

AdS
IIB string theory on $AdS_5 \times S^5$

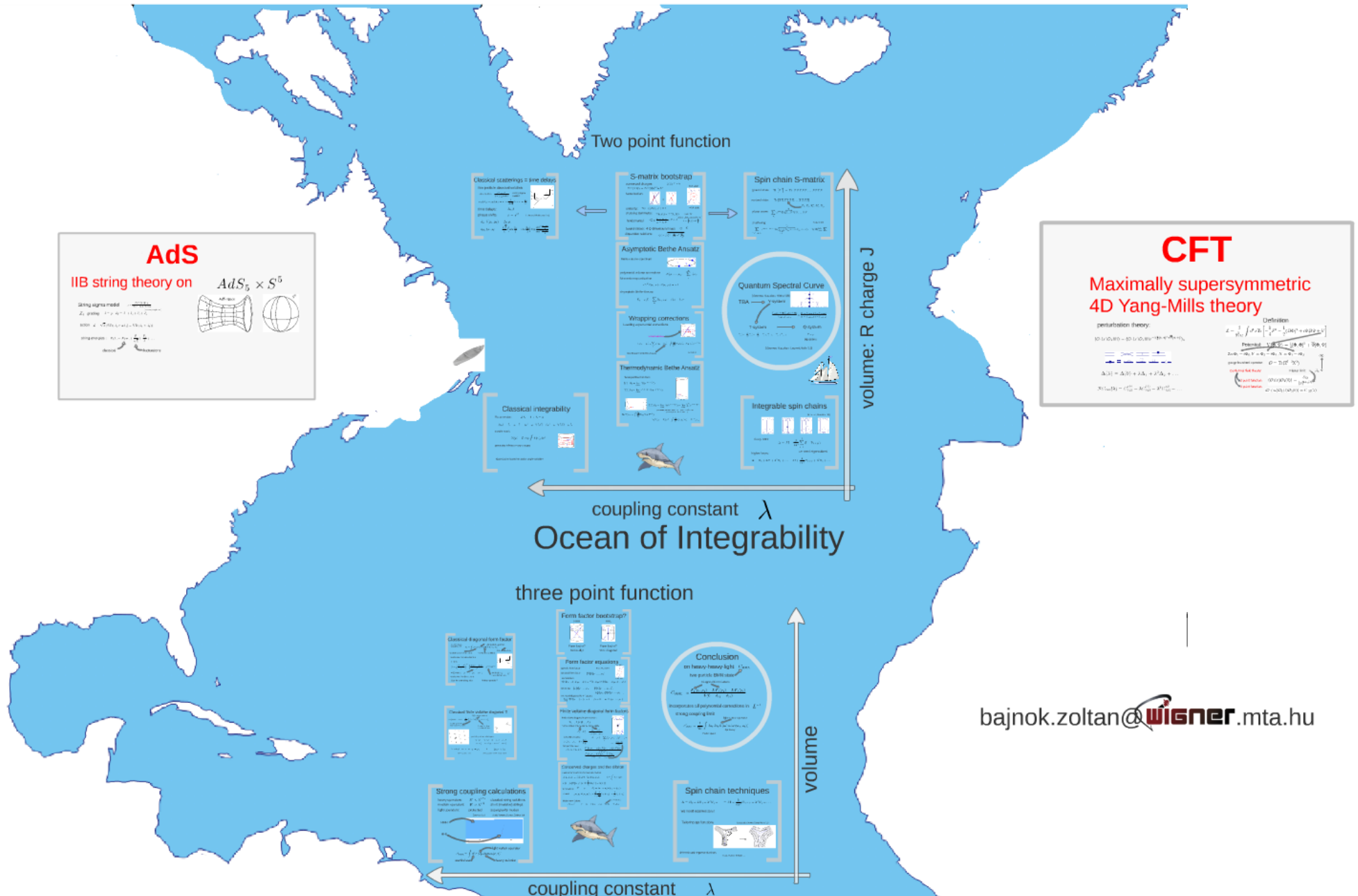
String sigma model
 Z_2 orbifold
 Wilson loops
 Integrability

CFT
Maximally supersymmetric
4D Yang-Mills theory

perturbation theory:
 $g^2 \ln^2 \mu$
 $g^2 \ln \mu$
 g^2
 $g^2 \ln^2 \mu$
 $g^2 \ln \mu$
 g^2

Definition:
 $\mathcal{L} = \int d^4x \text{Tr} \left[-\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi} \not{D} \psi + \dots \right]$
 Perturbative expansion:
 $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \dots$
 Wilson loop:
 $W_C = \text{Tr} \mathcal{P} \exp \left[i \oint_C A_\mu dx^\mu \right]$
 Integrability:
 $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \dots$

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AdS
IIB string theory on $AdS_5 \times S^5$

String sigma model
2. gauge theory
3. integrability

CFT
Maximally supersymmetric
4D Yang-Mills theory

perturbation theory

Definition

Two point function

Classical scattering = time delay

S-matrix bootstrap

Spin chain S-matrix

Asymptotic Bethe Ansatz

Quantum Spectral Curve

Wrapping corrections

Thermodynamic Bethe Ansatz

Classical integrability

Integrable spin chains

volume: R charge J

coupling constant λ

Ocean of Integrability

three point function

Classical diagonal form factor

Fermi factor bootstrap?

Conclusion on heavy-heavy light

Strong coupling calculations

Spin chain techniques

volume

coupling constant λ

CFT

Maximally supersymmetric 4D Yang-Mills theory

perturbation theory:

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(0) \rangle = \langle \mathcal{O}_1(x) \mathcal{O}_2(0) e^{-i(\frac{1}{4}[\Phi, \Phi]^2 + \bar{\Psi}[\Phi, \Psi])} \rangle_0$$



$$\Delta(\lambda) = \Delta(0) + \lambda \Delta_1 + \lambda^2 \Delta_2 + \dots$$

$$NC_{123}(\lambda) = C_{123}^{(0)} + \lambda C_{123}^{(1)} + \lambda^2 C_{123}^{(2)} + \dots$$

Definition

$$\mathcal{L} = \frac{2}{g_{YM}^2} \int d^4x \text{Tr} \left[-\frac{1}{4} F^2 - \frac{1}{2} (D\Phi)^2 + i \bar{\Psi} \not{D} \Psi + V \right]$$

Potential: $V(\Phi, \Psi) = \frac{1}{4} [\Phi, \Phi]^2 + \bar{\Psi}[\Phi, \Psi]$

$$Z = \Phi_5 + i\Phi_6, Y = \Phi_3 + i\Phi_4, X = \Phi_1 + i\Phi_2$$

gauge invariant operator: $\mathcal{O} = \text{Tr}(Z^{J-2} X^2)$

Conformal field theory:

Planar limit: $\lambda = g_{YM}^2 N$

2 point function $\langle \mathcal{O}_1(x) \mathcal{O}_2(0) \rangle = \frac{\delta_{12}}{|x|^{2\Delta_1(\lambda)}}$

3 point function $\langle \mathcal{O}_1(\infty) \mathcal{O}_2(1) \mathcal{O}_3(0) \rangle = C_{123}(\lambda)$

Definition

$$\mathcal{L} = \frac{2}{g_{YM}^2} \int d^4x \text{Tr} \left[-\frac{1}{4} F^2 - \frac{1}{2} (D\Phi)^2 + i\bar{\Psi} \not{D}\Psi + V \right]$$

Potential: $V(\Phi, \Psi) = \frac{1}{4} [\Phi, \Phi]^2 + \bar{\Psi} [\Phi, \Psi]$

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Conformal field theory:

2 point function

3 point function

Planar limit: $\lambda = g_{YM}^2 N$

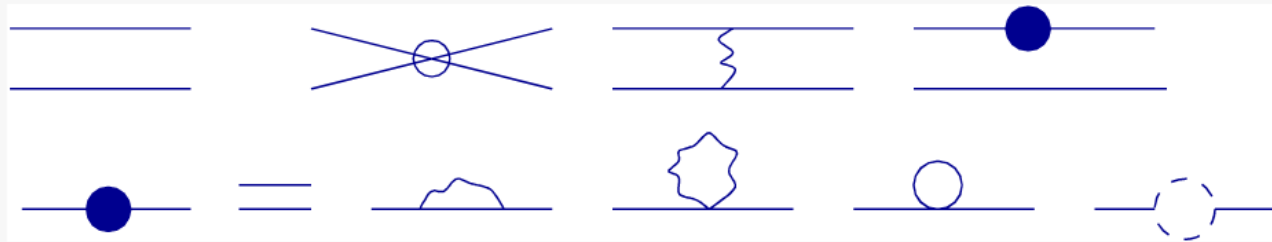
$$\langle \mathcal{O}_1(x) \mathcal{O}_2(0) \rangle = \frac{\delta_{12}}{|x|^{2\Delta_1(\lambda)}}$$

$$\langle \mathcal{O}_1(\infty) \mathcal{O}_2(1) \mathcal{O}_3(0) \rangle = C_{123}(\lambda)$$

∞

perturbation theory:

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(0) \rangle = \langle \mathcal{O}_1(x) \mathcal{O}_2(0) e^{-i(\frac{1}{4}[\Phi, \Phi]^2 + \overline{\Psi}[\Phi, \Psi])} \rangle_0$$



$$\Delta(\lambda) = \Delta(0) + \lambda \Delta_1 + \lambda^2 \Delta_2 + \dots$$

$$N C_{123}(\lambda) = C_{123}^{(0)} + \lambda C_{123}^{(1)} + \lambda^2 C_{123}^{(2)} + \dots$$

AdS

IIB string theory on

$$AdS_5 \times S^5$$

String sigma model $g \in \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$

[Metsaev, Tseytlin '98]

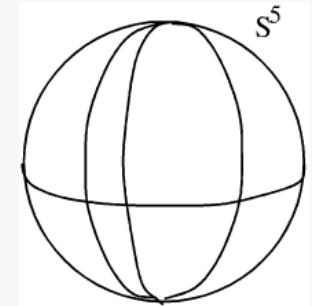
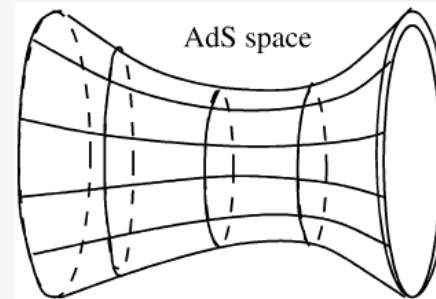
Z_4 grading $J = g^{-1}dg = J_1 + J_2 + J_3 + J_4$

action $\mathcal{L} = \sqrt{\lambda} (\text{STr}(J_2 \wedge *J_2) - \text{STr}(J_1 \wedge J_3))$

string energies : $E(\lambda) = E(\infty) + \frac{E_1}{\sqrt{\lambda}} + \frac{E_2}{\lambda} + \dots$

classical

fluctuations



String sigma model

$$g \in \frac{PSU(2, 2|4)}{SO(4, 1) \times SO(5)}$$

[Metsaev, Tseytlin '98]

$$Z_4 \text{ grading} \quad J = g^{-1} dg = J_1 + J_2 + J_3 + J_4$$

$$\text{action} \quad \mathcal{L} = \sqrt{\lambda} (\text{STr}(J_2 \wedge *J_2) - \text{STr}(J_1 \wedge J_3))$$

$$\text{string energies :} \quad E(\lambda) = E(\infty) + \frac{E_1}{\sqrt{\lambda}} + \frac{E_2}{\lambda} + \dots$$

classical



fluctuations

Two point function

Classical scatterings = time delays

two particle classical solution


one-particle $\frac{v_1 - v_2}{v_1 + v_2}$ (string sigma model)

time delays: $\Delta_{12} t$

phase shifts $S = e^{i\theta}$ (Hirota, Matsumoto '95)

$\partial_{p_1} \delta(p_1, p_2) = \Delta_{12} t$

$\partial_{p_2} \delta(p_1, p_2) = -\frac{\sqrt{5}}{4} (\cos \frac{p_1}{2} - \cos \frac{p_2}{2}) \log \frac{\sin \frac{p_1 + p_2}{2}}{\sin \frac{p_1 - p_2}{2}}$



S-matrix bootstrap

conserved charges: $\Delta(Q), S = 0$

factorization: $R_{12}(u, v) = R_{21}(v, u) R_{11}(u, v)$


unitarity: $S(p_1, v) S(p_2, v) = 1$

crossing symmetry: $R_{12}(u, v) = R_{21}^*(v, u)$

fundamental $R_{12}(u, v) = \frac{u-v}{u-v+1} R_{21}(v, u)$

bound-states: 4 Q dimensional reps $Q \in \mathbb{N}$

dispersion relations: $\epsilon(p) = \sqrt{p^2 - \frac{1}{4}} \cos \frac{p}{2}$



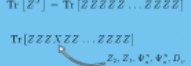
Spin chain S-matrix

ground state: $\text{Tr} [Z^L] = \text{Tr} [ZZZZ \dots ZZZZ]$

excited state: $\text{Tr} [ZZZYZZ \dots ZZZZ]$

plane wave: $\sum_{\alpha} e^{i\alpha x} \text{Tr} [Z_{\alpha} \dots Z_{\alpha} \dots Z]$

scattering: $\sum_{\alpha, \beta, \gamma, \delta} e^{i\alpha x + i\beta y} \text{Tr} [Z_{\alpha} Z_{\beta} Z_{\gamma} Z_{\delta} \dots Z] + S(12) \dots$




Asymptotic Bethe Ansatz

Finite volume spectrum

polynomial volume corrections $E(p_1, \dots, p_n) = \sum_i \epsilon(p_i)$

Momentum quantization $e^{i\sum_j p_j L} = 1$

Asymptotic Bethe Ansatz: $\epsilon_j = p_j L + \sum_i \theta(p_j, p_i) = (2n+1)\pi$




Wrapping corrections

Leading exponential corrections

virtual particles

modification of Bethe Ansatz



Thermodynamic Bethe Ansatz


torus partition function: $Z(L, N) = \frac{1}{L} \text{Tr} \text{Tr} e^{N \log T}$

$Z(L, N) = \int \prod_{\alpha} e^{N \log A_{\alpha}} e^{-L \log B_{\alpha}}$

$Z(L, N) = \int \prod_{\alpha} e^{N \log A_{\alpha}} e^{-L \log B_{\alpha}}$

$E_{\alpha}(L) = - \int \frac{d\mu}{2\pi} \log(1 + Y^{\mu})$

$\log Y^{\mu} = -R(\mu) + \int \frac{d\nu}{2\pi} \log(1 + Y^{\nu}) V^{\mu\nu}$

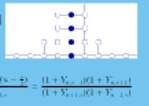


Quantum Spectral Curve

TBA \rightarrow Y-system \rightarrow T-system \rightarrow Q-system

equations

[Gromov, Kazakov, Leurent, Vain '12]



Classical integrability


Poisson connection: $A \wedge -A \wedge A = 0$

$A(p) = J_1 + p^{-1} J_2 + (p^2 + p^{-2}) J_3 + (p^4 + p^{-4}) J_4 + \dots$

transfer matrix $T(p) = \mathcal{P} \exp \int A(p) dx$

generates infinitely many charges

Quantization based on action-angle variable?

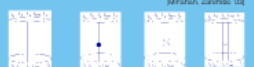


Integrable spin chains

1 loop XXX $\Delta = J L + \frac{1}{8\pi^2} \sum_{k=1}^L (1 - P_{k, k-1})$

Higher loops: we need eigenvalues!

$\Delta = B_0 + \lambda B_1 + \lambda^2 B_2 + \dots + J L + \frac{1}{8\pi^2} B_{XXX} + \lambda^3 B_3 + \dots$



volume: R charge J

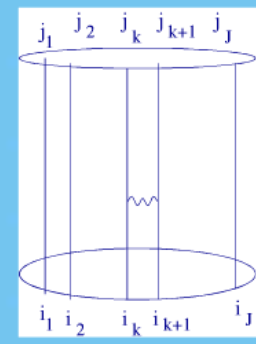
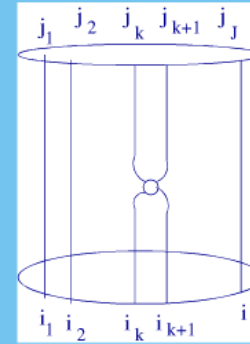
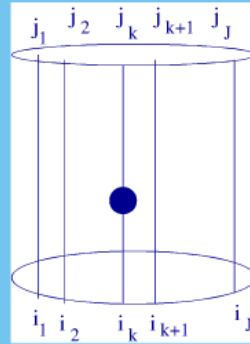
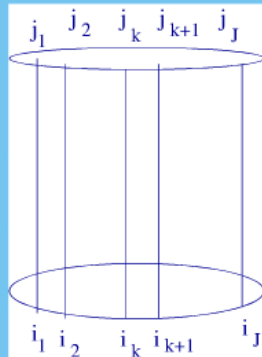
coupling constant λ

Ocean of Integrability



Integrable spin chains

[Minahan, Zarembo '03]



1 loop: XXX

$$\Delta = J \mathbb{I} + \frac{\lambda}{8\pi^2} \sum_{k=1}^J (\mathbb{I} - \mathbb{P}_{k,k+1})$$

higher loops:

we need eigenvalues!

$$\Delta = H_0 + \lambda H_1 + \lambda^2 H_2 + \dots = J \mathbb{I} + \frac{\lambda}{8\pi^2} H_{XXX} + \lambda^2 H_2 + \dots$$

Classical integrability

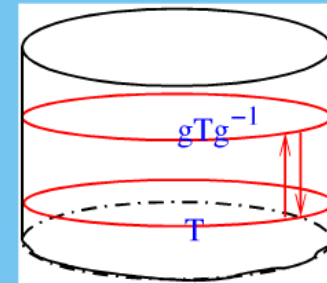
Flat connection: $dA - A \wedge A = 0$

$$A(\mu) = J_0 + \mu^{-1} J_1 + (\mu^2 + \mu^{-2}) J_2 / 2 + (\mu^2 + \mu^{-2}) J_2 / 2 + \mu J_3$$

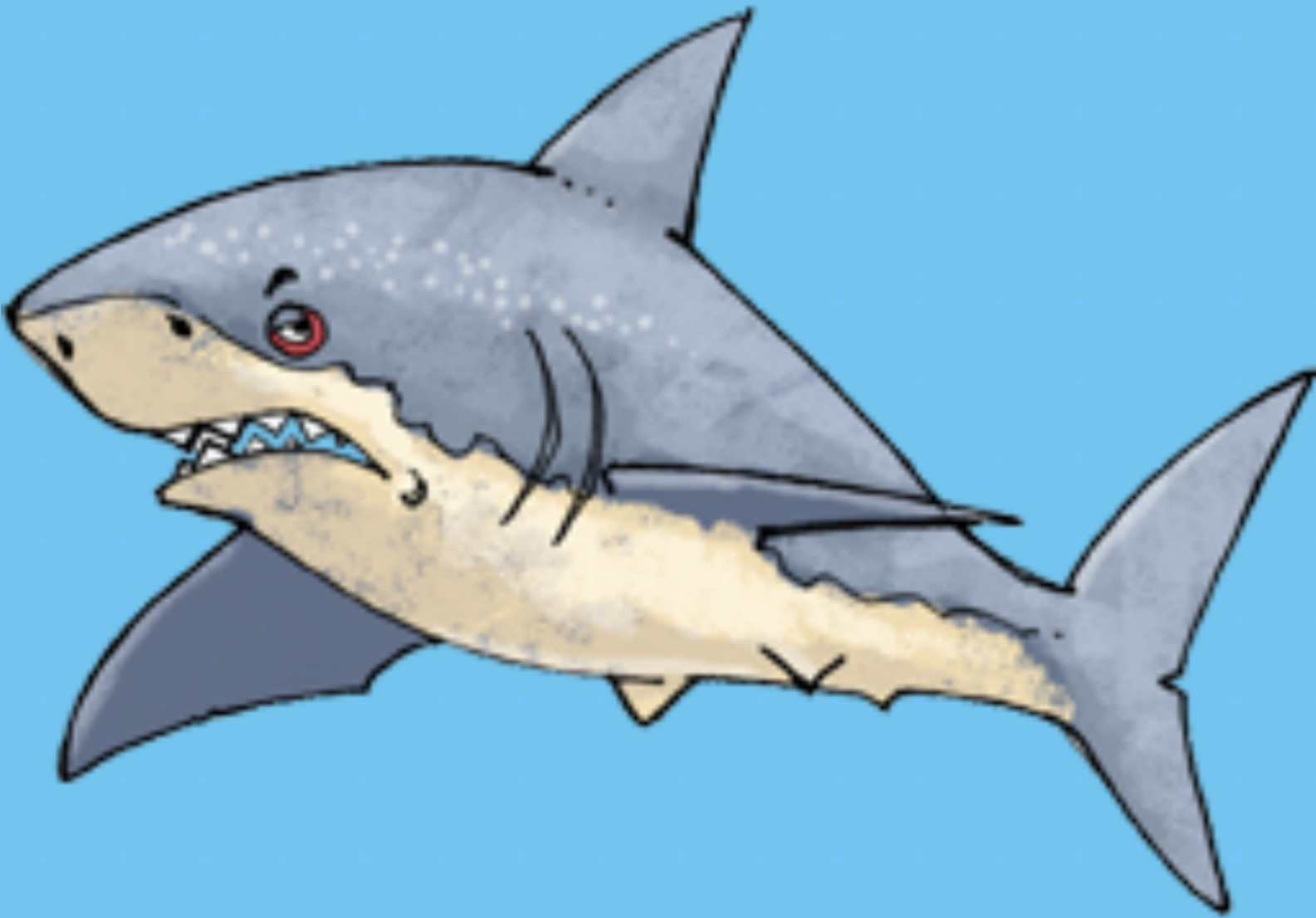
transfer matrix

$$T(\mu) = \mathcal{P} \exp \oint A(x)_\mu dx^\mu$$

generates infinitely many charges



Quantization based on action angle variable?



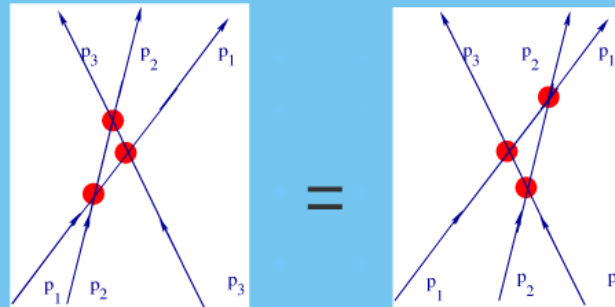
S-matrix bootstrap

conserved charges:

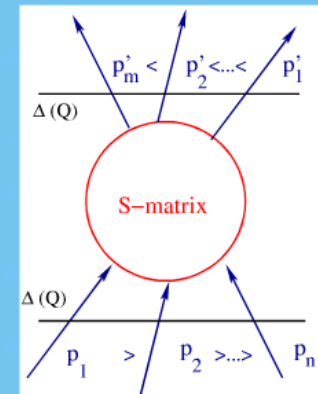
$$[\Delta(Q), S] = 0$$

$$PSU(2, 2|4) \rightarrow PSU(2|2)^{\otimes 2} \ltimes \mathbb{R}^3$$

factorization:



final state



unitarity: $S(p_1, p_2)S(p_2, p_1) = 1$

crossing symmetry:

$$S(p_1, p_2) = S^{c_1}(p_2, \bar{p}_1) \quad [\text{Janik '06}]$$

fundamental

$$S_{11}^{11} = \frac{u_1 - u_2 - i}{u_1 - u_2 + i} e^{i2\theta(p_1, p_2)} \quad [\text{Beisert, Eden, Staudacher '07}]$$

$$u = \frac{1}{2} \epsilon(p) \cot \frac{p}{2}$$

bound-states: 4 Q dimensional reps $Q \in \mathbb{N}$

dispersion relations:


$$\epsilon(p) = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2 \left(\frac{p}{2} \right)}$$

Spin chain S-matrix

ground state $\text{Tr} [Z^J] = \text{Tr} [ZZZZZ \dots ZZZZ]$

excited state: $\text{Tr} [ZZZXZZ \dots ZZZZ]$

$Z_2, Z_3, \Psi_a^\alpha, \Psi_a^{\dot{\alpha}}, D_\mu$



plane wave: $\sum_n e^{ipn} \text{Tr} (\overbrace{Z \dots Z}^n X Z \dots ZZ)$

scattering:

[Beisert '04]

$$\sum_{n_1 n_2 a_1 a_2} e^{ip_1 n_1 + ip_2 n_2} \text{Tr} (\underbrace{Z \dots Z}_{n_1} X_{a_1} \overbrace{Z \dots Z}^{n_2} X_{a_2} Z \dots Z) + S(12)_{a_1 a_2}^{b_1 b_2} \sum$$

Classical scatterings = time delays

two particle classical solution

sine-Gordon $\frac{e^{u_1} + e^{u_2}}{1 - u_{12}e^{u_1+u_2}}$ string sigma model

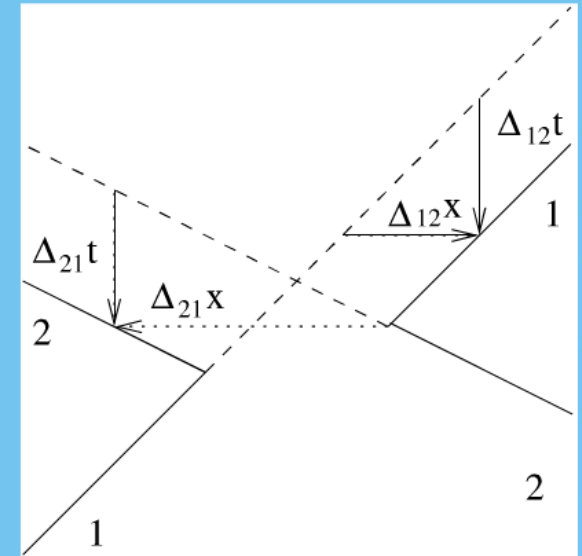
$$\cosh \theta_i x - \sinh \theta_i t = u_i = \frac{1}{\sin \frac{p_i}{2}} \left(\sigma - \tau \cos \frac{p_i}{2} \right)$$

time delays: $\Delta_{12}t$

phase shifts $S = e^{i\delta}$ [Hofman, Maldacena '06]

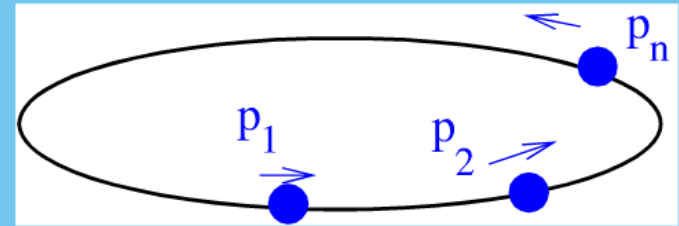
$$\partial_{E_1} \delta(p_1, p_2) = \Delta_{12}t$$

$$\delta_{HM}(p_1, p_2) = -\frac{\sqrt{\lambda}}{\pi} \left(\cos \frac{p_1}{2} - \cos \frac{p_2}{2} \right) \log \frac{\sin^2 \frac{p_1 - p_2}{4}}{\sin^2 \frac{p_1 + p_2}{4}}$$



Asymptotic Bethe Ansatz

Finite volume spectrum



polynomial volume corrections

$$E(p_1, \dots, p_n) = \sum_i^n \epsilon(p_i)$$

Momentum quantization

$$e^{ip_j L} S(p_j, p_1) \dots S(p_j, p_n) = -1$$

Asymptotic Bethe Ansatz:

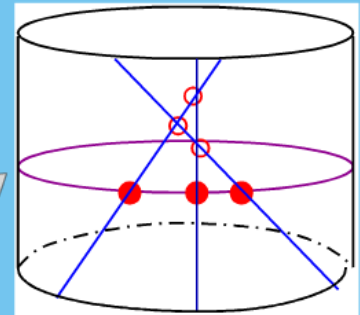
$$\Phi_j = p_j L + \sum_k \delta(p_j, p_k) = (2n + 1)i\pi$$

k

Wrapping corrections

Leading exponential corrections

virtual particles



$$E(p_1, \dots, p_n) = \sum_k E(p_k + \delta p_k) - \int \frac{dq}{2\pi} (S(q, p_1) \dots S(q, p_n) e^{-LE(q)})$$

modification of Bethe Ansatz

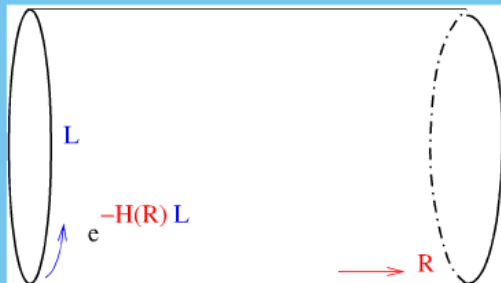
[BZ, Janik '08]

Thermodynamic Bethe Ansatz

Torus partition function:

$$Z(L, R) = \lim_{R \rightarrow \infty} \text{Tr}(e^{-H(L)R})$$

$$Z(L, R) = \lim_{R \rightarrow \infty} e^{-E_0(L)R} (1 + e^{-\Delta E R})$$



$$Z(L, R) = \lim_{R \rightarrow \infty} \text{Tr}(e^{-H(R)L}) = \lim_{R \rightarrow \infty} \sum_n e^{-E_n(L)R}$$

[Bombardelli, Fioravanti, Tateo '09] [Arutyunov, Frolov '09]

[Gromov, Kazakov, Kozak, Vieira '10]

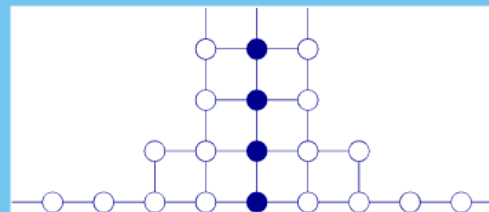
$$E_0(L) = - \int \frac{dp}{2\pi} \log(1 + Y^{-1})$$

$$\log Y(p) = E(p)L + \int \frac{dp'}{2\pi} \delta(p', p) \log(1 + Y^{-1}(p'))$$

Quantum Spectral Curve

[Gromov, Kazakov, Vieira '09]

TBA \longrightarrow Y-system



$$\frac{Y_{a,s}(u + \frac{i}{2})Y_{a,s}(u - \frac{i}{2})}{Y_{a+1,s}Y_{a-1,s}} = \frac{(1 + Y_{a,s-1})(1 + Y_{a,s+1})}{(1 + Y_{a+1,s})(1 + Y_{a-1,s})}$$

T-system



Q-system

$$T_{a,s}(u + \frac{i}{2})T_{a,s}(u - \frac{i}{2}) = T_{a-1,s}T_{a+1,s} + T_{a,s-1}T_{a,s+1}$$

$P - \mu$
equations

[Gromov, Kazakov, Leurent, Volin '12]



Two point function

Classical scatterings = time delays

two particle classical solution


one-particle $\frac{v_1 - v_2}{v_1 + v_2}$ (string sigma model)

time delays: $\Delta_{12} t$

phase shifts $S = e^{i\theta}$ (Hirota, Matsumoto '95)

$\partial_{p_1} \delta(p_1, p_2) = \Delta_{12} t$

$\partial_{p_2} \delta(p_1, p_2) = -\frac{\sqrt{5}}{4} (\cos \frac{p_1}{2} - \cos \frac{p_2}{2}) \log \frac{\sin \frac{p_1 + p_2}{2}}{\sin \frac{p_1 - p_2}{2}}$



S-matrix bootstrap

conserved charges: $\Delta(Q), S = 0$

factorization: $R_{12}(z) = R_{21}(z) R_{11}(z) R_{22}(z)$


unitarity: $S(p_1, v) S(p_2, v) = 1$

crossing symmetry: $R_{12}(z) = R_{21}^*(z)$

fundamental $R_{12}(z) = \frac{z - z_1}{z - z_2} \frac{z - z_3}{z - z_4}$

bound-states: 4 Q dimensional reps $Q \in \mathbb{N}$

dispersion relations: $\langle p \rangle = \sqrt{\psi^2 - \frac{1}{4}} \cos \left(\frac{p}{2} \right)$



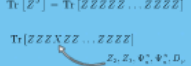
Spin chain S-matrix

ground state: $\text{Tr} [Z^L] = \text{Tr} [ZZZZ \dots ZZZZ]$

excited state: $\text{Tr} [ZZZYZZ \dots ZZZZ]$

plane wave: $\sum_{z_1, z_2, \dots, z_N} e^{i p z} \text{Tr} (Z_{z_1} \dots Z_{z_N})$

scattering: $\sum_{z_1, z_2, \dots, z_N} e^{i p z} \text{Tr} (Z_{z_1} \dots Z_{z_N})$




Asymptotic Bethe Ansatz

Finite volume spectrum

polynomial volume corrections $E(p_1, \dots, p_n) = \sum_i \epsilon_i(p_i)$

Momentum quantization $e^{i p L} S(p_1, p_2, \dots) = -1$

Asymptotic Bethe Ansatz: $\epsilon_j = p_j L + \sum_i \epsilon_i(p_i, p_j) = (2n+1)p_j$




Wrapping corrections

Leading exponential corrections

virtual particles

modification of Bethe Ansatz




Thermodynamic Bethe Ansatz

torus partition function: $Z(L, N) = \frac{1}{L} \text{Tr} \text{Tr} e^{N H}$

$Z(L, N) = \int \prod_{\alpha} e^{-N \epsilon_{\alpha}} \prod_{\alpha} (1 + e^{-\epsilon_{\alpha}})$

$\epsilon_{\alpha}(k) = - \int \frac{d\omega}{2\pi} \log(1 + Y^{\omega})$

$\log Y^{\omega} = - \log(1 + \int \frac{d\omega'}{2\pi} \log(1 + Y^{\omega'})) + Y^{\omega}$

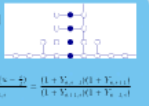


Quantum Spectral Curve

TBA \rightarrow Y-system \rightarrow T-system \rightarrow Q-system

equations: $Z_{\alpha}(k) + Z_{\alpha}(k + \frac{1}{2}) - Z_{\alpha}(k - \frac{1}{2}) - Z_{\alpha}(k + 1) + Z_{\alpha}(k - 1) = 0$

[Gromov, Kazakov, Leurent, Vain '12]



Classical integrability


Poisson connection: $A \wedge - A \wedge A = 0$

$A(p) = J_1 + p^{-1} J_2 + (p^2 + p^{-2}) J_3 + (p^4 + p^{-4}) J_4 + \dots$

transfer matrix $T(p) = \mathcal{P} \exp \int A(p) dx$

generates infinitely many charges

Quantization based on action-angle variable?

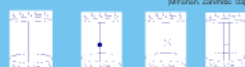


Integrable spin chains

1 loop XXX $\Delta = J L + \frac{1}{8\pi^2} \sum_{k=1}^L (1 - P_{k, k-1})$

Higher loops: we need eigenvalues!

$\Delta = B_0 + N B_1 + N^2 B_2 + \dots + J L + \frac{1}{8\pi^2} B_{XXX} + N^3 B_3 + \dots$



volume: R charge J

coupling constant λ

Ocean of Integrability

three point function

Classical diagonal form factor

Conjecture: $F_2 = \int d^2u_1 d^2u_2 \mathcal{V}[\text{solution}(u_1, u_2)]$ light vertex operator
 moduli space of solution heavy 2pt solution 1pt subtraction
 works well for sine-Gordon dilaton

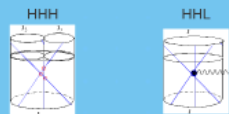
$$V = \left(\frac{t^2 + z^2}{z} \right)^{-1} \left[\frac{\partial \bar{\partial} \phi + \partial z \bar{\partial} z}{z^2} + \partial X^{\mu} \bar{\partial} X^{\mu} \right]$$

 ADS metric $ds^2 = z^{-2}(dt^2 + dx^2)$ embedding coordinates on S^5
 Use for something else! Vertex operator?

Classical finite volume diagonal ff

Conjecture: $F_{\text{vol}} = \frac{1}{\text{Vol}} \int d^2u_1 d^2u_2 \mathcal{V}[\text{solution}(u_1, u_2)]$ light vertex operator
 moduli space of finite volume solution heavy 2pt finite volume solution
 periodicity in the moduli space:
 $(u_1, u_2) \rightarrow (u_1, u_2) + (L, \Delta_{12})$
 $(u_1, u_2) \rightarrow (u_1, u_2) + (-\Delta_{12}, L)$
 $\text{Vol} = L(L - \Delta_{12} - \Delta_{21}) = p_2$ if $-\Delta_{12} \mp \Delta_{21}$
 sine-Gordon: OK strong sigma model: non-normalize!

Form factor bootstrap?



Form factor? Not really!
 Form factor? Yes: diagonal

Form factor equations

generic form factor (Kronecker vs. Dirac delta)
 crossed form factor $\langle 0 | \mathcal{V} | p_1, \dots, p_n \rangle$
 permutation $\langle 0 | \mathcal{V} | p_1, \dots, p_n \rangle = S(p_1, p_2, \dots, p_n) \langle 0 | \mathcal{V} | p_1, \dots, p_n \rangle$
 crossing $\langle p | \mathcal{V} | p_1, \dots, p_n \rangle = \langle 0 | \mathcal{V} | p_1, \dots, p_n, \bar{p} \rangle + \langle p | p_n \rangle \langle 0 | \mathcal{V} | p_1, \dots, p_{n-1} \rangle$
 we need diagonal form factors:
 $\lim_{z \rightarrow 0} \langle 0 | \mathcal{V} | p_1, \dots, p_n, \bar{p}_n + \epsilon, \dots, \bar{p}_1 + \epsilon \rangle = F_n(p_1, \dots, p_n)$

Finite volume diagonal form factors

finite volume diagonal matrix element
 $\langle I_1, \dots, I_n | \mathcal{V} | I_1, \dots, I_n \rangle$
 normalization: Kronecker vs. Dirac delta
 $\langle I_1, \dots, I_n | I_n \rangle = \frac{\langle I_1, \dots, I_n \rangle}{\sqrt{\langle I_1, \dots, I_n | I_n \rangle}}$
 density of states: $\phi_j = p_j L + \sum_{i=1}^n \delta_{ij} p_i = (2j + 1)\pi x$
 $\langle p_1, \dots, p_n | \dots \rangle = \det \left[\frac{\partial \phi_j}{\partial p_i} \right]_{i,j=1}^n$
 two particle case
 $\langle p_1, p_2 | \mathcal{V} | p_1, p_2 \rangle = \frac{F_2(p_1, p_2) + p_1(p_1) F_1(p_2) + p_2(p_2) F_1(p_1)}{L(L + \phi_{12} + \phi_{21})}$

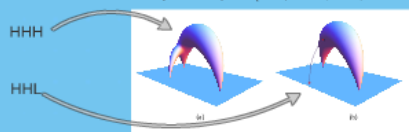
Conserved charges and the dilaton

Diagonal form factors of conserved charges
 $Q(p_1, p_1)_L = (o(p_1) + o(p_1)) | p_1, p_1 \rangle_L \quad Q = \int_V \psi \epsilon \cdot \partial \phi$
 $\langle p_1, p_2 | \mathcal{V} | p_1, p_2 \rangle_L = \frac{1}{L} (o(p_1) + o(p_2))$
 form factors: $F_1 = o_1 \quad F_2 = (o_1 + o_2)(\phi_{12} + \phi_{21})$
 Dilaton: $\langle p_1, p_2 | \mathcal{D} | p_1, p_2 \rangle_L = \frac{1}{L} \frac{d}{d\tau} (\epsilon_1 + \epsilon_2) = \frac{1}{L} (\epsilon_1' + \epsilon_2')$
 dilaton form factors:
 $F_1 = \epsilon' \quad F_2 = (\epsilon_1' + \epsilon_2')(\phi_{12} + \phi_{21}) + \psi_{12} \partial_{\tau_1} \psi_2 + \psi_{21} \partial_{\tau_2} \psi_1$



Strong coupling calculations

heavy operators: $E \propto \lambda^{1/2}$ classical string solutions
 medium operators: $E \propto \lambda^{1/4}$ short (massive) strings
 light operators: protected supergravity modes
 [Zarembo '10] [Costa, Monteiro, Sams, Zoccarato '10]



$C_{HHL} = \int d^2\sigma \mathcal{V}[\text{solution}(\sigma, \tau)]$
 light vertex operator
 worldsheat heavy solution

Conclusion

on heavy-heavy-light C_{HHL}
 two particle BMN state

Diagonal form factors

$$C_{HHL} = \frac{F_2(p_1, p_2) + L F_1(p_2) + L F_1(p_1)}{L(L + \phi_{12} + \phi_{21})}$$

incorporates all polynomial corrections in L^{-1}
 strong coupling limit

light vertex operator

$$C_{HHL} = \frac{1}{\text{Vol}} \int d^2u_1 d^2u_2 \mathcal{V}[\text{solution}(u_1, u_2)]$$

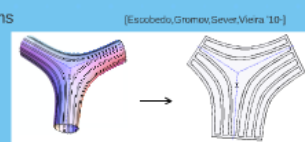
 moduli space 2pt heavy

Spin chain techniques

$$\Delta = H_0 + \lambda H_1 + \lambda^2 H_2 + \dots = J \square + \frac{\lambda}{8\pi^2} H_{XX} + \lambda^2 H_2 + \dots$$

we need eigenvectors!

Tailoring 3pt functions



determinant representations

[Foda, Kostov, Serban, ...]

volume

coupling constant λ

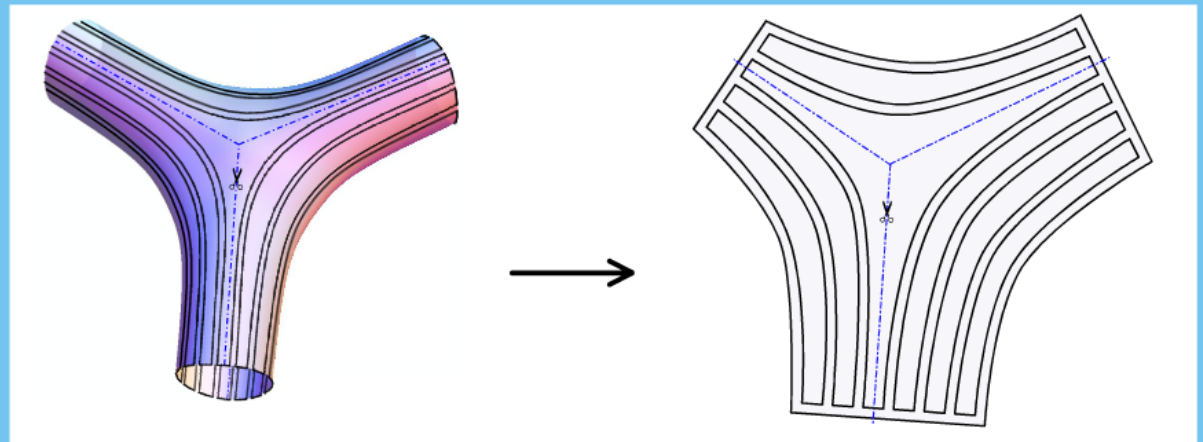
Spin chain techniques

$$\Delta = H_0 + \lambda H_1 + \lambda^2 H_2 + \dots = J \mathbb{I} + \frac{\lambda}{8\pi^2} H_{XXX} + \lambda^2 H_2 + \dots$$

we need eigenvectors!

Tailoring 3pt functions

[Escobedo, Gromov, Sever, Vieira '10-]



determinant representations

[Foda, Kostov, Serban, ...]

Strong coupling calculations

heavy operators: $E \propto \lambda^{1/2}$

medium operators: $E \propto \lambda^{1/4}$

light operators: protected

classical string solutions

short (massive) strings

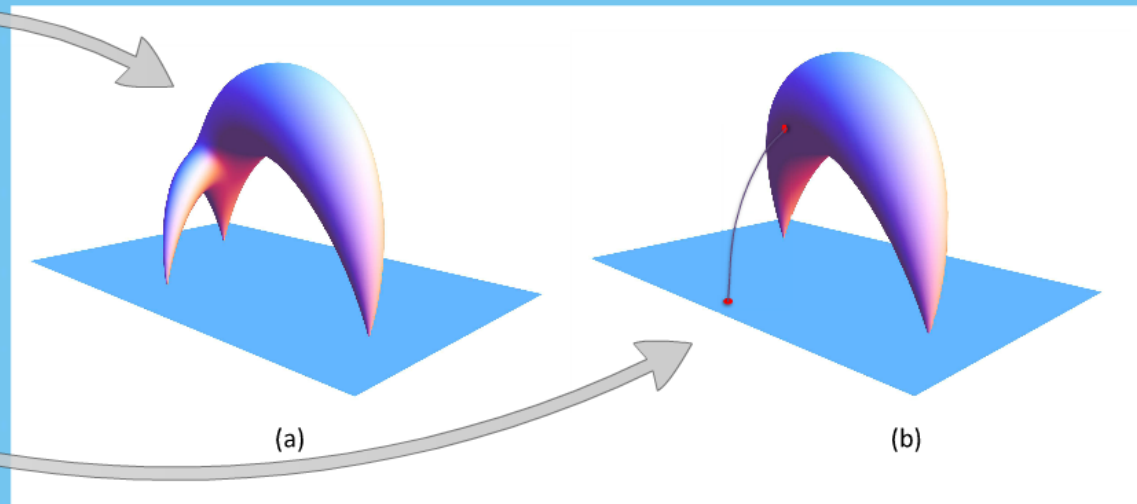
supergravity modes

[Zarembo '10]

[Costa, Monteiro, Santos, Zoakos '10]

HHH

HHL



$$C_{HHL} = \int d^2\sigma \mathcal{V}[\text{solution}(\sigma, \tau)]$$

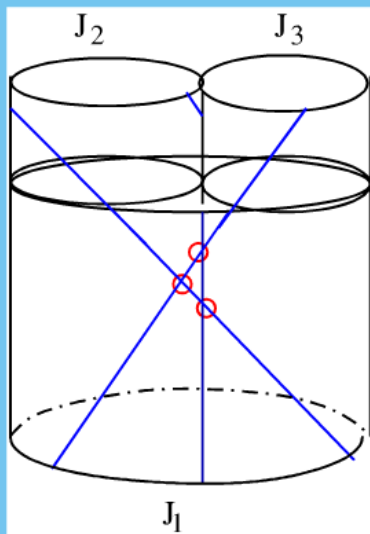
worldsheet

light vertex operator

heavy solution

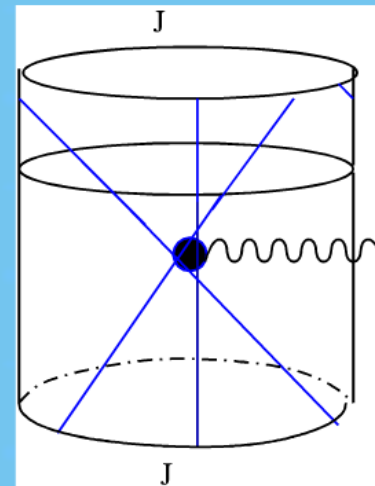
Form factor bootstrap?

HHH



Form factor?
Not really!

HHL



Form factor?
Yes: diagonal

Form factor equations

generic form factor

[Klose, McLoughlin]

crossed form factor

$$\langle 0 | \mathcal{V} | p_1, \dots, p_n \rangle$$

permutation

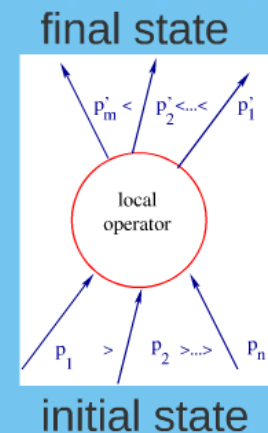
$$\langle 0 | \mathcal{V} | p_1, \dots, p_i, p_{i+1}, \dots, p_n \rangle = S(p_i, p_{i+1}) \langle 0 | \mathcal{V} | p_1, \dots, p_{i+1}, p_i, \dots, p_n \rangle$$

crossing

$$\begin{aligned} \langle p | \mathcal{V} | p_1, \dots, p_n \rangle &= \langle 0 | \mathcal{V} | p_1, \dots, p_n, \bar{p} \rangle \\ &+ \langle p | p_n \rangle \langle 0 | \mathcal{V} | p_1, \dots, p_{n-1} \rangle \end{aligned}$$

we need diagonal form factors:

$$\lim_{\epsilon \rightarrow 0} \langle 0 | \mathcal{V} | p_1, \dots, p_n, \bar{p}_n + \epsilon, \dots, \bar{p}_1 + \epsilon \rangle = F_n(p_1, \dots, p_n)$$



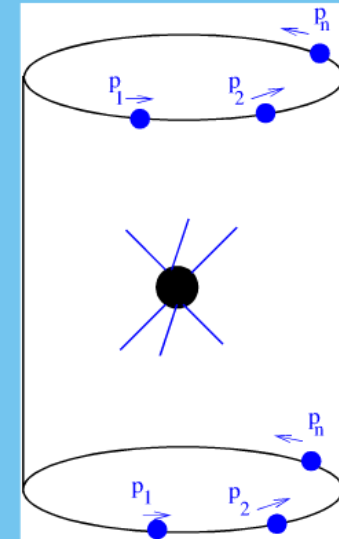
Finite volume diagonal form factors

finite volume diagonal matrix element

$$\langle I_1, \dots, I_n | \mathcal{V} | I_1, \dots, I_n \rangle$$

normalization: Kronecker vs. Dirac delta

$$|I_1, \dots, I_n \rangle = \frac{|p_1, \dots, p_n \rangle}{\sqrt{\rho_n(p_1, \dots, p_n)}}$$



density of states:

$$\rho_n(p_1, \dots, p_n) = \det \left[\frac{\partial \Phi_j}{\partial p_i} \right]$$

$$\Phi_j = p_j L + \sum_k \delta(p_j, p_k) = (2I_j + 1)i\pi$$

$$\phi_{kl} = \frac{\partial \delta(p_k, p_l)}{\partial p_k}$$

two particle case

$$\begin{aligned} {}_L \langle p_2, p_1 | \mathcal{V} | p_1, p_2 \rangle_L &= \frac{F_2(p_1, p_2) + \rho_1(p_1)F_1(p_2) + \rho_1(p_2)F_1(p_1)}{\rho_2(p_1, p_2)} \\ &= \frac{F_2(p_1, p_2) + L F_1(p_2) + L F_1(p_1)}{L(L + \phi_{12} + \phi_{21})} \end{aligned}$$

Conserved charges and the dilaton

Diagonal form factors of conserved charges

$$Q|p_2, p_1\rangle_L = (o(p_1) + o(p_2))|p_2, p_1\rangle_L \quad Q = \int_0^L \mathcal{V}(x, t) dx$$

$${}_L\langle p_1, p_2 | \mathcal{V} | p_2, p_1 \rangle_L = \frac{1}{L} (o(p_1) + o(p_1))$$

form factors: $F_1 = o_1 \quad F_2 = (o_1 + o_2)(\phi_{12} + \phi_{21})$

Dilaton: ${}_L\langle p_1, p_2 | \mathcal{D} | p_2, p_1 \rangle_L = \frac{1}{L} \frac{d}{d\frac{\sqrt{\lambda}}{\pi}} (\epsilon_1 + \epsilon_2) = \frac{1}{L} (\epsilon'_1 + \epsilon'_2)$

dilaton form factors:

$$F_1 = \epsilon' \quad F_2 = (\epsilon'_1 + \epsilon'_2)(\phi_{12} + \phi_{21}) + \psi_{12} \partial_p \epsilon_2 + \psi_{21} \partial_p \epsilon_1$$

$\psi_{ij} = \delta(p_i, p_j)'$

Classical diagonal form factor

[ZB, Janik, Wereszczynski]

Conjecture:

$$F_2 = \int du_1 du_2 \mathcal{V}[\text{solution}(u_1, u_2)] - \dots$$

moduli space of solution

heavy 2pt solution

1pt subtraction

works well for sine-Gordon

dilaton

$$\mathcal{V} = \left(\frac{x^2 + z^2}{z} \right)^{-4} \left[\frac{\partial x \bar{\partial} x + \partial z \bar{\partial} z}{z^2} + \partial X^K \bar{\partial} X^K \right]$$

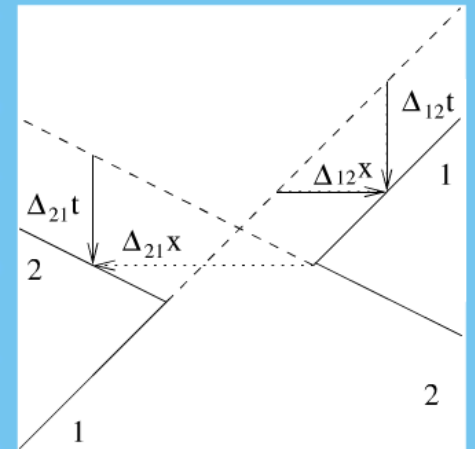
AdS metric $ds^2 = z^{-2}(dz^2 + dx^2)$

works well for the dilaton

embedding coordinates on S^5

Use for something else!

Vertex operator?



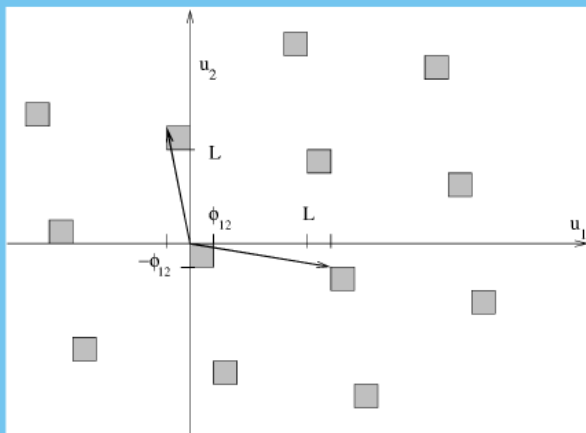
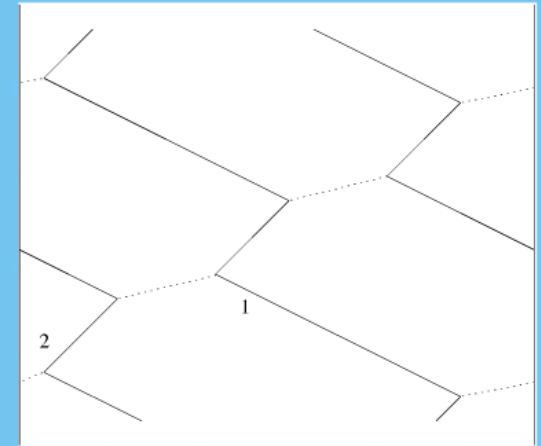
Classical finite volume diagonal ff

Conjecture: $C_{HHL} = \frac{1}{\text{Vol}} \int du_1 du_2 \mathcal{V}[\text{solution}(u_1, u_2)]$

moduli space of finite volume solution

light vertex operator

heavy 2pt finite volume solution



periodicity in the moduli space:

$$(u_1, u_2) \rightarrow (u_1, u_2) + (L - \Delta_{12}x, -\Delta_{21}x)$$

$$(u_1, u_2) \rightarrow (u_1, u_2) + (-\Delta_{12}x, L - \Delta_{21}x)$$

$$\text{Vol} = L(L - \Delta_{12}x - \Delta_{21}x) = \rho_2 \quad \text{if} \quad -\Delta_{12}x \leftrightarrow \phi_{12}$$

sine-Gordon : OK

string sigma model: renormalize L

Conclusion

on heavy-heavy-light C_{HHL}
two particle BMN state

Diagonal form factors

$$C_{HHL} = \frac{F_2(p_1, p_2) + L F_1(p_2) + L F_1(p_1)}{L(L + \phi_{12} + \phi_{21})}$$

incorporates all polynomial corrections in L^{-1}
strong coupling limit

light vertex operator

$$C_{HHL} = \frac{1}{\text{Vol}} \int_{\text{moduli space}} du_1 du_2 \mathcal{V}[\text{solution}(u_1, u_2)]_{\text{2pt heavy}}$$