Scattering Amplitudes in Twistor Space

Arthur Lipstein
Mathematical Institute
University of Oxford

18/7/2014

Based on

1212.6228, 1307.1443 (Lipstein, Mason)

1404.6219, 1406.1462 (Geyer, Lipstein, Mason)

Spinor-Helicity

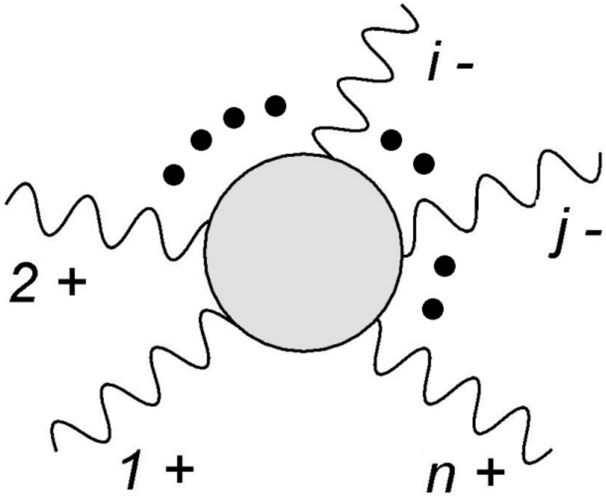
- 4d null momentum:

$$p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

- Expressing amplitudes in terms of these spinors leads to very simple expressions.

MHV Amplitudes

At tree-level:



$$A_n = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

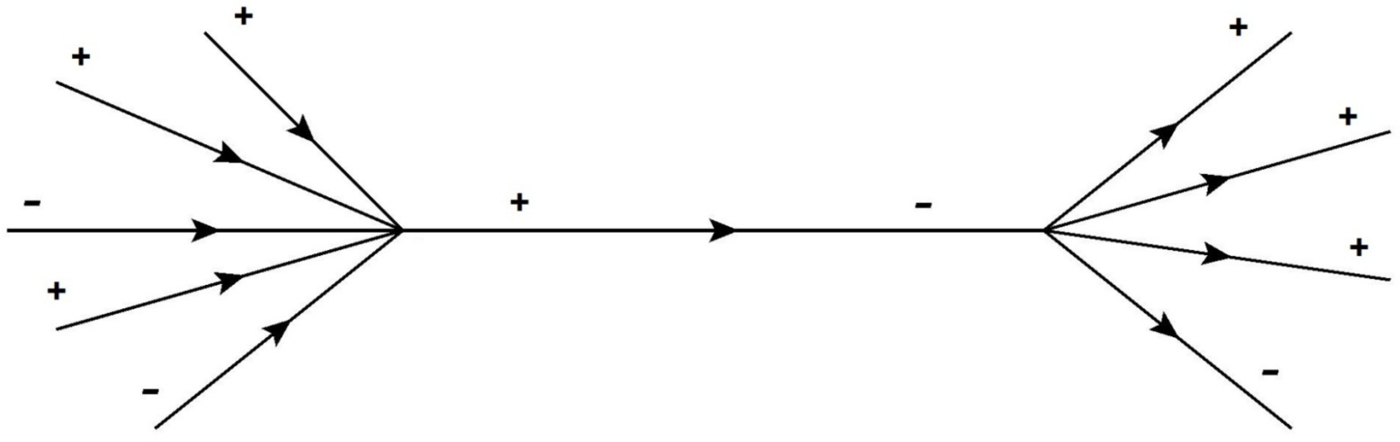
(Parke,Taylor)

where $\langle ij \rangle = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$

CSW Formalism

- Use tree-level MHV amplitudes as Feynman vertices for constructing tree-level non-MHV amplitudes.

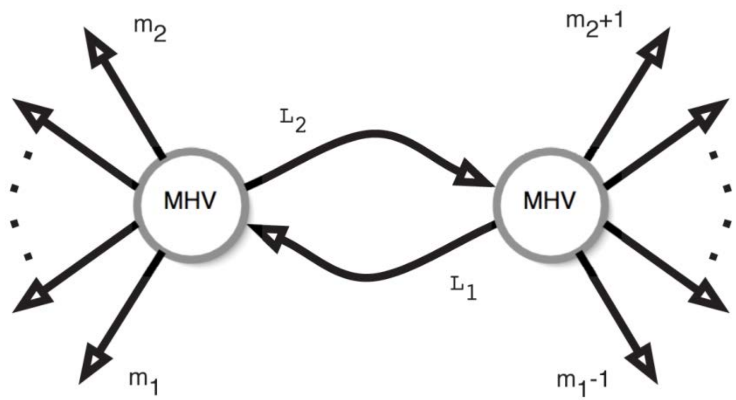
  (Cachazo,Svrcek,Witten)
- Example: NMHV amplitude



(super) CSW Formalism

- Use tree-level MHV superamplitudes as Feynman vertices to construct tree-level non-MHV amplitudes

- Can also use these vertices to construct loop amplitudes (Brandhuber,Spence,Travaglini)



Twistor String Theory

- The simplicity of MHV amplitudes and the CSW formalism suggests a deeper mathematical structure.

- Is there a way to reformulate Yang-Mills theory to make this structure manifest?

- Witten: N=4 SYM is equivalent to string theory with target space $CP^{3|4}$

Twistors

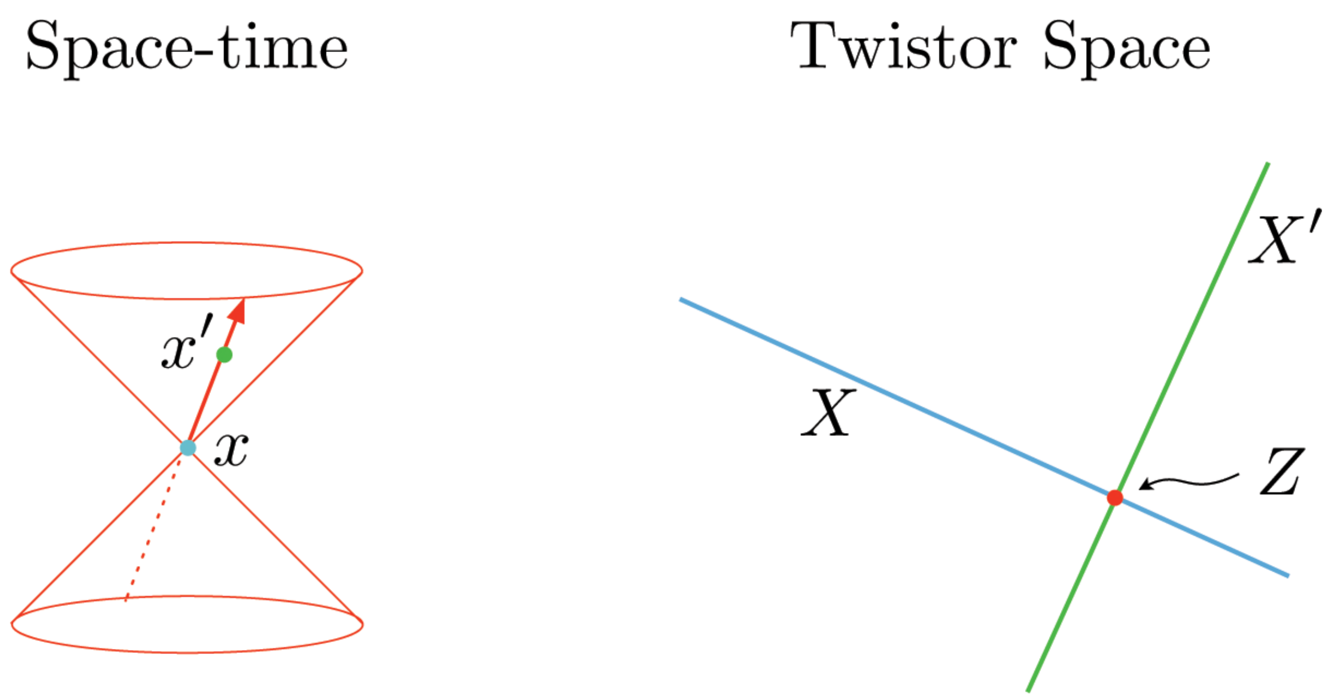
- Twistors: (Penrose)

$$\begin{pmatrix} Z^A \\ \chi^a \end{pmatrix}, \;\; Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot\alpha} \end{pmatrix}$$

- Incidence relations:

$$\mu^{\dot\alpha} = -i x^{\dot\alpha\alpha}\lambda_\alpha, \;\; \chi^a = -i\theta^{a\alpha}\lambda_\alpha$$

Spacetime vs Twistor Space



Point in spacetime ⟷ CP^1 in twistor space

Point in twistor space ⟷ null ray in spacetime

N=4 sYM in Twistor Space

- Superfield:

$$\mathcal{A} = g^+ + \chi^a \tilde{\psi}_a + \frac{1}{2}\chi^a\chi^b\phi_{ab} + \epsilon_{abcd}\chi^a\chi^b\chi^c\left(\frac{1}{3!}\psi^d + \frac{1}{4!}\chi^d g^-\right)$$

- Twistor action: (Boels/Mason/Skinner)

$$S[\mathcal{A}] = \frac{i}{2\pi}\int D^{3|4}Z\,\mathrm{Tr}\left(\mathcal{A}\wedge\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) \quad \longleftarrow \text{ self-dual sector}$$

$$+ g^2\int d^{4|8}x \log\det\left(\left(\bar{\partial}+\mathcal{A}\right)\big|_X\right) \quad \longleftarrow \text{ MHV expansion}$$

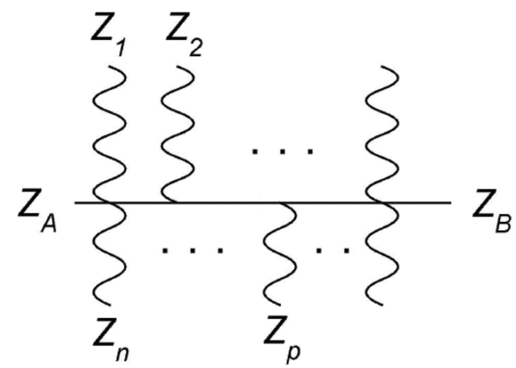
- Feynman rules correspond to CSW formalism!

Feynman Rules

- Expanding logdet term gives infinite sum of MHV vertices.

- MHV Vertices:

$$\int_{\mathbb{M}\times(\mathbb{CP}^1)^n} \frac{\mathrm{d}^{4|4}Z_A\,\mathrm{d}^{4|4}Z_B}{\mathrm{Vol}\ \mathrm{GL}(2)} \prod_{i=1}^{n} \frac{D\sigma_i}{(\sigma_i\,\sigma_{i+1})}$$



where $\sigma = (\sigma^0, \sigma^1)$ are homogeneous coordinates on (Z_A,Z_B)

$Z(\sigma_i) = \sigma_i^0 Z_A + \sigma_i^1 Z_B$ is the i'th insertion point, and

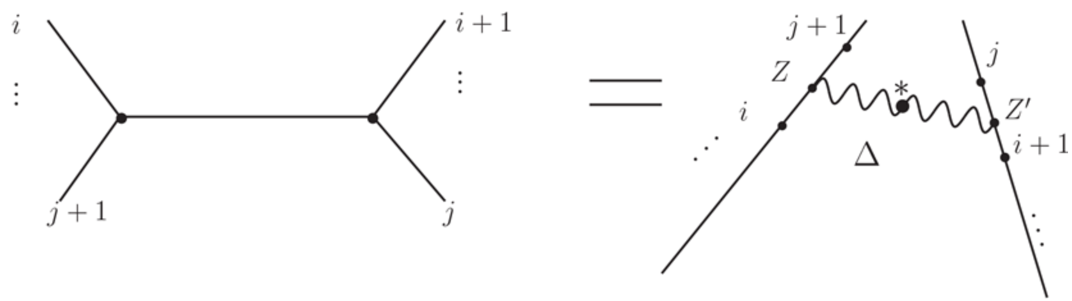
$$(\sigma_i\,\sigma_j) = \sigma_i^0\sigma_j^1 - \sigma_i^1\sigma_j^0\,,\quad D\sigma = (\sigma d\sigma)$$

Feynman Rules

- Axial gauge: $\bar{Z}_* \cdot \mathcal{A} = 0$
- Propagator: $\Delta(Z, Z') = \frac{1}{2\pi i} \int_{\mathbb{C}^2} \frac{\mathrm{d}u}{u} \frac{\mathrm{d}v}{v} \bar{\delta}^{4|4}(Z + uZ_* + vZ')$

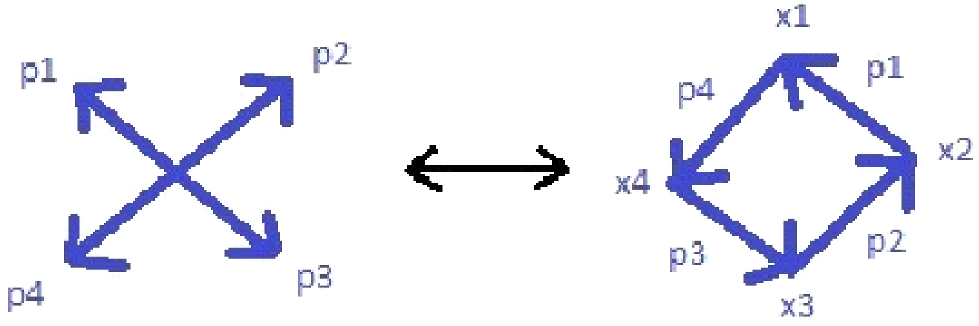
where $\delta^{4|4}(Z) = \Pi_{A=1}^{4} \bar{\delta}\left(Z^A\right) \Pi_{a=1}^{4} \chi^a$

- NMHV:



Dual Conformal Symmetry

- Dual variables: $x_i - x_{i+1} = p_i$



- Tree-level amplitudes and loop integrands transform covariantly when

$$x_i \to x_i^{-1}$$

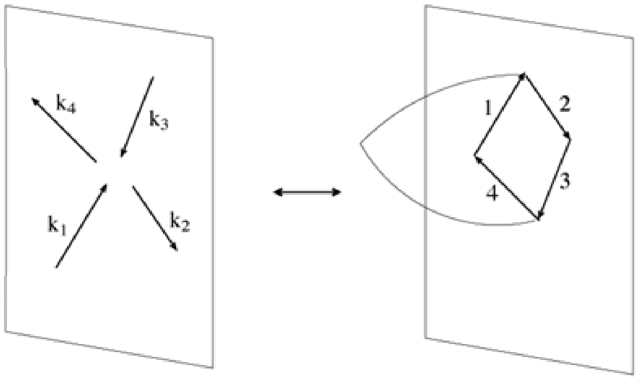
- Can be extended to dual superconformal symmetry by defining fermionic dual variables:

$$\theta_i - \theta_{i+1} = q_i$$

(Drummond,Henn,Korchemsky,Smirnov,Sokatchev)

Amplitude/Wilson Loop Duality

- Alday/Maldacena: Amplitudes mapped into null polygonal Wilson loops by T-duality:



- To compute an amplitude at strong coupling, just compute the area of a "soap bubble" in string theory.

- Remarkably, this duality extends to weak coupling! (Brandhuber/Heslop/Travaglini)

Momentum Twistors

- Make dual conformal symmetry manifest:

$$\begin{pmatrix} Z^A \\ \chi^a \end{pmatrix}, \quad Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot\alpha} \end{pmatrix}$$

$$\mu^{\dot\alpha} = -i x^{\dot\alpha\alpha}\lambda_\alpha, \quad \chi^a = -i\theta^{a\alpha}\lambda_\alpha$$

where (x,θ) now live in region momentum space.

- Momentum conservation automatic (Hodges)

R-invariants

- In momentum twistor space, NMHV amplitudes are simple:

$$\frac{A^{NMHV}}{A^{MHV}} = \sum_{i<j} [*i-1ij-1j]$$

where (Mason/Skinner)

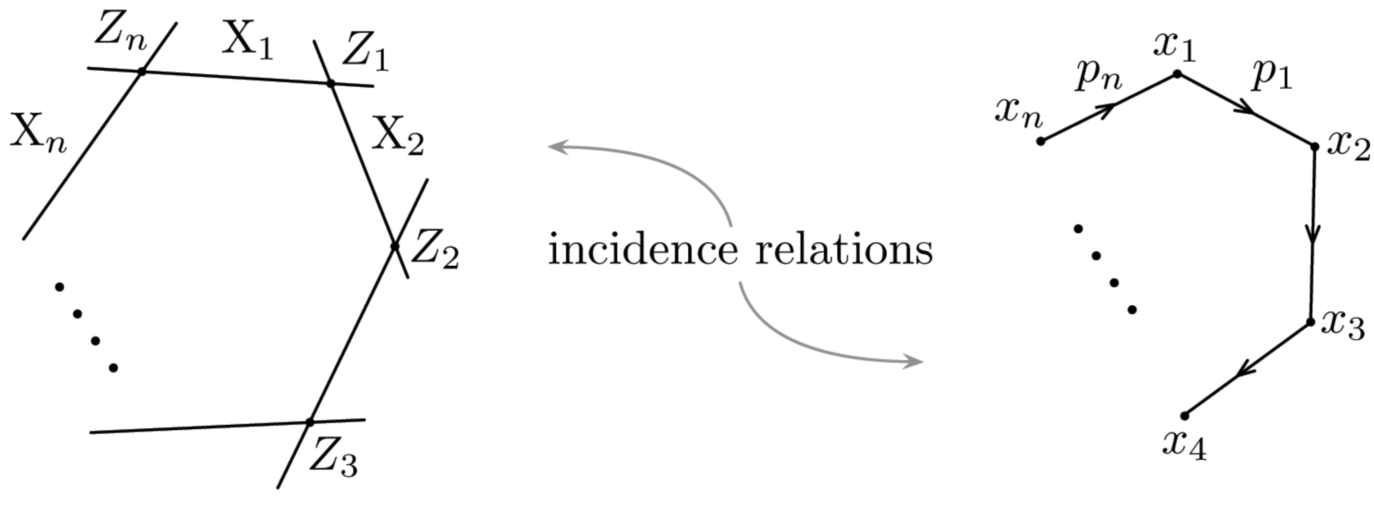
$$[*i-1ij-1j] = \frac{\langle *i-1ij-1\rangle \chi_j + cyclic}{\langle *i-1ij-1\rangle \times cyclic}$$

$$\langle ijkl\rangle = \epsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D$$

- N^kMHV amplitudes involve products of k such “R-invariants.” (Drummond/Henn;Bullimore/Mason/Skinner)

Twistor Wilson Loop

- Null polygon in spacetime corresponds to polygon in twistor space:



- Expectation value of the twistor Wilson loop computes planar S-matrix! (Mason/Skinner;Caron-Huot)

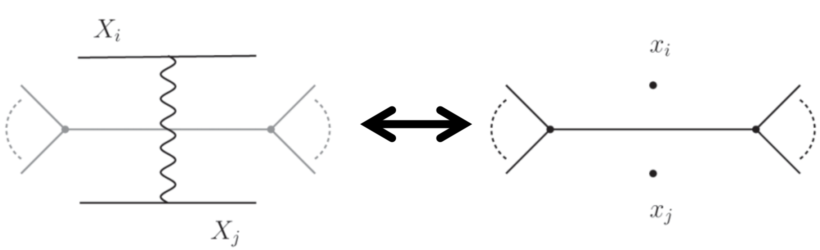
Planar Duality

- Amplitude Diagrams vs Wilson loop diagrams:

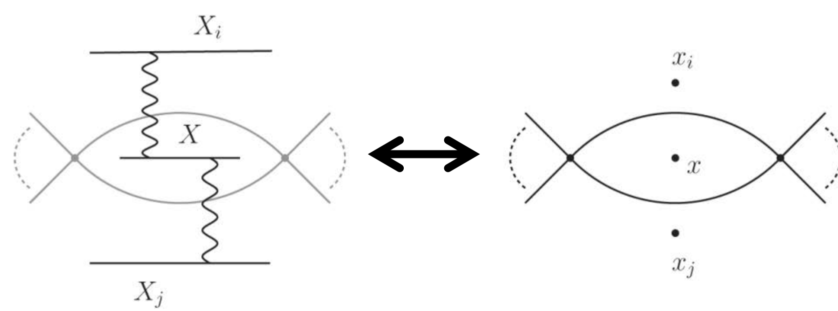
Amplitude	Wilson Loop
# of legs	# of sides
# of loops	# of MHV vertices
MHV degree	# of propagators – 2 x (# of MHV vertices)

- Examples:

Tree-level NMHV:



1-loop MHV:



Dlog Form

- Lipstein/Mason: Feynman rules for twistor Wilson loop are extremely simple and reveal new mathematical structure of loop amplitudes.

- In particular, the loop integrands can be expressed as a product of exterior derivatives of logarithms of rational functions, with external data encoded in the integration contour.

- We also found a systematic method for doing the integrals, and obtained a new expression for 1-loop MHV amplitudes.

Example: 1-loop MHV amplitude

$$K_{ij} = -\frac{1}{4\pi^2}\int \mathrm{d}\ln s_0\ \mathrm{d}\ln t_0\ \mathrm{d}\ln s\ \mathrm{d}\ln t$$

where

$$s_0 = \bar{s}_0$$

$$t_0 = \bar{t}_0$$

$$s = -\frac{\bar{t}\,(a_{i-1j} - v) + a_{i-1j-1} - v}{\bar{t}\,(a_{i\,j} - v) + a_{ij-1} - v} \qquad v = s_0 - t_0$$

Poles in s_0 and t_0 are real, so require regularization. This can be achieved using the Feynman iε prescription.

Generic Diagrams

- For diagrams invovling non-adjacent edges,

$$K_{ij} = \mathrm{Li}_2\left(\frac{a_{ij}}{v_*}\right) + \mathrm{Li}_2\left(\frac{a_{i-1j-1}}{v_*}\right) - \mathrm{Li}_2\left(\frac{a_{i-1j}}{v_*}\right) - \mathrm{Li}_2\left(\frac{a_{ij-1}}{v_*}\right) + c.c.$$

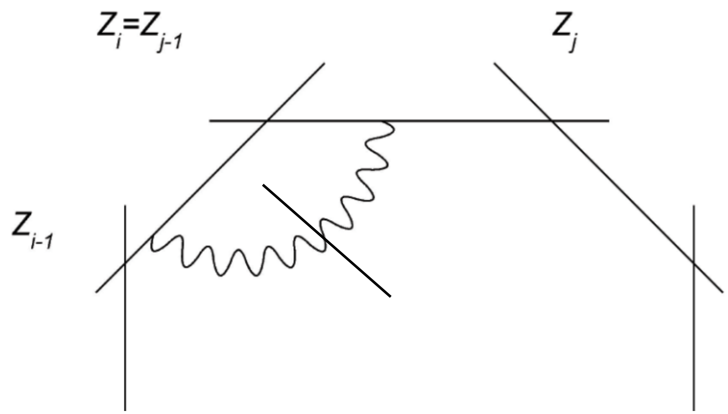
where $$v_* = \frac{a_{ij}a_{i-1j-1} - a_{ij-1}a_{i-1j}}{a_{ij} + a_{i-1j-1} - a_{i-1j} - a_{ij-1}}$$

- Dual conformal symmetry manifest

- Nontrivially agrees with previous results (Brandhuber/Spence/Travaglini)

Divergent Diagrams

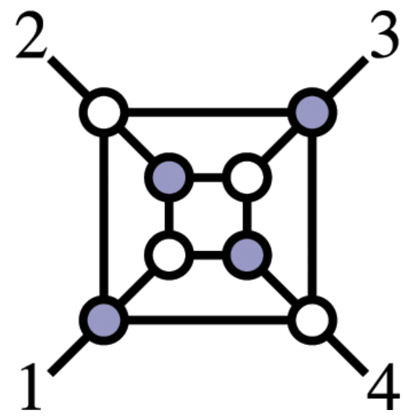
- Diagram is divergent when i=j-1:



- To regulate, take $i\varepsilon \longrightarrow m^2$ in Feynman $i\varepsilon$ prescription

On-Shell Diagrams

- Another dlog form for planar loop integrands of N=4 sYM follows from using on-shell diagrams:



- Integration variables correspond to BCFW shifts
- Diagrams correspond to cells of positive Grassmannian

(Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka)

Summary

- Combining insights from AdS/CFT and twistor string theory has lead to powerful new techniques for computing amplitudes of N=4 super-Yang-Mills, which have revealed new symmetries and dualities, and new mathematical structures at loop level.

- Question: Can these ideas be extended to other theories? In particular, can they make contact with real-world physics?

Extension to Gravity

- Hodges formula for tree-level MHV:

$$\mathcal{M}_{n,0} = \langle i,j\rangle^8 \det{}'(\mathrm{H})\, \delta^4\left(\sum_i p_i\right)$$

$$\mathrm{H}_{ij} = \frac{[i,j]}{\langle i,j\rangle} \qquad \text{for } i \neq j, \qquad\qquad \mathrm{H}_{ii} = -\sum_{j\neq i} \frac{[i,j]}{\langle i,j\rangle} \frac{\langle a,j\rangle\langle b,j\rangle}{\langle a,i\rangle\langle b,i\rangle}$$

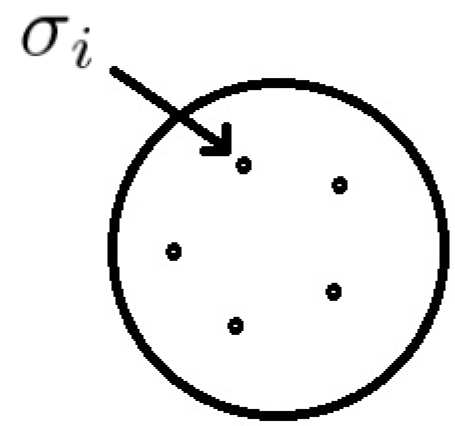
- Skinner: N=8 SUGRA is equivalent to string theory with target space $CP^{3|8}$

Scattering Equations

external momentum

$$\sum_{i \neq j} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0$$

point on 2-sphere



- Gross/Mende: These equations arise from the tensionless limit of string amplitudes
- Cachazo/He/Yuan: They also arise in the amplitudes of massless point particles!

Ambitwistor Strings

- Mason/Skinner: Amplitudes of complexified massless point particles can be computed using a chiral, infinite tension limit of the RNS string:

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu + \frac{e}{2} P_\mu P^\mu + \ldots$$

- Correlation functions of vertex operators reproduce the CHY formulae!

- Critical in d=26 (bosonic) and d=10 (superstring)

4d Ambitwistor Space

- Twistors/Dual Twistors:

$$Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot{\alpha}} \\ \chi^a \end{pmatrix}, \quad W_A = \begin{pmatrix} \tilde{\mu}^\alpha \\ \tilde{\lambda}_{\dot{\alpha}} \\ \tilde{\chi}_a \end{pmatrix}$$

- Incidence Relations:

$$\mu^{\dot{\alpha}} = i(x^{\alpha\dot{\alpha}} + i\theta^{a\alpha}\tilde{\theta}_a^{\dot{\alpha}})\lambda_\alpha \qquad \chi^a = \theta^{a\alpha}\lambda_\alpha$$

$$\tilde{\mu}^\alpha = -i(x^{\alpha\dot{\alpha}} - i\theta^{a\alpha}\tilde{\theta}_a^{\dot{\alpha}})\tilde{\lambda}_{\dot{\alpha}} \qquad \tilde{\chi}_a = \tilde{\theta}_a^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}$$

4d Ambitwistor Strings

- Action for YM:

$$S = \int_\Sigma W \cdot \bar{\partial} Z - Z \cdot \bar{\partial} W ...$$

- Action for gravity (N=2 worldsheet susy):

$$S = \int_\Sigma W \cdot \bar{\partial} Z - Z \cdot \bar{\partial} W + \rho \bar{\partial} \tilde{\rho} + \tilde{\rho} \bar{\partial} \rho ...$$

Vertex Operators

- (super)Yang-Mills: $\mathcal{V}_a = \int \frac{\mathrm{d}s_a}{s_a} \bar{\delta}^{2|\mathcal{N}}(\lambda_a - s_a\lambda | \eta_a - s_a\chi) \mathrm{e}^{is_a[\mu\,\tilde{\lambda}_a]} J \cdot t_a$

$$\widetilde{\mathcal{V}}_a = \int \frac{\mathrm{d}s_a}{s_a} \bar{\delta}^2(\tilde{\lambda}_a - s_a\tilde{\lambda}) \mathrm{e}^{is_a\left(\langle\tilde{\mu}\,\lambda_a\rangle + \tilde{\chi}_I \eta_a^I\right)} J \cdot t_a$$

- (super)gravity: $\mathcal{V}_h = \int \left[W, \frac{\partial h}{\partial Z}\right] + \left[\tilde{\rho}, \frac{\partial}{\partial Z}\right] \rho \cdot \frac{\partial h}{\partial Z}$

$$\widetilde{\mathcal{V}}_{\tilde{h}} = \int \left\langle Z, \frac{\partial \tilde{h}}{\partial W} \right\rangle + \left\langle \rho, \frac{\partial}{\partial W} \right\rangle \tilde{\rho} \cdot \frac{\partial \tilde{h}}{\partial W}$$

where $h_a = \int \frac{\mathrm{d}s_a}{s_a^3} \bar{\delta}^{2|\mathcal{N}}(\lambda_a - s_a\lambda | \eta_a - s_a\chi) \mathrm{e}^{is_a[\mu\,\tilde{\lambda}_a]}$

$$\tilde{h}_a = \int \frac{\mathrm{d}s_a}{s_a^3} \bar{\delta}^2(\tilde{\lambda}_a - s_a\tilde{\lambda}) \mathrm{e}^{is_a\left(\langle\tilde{\mu}\,\lambda_a\rangle + \tilde{\chi}_I \eta_a^I\right)} .$$

$$\langle Z_1 Z_2 \rangle \equiv \langle \lambda_1 \lambda_2 \rangle \,, \quad [W_1 W_2] \equiv \left[\tilde{\lambda}_1 \tilde{\lambda}_2\right]$$

Path Integral

- Consider N^kMHV amplitude
- Bringing exponentials into the action gives

$$\int_\Sigma \sum_{i=1}^{k} is_i(\langle\tilde{\mu}\lambda_i\rangle+\tilde{\chi}\cdot\eta_i)\bar{\delta}(\sigma-\sigma_i)+\sum_{p=k+1}^{n} is_p[\mu\,\tilde{\lambda}_p]\bar{\delta}(\sigma-\sigma_p)$$

- Equations of motion:

$$\bar{\partial}_\sigma Z=\bar{\partial}\left(\lambda,\mu,\chi\right)=\sum_{i=1}^{k} s_i\left(\lambda_i,0,\eta_i\right)\bar{\delta}\left(\sigma-\sigma_i\right),$$

$$\bar{\partial}_\sigma W=\bar{\partial}\left(\tilde{\mu},\tilde{\lambda},\tilde{\chi}\right)=\sum_{p=k+1}^{n} s_p\left(0,\tilde{\lambda}_p,0\right)\bar{\delta}(\sigma-\sigma_p)$$

- Solution:

$$Z(\sigma)=(\lambda,\mu,\chi)=\sum_{i=1}^{k}\frac{s_i\left(\lambda_i,0,\eta_i\right)}{\sigma-\sigma_i}$$

$$W(\sigma)=\left(\tilde{\mu},\tilde{\lambda},\tilde{\chi}\right)=\sum_{p=k+1}^{n}\frac{s_p\left(0,\tilde{\lambda}_p,0\right)}{\sigma-\sigma_p}$$

Amplitudes

- Geyer/Lipstein/Mason: 4d ambitwistor string theory gives rise to new tree-level formulae for gauge and gravity amplitudes with any amount of supersymmetry!

$$\mathcal{A} = \int \frac{1}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \prod_{i=1}^{k} \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i)) \prod_{p=k+1}^{n} \bar{\delta}^{2|\mathcal{N}}(\lambda_p - \lambda(\sigma_p), \eta_p - \chi(\sigma_p))$$

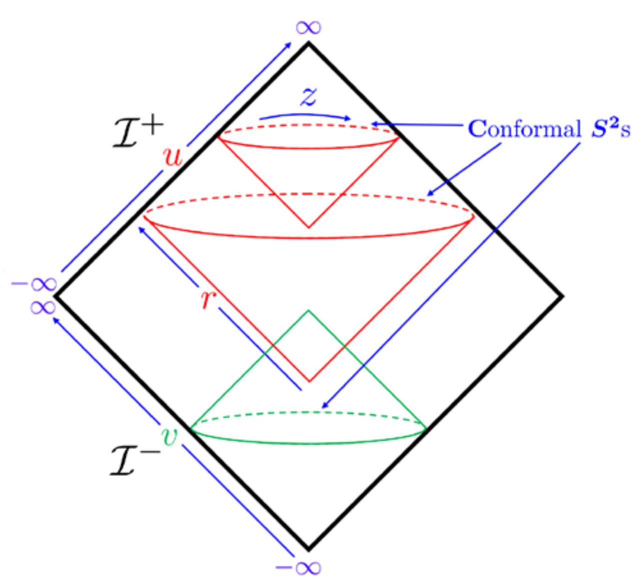
$$\mathcal{M} = \int \frac{\prod_{a=1}^{n} \mathrm{d}^2\sigma_a}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \det{}'(\mathcal{H}) \prod_{i=1}^{k} \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i)) \prod_{p=k+1}^{n} \bar{\delta}^{2|\mathcal{N}}(\lambda_p - \lambda(\sigma_p), \eta_p - \chi(\sigma_p))$$

- Furthermore, these formulae are much simpler than previous twistor string formulae.

BMS Symmetry

- Strominger conjectured that diag(BMS$^+$ x BMS$^-$) is a symmetry of the 4d gravitational S-matrix:

$$\langle out|\, B^+\mathcal{S} - \mathcal{S}B^- \,|in\rangle = 0$$



Soft Limits

- The Ward-identities associated with BMS symmetry correspond to soft graviton theorems:

$$\lim_{k_{n+1}\to 0} \mathcal{M}_{n+1} = \left(S^{(0)} + S^{(1)}\right)\mathcal{M}_n$$

where

$$S^{(0)} = \sum_{a=1}^{n} \frac{(\epsilon \cdot k_a)^2}{s \cdot k_a}$$ ← supertranslations

$$S^{(1)} = \frac{\epsilon_{\mu\nu} k_a^{\mu} s_{\lambda} J_a^{\lambda\nu}}{s \cdot k_a}$$ ← superrotations

Soft Limits and BMS Symmetry

- Ambitwistor string theory makes the relation between BMS symmetry and soft limits completely transparent, and implies an extension to gravity and Yang-Mills theory in arbitrary dimensions!
 (Adamo/Casali/Skinner;Geyer/Lipstein/Mason)

- Key idea: BMS generators correspond to leading and subleading terms in the Taylor expansion of vertex operators

- In 4d, BMS symmetry can be extended to Virasoro symmetry. Hence, there is an ∞ dimensional symmetry lurking in the tree-level S-matrix of pure Yang-Mills theory and Einstein gravity!

Summary

- Complexified massless point-particles can be formulated as ambitwistor string theories.
- 4d ambitwistor string theory gives rise to new tree-level formulae for gauge and gravity amplitudes with any amount of supersymmetry
- Ambitwistor string theory provides new insight into BMS symmetries and their relationship to soft limits

Future Directions

- N=4 super-Yang-Mills:
  - higher loops from dlog form
  - cluster polylogarithms
  - deformation by spectral parameter
  - collinear/Regge limit

- Ambitwistor Strings:
  - loops
  - massive particles
  - ABJM/BLG
  - AdS/dS background

Thank You