left(g^2 - \check{g}^2\right)\zeta(3)\right]$$
$$+96g^4\Big[-26g^4 + 6\zeta(3)g^4 - 15\zeta(5)g^4 + \left(g^2 - \check{g}^2\right)\Big(12g^2\zeta(3)$$
$$+5\left(3g^2 + \check{g}^2\right)\zeta(5)\Big)\Big] + 16g^4\Big[948g^6 + 432g^6\zeta(3)$$
$$-324g^6\zeta(3)^2 - 540g^6\zeta(5) + 1890g^6\zeta(7)$$
$$-3\left(g^2 - \check{g}^2\right)\Big[\left(8g^4 + 5g^2\check{g}^2 + \check{g}^4\right)35\zeta(7)$$
$$-g^2\left(4\check{g}^2 + g^2\left(12 - \zeta(2)\right)\right)20\zeta(5)$$
$$-\left(2g^4 - g^2\check{g}^2 + \check{g}^4\right)\left(6\zeta(3)\right)^2 + 42g^4\left(6\zeta(3)\right)\Big]\Big] + \cdots$$

Outside the sector

Bifundamental hypermultiplet in the ϕ vacuum

$$\cdots \phi\phi Q\check{\phi}\check{\phi}\cdots$$

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[Gadde, Rastelli, 2010]

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[Gadde, Rastelli, 2010]

$$\mathbf{g} = f(g, \check{g})^{\frac{1}{2}} \qquad \check{\mathbf{g}} = \check{f}(g, \check{g})^{\frac{1}{2}} = f(\check{g}, g)^{\frac{1}{2}}$$

Future

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- Mass terms for the hypermultiplets
  $\Longrightarrow$ asymptotically conformal quiver theories
  $\Longrightarrow$ No additional UV divergences
- Check in other observables: Cusp anomalous dimension, scattering amplitudes, Wilson loops, ...
  [Leoni, Mauri, Santambrogio, 2014]

Thank you

The $SU(2,1|2)$ sector of $\mathcal{N} = 2$ SCFT's

Why the sector is closed to all loops?

- For $g = 0$:

  all the fields $\boxed{\phi\,,\,\lambda^{\mathcal{I}}_{+}\,,\,\mathcal{D}_{+\dot\alpha}}$ obey $\boxed{\Delta = 2j - r}$

  while all the rest of the fields: $Q\,,\,\tilde{Q}\,,\,\psi\,,\,\tilde{\psi}\,,\,\bar{\phi}\,,\,\lambda^{\mathcal{I}}_{-}\,,\,\bar{\lambda}_{\mathcal{I}\,\dot\alpha}\,,\,\mathcal{D}_{-\dot\alpha}$

  violate (only in one direction) the equality: $\Delta > 2j - r$ by at least $1/2$

- In perturbation theory $g << 1$ the radiative corrections in $\Delta\,(\lambda)$, $j\,(\lambda)$ and $r\,(\lambda)$ will never be bigger that $1/2$!

The λ expansion is believed to converge ('t Hooft) .

This sector is closed for any finite value of λ in the planar limit !

Weak coupling expansion

$$g_k = \kappa_k g, \qquad \kappa_k \text{ fixed}$$

Densities widths

$$\mu_k = g_k \left(1 + \sum_{i=1}^{P+1} \mathbf{A}_{k;i}\, g_k^i \right)$$

Moments of the densities

$$\int_{-\mu_k}^{\mu_k} \rho_k(x) x^{2i} dx = g_k^i \left(C_i + \sum_{j=1}^{P+1-i} \mathbf{B}_{k;2i;j}\, g_k^j \right)$$