



NON-PERTURBATIVE DYSON-SCHWINGER EQUATIONS AND INTEGRABILITY

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Integrability in Gauge & String Theory





DYSON-SCHWINGER EQUATIONS

INVARIANCE OF (PATH) INTEGRAL

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{1}{Z} \int_{\Gamma} D\Phi e^{-\frac{1}{\hbar} S[\Phi]} \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n)$$

UNDER "SMALL" DEFORMATIONS
OF THE INTEGRATION CONTOUR

$$\Phi \longrightarrow \Phi + \delta\Phi$$





DYSON-SCHWINGER EQUATIONS

QUANTUM EQUATIONS OF MOTION

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \delta S[\Phi] \rangle = \hbar \sum_{i=1}^n \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_{i-1}(x_{i-1}) \delta \mathcal{O}_i(x_i) \mathcal{O}_{i+1}(x_{i+1}) \dots \mathcal{O}_n(x_n) \rangle$$





DYSON-SCHWINGER EQUATIONS

WITH SOME LUCK

=

GOOD CHOICE OF (POSSIBLY NON-LOCAL) OBSERVABLES

$\mathcal{O}_i(x)$

AND IN SOME LIMIT (CLASSICAL, PLANAR, ...)

THE DS EQUATIONS FORM A CLOSED SYSTEM





FOR EXAMPLE

$$\hbar \longrightarrow 0$$

CLASSICAL LIMIT

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \delta S[\Phi] \rangle = \hbar (\dots) \rightarrow 0$$

$$\Leftrightarrow \delta S[\Phi] = 0$$





GAUGE THEORY

$$\Phi \longrightarrow A = A_\mu dx^\mu \in \text{Lie}U(N)$$

$$\frac{1}{\hbar}S[\Phi] \longrightarrow S_{YM}[A] = -\frac{1}{4g^2} \int_{\mathbb{R}^4} \text{tr} F_A \wedge \star F_A$$

$$\mathcal{O}_i(x_i) \longrightarrow W_R(\gamma) = \text{tr}_R P \exp \oint_\gamma A$$

$$\mathcal{W}(\gamma) = \frac{1}{N} \langle W_\square(\gamma) \rangle$$





GAUGE THEORY: PLANAR LIMIT

$$N \longrightarrow \infty, \quad g^2 \rightarrow 0,$$

$$\text{FINITE} \quad \lambda = g^2 N$$

$$\begin{aligned} \Delta_\gamma \mathcal{W}(\gamma) &= \frac{g^2}{N} \langle W_\square(\gamma) \delta S_{YM}[A] \rangle = \\ &= \lambda \delta_{\gamma=\gamma_1 \star \gamma_2} \mathcal{W}(\gamma_1) \mathcal{W}(\gamma_2) + \frac{1}{N^2} \text{correctons} \end{aligned}$$

MAKEENKO-MIGDAL LOOP EQUATIONS





GAUGE THEORY: MATRIX MODEL

$$\Phi \in \text{Lie}U(N)$$

$$\frac{1}{\hbar}S[\Phi] = \frac{1}{\hbar}\text{tr}V(\Phi)$$

$$V(X) = v_p X^p + v_{p-1} X^{p-1} + \dots + v_1 X + v_0$$

$$\mathcal{O}(x) = \frac{1}{N}\text{tr}\square\left(\frac{1}{x - \Phi}\right)$$





MATRIX MODEL

PLANAR LIMIT: $\lambda = \hbar N$ FIXED

$$\hbar \rightarrow 0, N \rightarrow \infty$$

DS EQUATIONS \implies LOOP EQUATIONS

$$y(x)^2 = V'(x)^2 + g_{p-2}(x)$$

$$y(x) = \langle \mathcal{O}(x) \rangle + V'(x)$$

$g_{p-2}(x) =$ DEGREE $p - 2$ POLYNOMIAL IN x





QFT PATH INTEGRAL INVOLVES SUMMATION OVER TOPOLOGICAL SECTORS





FOR EXAMPLE, IN GAUGE THEORY

$$Z = \sum_{n \in \mathbb{Z}} e^{in\theta} \int_{\mathcal{A}_n} \left[\frac{DA}{\text{Vol}(\mathcal{G}_n)} \right] e^{-S_{\text{YM}}[A]}$$
$$-\frac{1}{8\pi^2} \int \text{tr} F_A \wedge F_A = n, \quad A \in \mathcal{A}_n$$





NON-PERTURBATIVE DS EQUATIONS

IDENTITIES DERIVED BY

LARGE “DEFORMATIONS” OF THE PATH INTEGRAL CONTOUR

$$A \in \mathcal{A}_n \longrightarrow A + \delta A \in \mathcal{A}_{n+1}$$

GRAFTING A POINT-LIKE INSTANTON





COMPATIBILITY OF PERTURBATIVE

expansion in \hbar, g^2, \dots

AND NON-PERTURBATIVE CONTRIBUTIONS

expansion in $e^{-\frac{1}{\hbar}}, e^{-\frac{1}{g^2}}, \dots$

Resurgence, trans-series, ...

A.Voros, J.Zinn-Justin, ...

Exact β -functions in SYM,

Novikov-Shifman-Vainshtein, Zakharov, 1983





TESTING GROUNDS





SUPERSYMMETRIC

GAUGE THEORIES

SIGMA MODELS





$\mathcal{N} = 2$ **SUPERSYMMETRIC**

FOUR DIMENSIONAL GAUGE THEORIES

TWO DIMENSIONAL SIGMA MODELS





$\mathcal{N} = 2$ SUPERSYMMETRIC GAUGE THEORIES

VECTOR MULTIPLETS

$\mathbf{A} = \begin{matrix} \phi \\ \lambda \\ A \end{matrix}$ *complex adjoint scalar*
two Weyl adjoint fermions
gauge field

adjoint = $Lie(G)$





$\mathcal{N} = 2$ SUPERSYMMETRIC GAUGE THEORIES

HYPERMULTIPLETS

in representation $R \in \text{Rep}(G)$

$\mathbf{M} = M$ ψ \tilde{M} *two complex scalars, in R and R^**
 $\tilde{\psi}^\dagger$ *two Weyl fermions, in R and R^**





CLASS OF SPECIFIC GAUGE THEORIES





$\mathcal{N} = 2$ SUPERSYMMETRIC GAUGE THEORIES

QUIVER THEORIES

$$G = U(N_1) \times \dots \times U(N_r)$$

$$R = \bigoplus_{\langle ij \rangle} (N_i, \bar{N}_j) \bigoplus_i (N_i, \bar{M}_i)$$





$\mathcal{N} = 2$ SUPERSYMMETRIC QUIVER THEORIES

MATTER IN FUNDAMENTAL

$$R \ni (N_i, \bar{M}_i), \quad M_i = \text{multiplicity space}$$

MATTER IN BI-FUNDAMENTAL

$$R \ni (N_i, \bar{N}_j), \quad i \neq j$$

MATTER IN ADJOINT

$$R \ni (N_i, \bar{N}_i)$$





$\mathcal{N} = 2$ QUIVER THEORIES

DATA

$\gamma =$ oriented graph, $s, t : \text{Edge}_\gamma \rightarrow \text{Vert}_\gamma$



$$G = \times_{i \in \text{Vert}_\gamma} U(N_i)$$

$$R = \bigoplus_{e \in \text{Edge}_\gamma} \text{Hom}(N_{s(e)}, N_{t(e)}) \bigoplus_{i \in \text{Vert}_\gamma} \text{Hom}(M_i, N_i)$$





$\mathcal{N} = 2$ QUIVER THEORIES

PARAMETERS

MASSES

$$\mathfrak{M}_i \in \text{End}(M_i), \quad i \in \text{Vert}_\gamma$$

$$m_e \in \mathbb{C}, \quad e \in \text{Edge}_\gamma$$





$\mathcal{N} = 2$ QUIVER THEORIES

PARAMETERS

GAUGE COUPLINGS

$$\tau_i = \frac{\vartheta_i}{2\pi} + \frac{4\pi i}{g_i^2}$$

AMPLITUDE OF INSTANTON IN THE $U(N_i)$

$$q_i = \exp 2\pi i \tau_i, \quad i \in \text{Vert}_\gamma$$





$\mathcal{N} = 2$ QUIVER THEORIES

$\mathcal{N} = 2$ SUPERCONFORMAL FIXED POINT

IN THE UV

iff $\gamma = A, D, E$ or $\hat{A}, \hat{D}, \hat{E}$





$\mathcal{N} = 2$ ASYMPTOTICALLY CONFORMAL QUIVER THEORIES

$$\gamma = A, D, E \quad \text{or} \quad \widehat{A}, \widehat{D}, \widehat{E}$$

QUIVER GROUP G_γ

SIMPLE LIE (KAC-MOODY) GROUP

! NOT AN APPARENT SYMMETRY OF THE THEORY !





WE SHALL NOW SEE THAT

G_γ

MORE PRECISELY ITS QUANTUM DEFORMATION

$Y(\mathfrak{g}_\gamma), U_q(\widehat{\mathfrak{g}}_\gamma), \dots$

IS PRESENT AS A SYMMETRY PRINCIPLE
ORGANIZING THE DYSON-SCHWINGER EQUATIONS





$\mathcal{N} = 2$ QUIVER THEORIES

SUBJECT TO DEFORMATIONS:

FOR BETTER CONTROL





$\mathcal{N} = 2$ QUIVER THEORIES

Ω -DEFORMATION:

$$\Phi \longrightarrow \Phi + v^\mu \nabla_\mu$$

$$\nabla_\mu = \partial_\mu + A_\mu$$

$$v^\mu \partial_m = \epsilon_1 (x^2 \partial_{x^1} - x^1 \partial_{x^2}) + \epsilon_2 (x^4 \partial_{x^3} - x^3 \partial_{x^4})$$





$\mathcal{N} = 2$ QUIVER THEORIES

Ω -DEFORMATION:

Generator \mathcal{R}_3 of the $SU(2)$ R-symmetry group is shifted

$$\mathcal{R}_3 \longrightarrow \mathcal{R}_3 + \mathcal{J}_3^R$$

where \mathcal{J}_3^R is the generator
of the $SU(2)_R \subset$
rotation group $Spin(4) = SU(2)_L \times SU(2)_R$





$\mathcal{N} = 2$ QUIVER THEORIES

NONCOMMUTATIVE DEFORMATION

$$\mathbb{R}^4 \longrightarrow \mathbb{R}_\theta^4$$

$$x^\mu \longrightarrow \hat{x}^\mu$$

$$[\hat{x}^\mu, \hat{x}^\nu] = -i\theta^{\mu\nu}$$

\hat{x}^μ ARE OPERATORS IN \mathcal{H}

$\mathcal{H} =$ REPRESENTATION OF \mathbb{R}_θ^4





$\mathcal{N} = 2$ THEORIES

SUBJECT TO Ω, θ -DEFORMATIONS

CAN BE MAPPED TO "MATRIX MODELS"

θ -N.Seiberg, 1999; Ω -NN, A.Okounkov, 2003

$$\widehat{\mathbf{X}}^\mu = \widehat{x}^\mu + i\theta^{\mu\nu} A_\nu(\widehat{x}), \quad (F_A)_{\mu'\nu'} \theta^{\mu\mu'} \theta^{\nu\nu'} = [\widehat{\mathbf{X}}^\mu, \widehat{\mathbf{X}}^\nu] + i\theta^{\mu\nu}$$

$$\Phi(x) \longrightarrow \widehat{\Phi} = \Phi(\widehat{x}) + \frac{1}{2} \tilde{\Omega}_{\mu\nu} \widehat{\mathbf{X}}^\mu \widehat{\mathbf{X}}^\nu$$





$\mathcal{N} = 2$ THEORIES

SUBJECT TO Ω, θ -DEFORMATIONS

CAN BE MAPPED TO "MATRIX MODELS"

θ -N.Seiberg, 1999; Ω -NN, A.Okounkov, 2003

$$S \longrightarrow \frac{1}{4g^2} \sum_{\mu < \nu} \text{Tr}_{\mathcal{H}} \left([\hat{\mathbf{X}}^\mu, \hat{\mathbf{X}}^\nu] + i\theta^{\mu\nu} \right)^2 +$$
$$\sum_{\mu} \text{Tr}_{\mathcal{H}} \left([\hat{\mathbf{X}}^\mu, \hat{\Phi}] + \Omega_{\nu}^{\mu} \hat{\mathbf{X}}^{\nu} \right) \left([\hat{\mathbf{X}}^\mu, \hat{\Phi}^\dagger] + \bar{\Omega}_{\nu}^{\mu} \hat{\mathbf{X}}^{\nu} \right) +$$
$$\text{Tr}_{\mathcal{H}} [\hat{\Phi}, \hat{\Phi}^\dagger]^2 + \textit{fermions}$$





OBSERVABLES FOR DS EQUATIONS

OBSERVABLE $Y(x)$

IN FOUR DIMENSIONAL $U(N)$ GAUGE THEORY

$$Y(x) \sim \det_{\mathbb{C}^N}(x - \Phi) \sim \prod_{\alpha=1}^N (x - a_{\alpha})$$

NAIVELY





OBSERVABLES FOR DS EQUATIONS

$Y(x)$ IN FOUR DIMENSIONS

MORE PRECISELY

$$Y(x) = \frac{\text{Det}_{\mathcal{H}}(x - \hat{\Phi} - \epsilon_1) \text{Det}_{\mathcal{H}}(x - \hat{\Phi} - \epsilon_2)}{\text{Det}_{\mathcal{H}}(x - \hat{\Phi}) \text{Det}_{\mathcal{H}}(x - \hat{\Phi} - \epsilon_1 - \epsilon_2)}$$





$$Y(x) = \frac{\text{Det}_{\mathcal{H}}(x - \hat{\Phi} - \epsilon_1) \text{Det}_{\mathcal{H}}(x - \hat{\Phi} - \epsilon_2)}{\text{Det}_{\mathcal{H}}(x - \hat{\Phi}) \text{Det}_{\mathcal{H}}(x - \hat{\Phi} - \epsilon_1 - \epsilon_2)}$$

$$\hat{\Phi} \in \text{End}(\mathcal{H})$$

RATIONAL FUNCTION OF DEGREE N

UNLIKE THE NAIVE $\det_{\mathbb{C}^N}(x - \Phi)$ IT HAS POLES





FOR QUIVER GAUGE THEORY

$$G = U(N_1) \times \dots \times U(N_r)$$

$$G = \times_{i \in \text{Vert}_\gamma} U(N_i)$$

$$\mathcal{H} = \bigoplus_{i \in \text{Vert}_\gamma} \mathcal{H}_i, \quad \widehat{\Phi} = \bigoplus_{i \in \text{Vert}_\gamma} \widehat{\Phi}_i$$

$$\mathbf{Y}(\mathbf{x}) \longrightarrow (\mathbf{Y}_i(\mathbf{x}))_{i \in \text{Vert}_\gamma}$$





MAIN CLAIM





MAIN CLAIM

THERE EXIST

LAURENT POLYNOMIALS (SERIES FOR AFFINE γ)

$$\mathcal{X}_i(x) = Y_i(x + \epsilon_1 + \epsilon_2) + \dots$$

in $Y_j(x + \text{linear combinations of } m_e, \epsilon_1, \epsilon_2)$'s, such that

$$\langle \mathcal{X}_i(x) \rangle = \text{POLYNOMIAL IN } x$$





MAIN CLAIM

$$\mathcal{X}_i(x) = Y_i(x + \epsilon_1 + \epsilon_2) + \dots$$

COEFFICIENTS = PRODUCTS OF

$q_j, P_j(x + \text{linear combinations of } m_e, \epsilon_1, \epsilon_2), \quad j \in \text{Vert}_\gamma$

$$P_j(x) = \mathbf{det}_{M_j}(x - \mathfrak{M}_j)$$

ENCODE FUNDAMENTAL MASSES





WE CALL $\mathcal{X}_i(x)$

THE FUNDAMENTAL qq -CHARACTERS





$\chi_i(x)$ BECOME

THE FUNDAMENTAL CHARACTERS OF G_γ

IN THE LIMIT $\epsilon_1, \epsilon_2 \rightarrow 0$





$$\mathcal{X}_i(x) = g_\infty(x)^{-\lambda_i} \text{Tr}_{V_i} g(x)$$

$$g(x) = \prod_{j \in \text{Vert}_\gamma} Y_j(x)^{\alpha_j^\vee} (\mathbf{q}_j P_j(x))^{-\lambda_j^\vee} \in \mathbb{C}G_\gamma$$

$\epsilon_1, \epsilon_2 \rightarrow 0$

NN, V.Pestun, 2012





THE DS EQUATIONS IN THE LIMIT $\epsilon_1, \epsilon_2 \rightarrow 0$

BECOME ALGEBRAIC EQUATIONS

$$\mathcal{X}_i(x) = g_\infty(x)^{-\lambda_i} \text{Tr}_{V_i} g(x) = \text{POLYNOMIAL IN } x$$

SEIBERG-WITTEN GEOMETRY

NN, V.Pestun, 2012





IN THE LIMIT $\epsilon_2 \rightarrow 0$, $\epsilon_1 = \hbar$ FINITE

$\mathcal{X}_i(x)$ BECOME

NN, V.Pestun, S.Shatashvili, 2013

THE FUNDAMENTAL q -CHARACTERS OF $Y(\mathfrak{g}_\gamma)$
H.Knight

FIVE DIMENSIONAL THEORY ON $S^1 \times \mathbb{R}^4$

PRODUCES q -CHARACTERS OF $U_q(\widehat{\mathfrak{g}}_\gamma)$
E.Frenkel, N.Reshetikhin





GENERAL ϵ_1, ϵ_2 CASE

USE NAKAJIMA'S QUIVER VARIETIES

$$\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v})$$





NAKAJIMA VARIETY

MATHEMATICALLY

$$\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v}) = T^* \left(\bigoplus_{e \in \text{Edge}_\gamma} \text{Hom}(V_{s(e)}, V_{t(e)}) \bigoplus_{i \in \text{Vert}_\gamma} \text{Hom}(V_i, W_i) \right) // \times_{i \in \text{Vert}_\gamma} \text{GL}(V_i)$$





NAKAJIMA VARIETY

PHYSICALLY

$$\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v}) =$$

HIGGS BRANCH OF THE $\mathcal{N} = 4$
THREE DIMENSIONAL QUIVER GAUGE THEORY
WITH GAUGE GROUP

$$G_v = \times_{i \in \text{Vert}_\gamma} U(V_i)$$

w_i FUNDAMENTALS FOR $U(V_i)$
BI-FUNDAMENTALS IN $(V_{s(e)}, V_{t(e)}^*)$





FORMULA FOR qq -CHARACTERS

$$\chi_{\mathbf{w}}(x) = \sum_{\mathbf{v}} \prod_{i \in \text{Vert}_{\gamma}} q_i^{\nu_i} \int_{\mathfrak{M}_{\gamma}(\mathbf{w}, \mathbf{v})} e_{-\varepsilon_2}(T\mathfrak{M}_{\zeta}(\mathbf{v}, \mathbf{w})) l[\mathcal{F}]$$

WHERE

$$l[\mathcal{F}] = \prod_{i \in \text{Vert}_{\gamma}} \prod_{\kappa} P_i(x + \nu_{i, \kappa}) \frac{\prod_{\kappa_+} Y_i(x + \xi_{i, \kappa_+}^+)}{\prod_{\kappa_-} Y_i(x + \xi_{i, \kappa_-}^-)}$$

$$\text{Ch}(V_i) = \sum_{\kappa} e^{\nu_{i, \kappa}}, \quad \text{Ch}(\mathcal{C}_i) = \sum_{\kappa_+} e^{\xi_{i, \kappa_+}^+} - \sum_{\kappa_-} e^{\xi_{i, \kappa_-}^-}$$

$\mathcal{C}_i =$ TAUTOLOGICAL COMPLEXES ON $\mathfrak{M}_{\gamma}(\mathbf{w}, \mathbf{v})$





FORMULA FOR qq -CHARACTERS

$\mathcal{X}_{\mathbf{w}}(x)$ = PARTITION FUNCTION

OF A POINT-LIKE DEFECT $\mathcal{D}_{\mathbf{w}}(x)$

FOR SOME (ALL?) THEORIES

$\mathcal{D}_{\mathbf{w}}(x)$ CAN BE ENGINEERED

USING INTERSECTING BRANES





EXAMPLES: $U(N)$ THEORIES

A_1 CASE: $N_c = N$, $N_f = 2N$

FUNDAMENTAL qq -CHARACTER

$$\mathcal{X}_1(x) = Y(x + \epsilon_1 + \epsilon_2) + qP(x)Y(x)^{-1}$$





EXAMPLES: $U(N)$ THEORIES

A_1 CASE: $N_c = N$, $N_f = 2N$

GENERAL qq -CHARACTER

$$\vec{\nu} = (\nu_1, \dots, \nu_w)$$

$$\chi_{\vec{\nu}}(x) = \sum_{[w]=I\cup J} q^{\#J} \prod_{i \in I, j \in J} \frac{(\nu_i - \nu_j + \epsilon_1)(\nu_i - \nu_j + \epsilon_2)}{(\nu_i - \nu_j)(\nu_i - \nu_j + \epsilon_1 + \epsilon_2)} \times$$
$$\prod_{j \in J} \frac{P(x + \nu_j)}{Y(x + \nu_j)} \times \prod_{i \in I} Y(x + \epsilon_1 + \epsilon_2 + \nu_i)$$





EXAMPLES: $U(N)$ THEORIES

\hat{A}_0 CASE: $\mathcal{N} = 2^*$ $U(N)$ THEORY

μ MASS OF THE ADJOINT HYPER





FUNDAMENTAL qq -CHARACTER

$$\begin{aligned} \mathfrak{X}_1(x) = & \sum_{\lambda} q^{|\lambda|} \prod_{\square \in \lambda} \frac{(\mu h_{\square} + (\epsilon_1 + \epsilon_2) a_{\square} + \epsilon_1)(\mu h_{\square} + (\epsilon_1 + \epsilon_2) a_{\square} + \epsilon_2)}{(\mu h_{\square} + (\epsilon_1 + \epsilon_2) a_{\square})(\mu h_{\square} + (\epsilon_1 + \epsilon_2)(a_{\square} + 1))} \\ & \times \prod_{\square \in \partial_+ \lambda} Y(x + \sigma_{\square} + \epsilon_1 + \epsilon_2) \\ & \times \prod_{\square \in \partial_- \lambda} Y(x + \sigma_{\square})^{-1} \end{aligned}$$





$a_{\square}, l_{\square}, h_{\square}$ are the arm-length, the leg-length and the hook-length
of the box $\square = (i, j) \in \lambda$

$$a_{\square} = \lambda_i - j, \quad l_{\square} = \lambda_j^t - i, \quad h_{\square} = a_{\square} + l_{\square} + 1,$$

$$\sigma_{\square} = \mu(i - j) + (\epsilon_1 + \epsilon_2)(1 - j)$$





APPLICATIONS

BPS/CFT CORRESPONDENCE

NN, 2002-2004

Correlators of chiral observables
in four dimensional supersymmetric theories
are **holomorphic blocks (form-factors)**
of some **conformal field theory**
(or a massive integrable deformation thereof)
in two dimensions





APPLICATIONS

THE NONPERTURBATIVE DS EQUATIONS

CAN BE USED TO DEMONSTRATE





Z-FUNCTIONS

OF A-TYPE QUIVER THEORIES

FOR SPECIAL MASSES

OBEY THE BPZ-TYPE EQUATIONS

\implies Alday-Gaiotto-Tachikawa dictionary





Z-FUNCTIONS

IN THE PRESENCE OF SURFACE DEFECTS

Y_i -observables fraction: $Y_i \rightarrow Y_{i,\alpha}, \alpha = 1, \dots, n_i$

OF A-TYPE QUIVER THEORIES

OBEY THE KZ-TYPE EQUATIONS

Use Nakajima realization, cf. Kanno-Tachikawa



APPLICATIONS

QUANTUM INTEGRABILITY

Surface operators

in $U(N)$ gauge theories

Surface operators

in $U(N)$ gauge theories

Codimension two defect in spacetime

Surface operators

in $U(N)$ gauge theories

Codimension two defect in spacetime

$$A(z, \bar{z}, w, \bar{w}) \longrightarrow J(z, \bar{z}) d \arg w + \text{regular}, \quad w \rightarrow 0$$

Surface operators

$$A(z, \bar{z}, w, \bar{w}) \longrightarrow J(z, \bar{z})d\arg w + \text{regular}, \quad w \rightarrow 0$$

Kronheimer-Mrowka, 1988

$$J(z, \bar{z}) = g(z, \bar{z})J_0g(z, \bar{z})^{-1}$$

The conjugacy class, J_0 , is fixed for the defect

The representative $J(z, \bar{z})$ may vary \implies

Four dimensional gauge theory is coupled to two dimensional sigma model



Four dimensional gauge theory is coupled

to two dimensional sigma model

The target space is the adjoint orbit \mathcal{O}_{J_0} of J_0

Geometry and Topology of \mathcal{O}_{J_0}

depends on the multiplicities of the eigenvalues of J_0

Partitions of $N = n_1 + \dots + n_k$

Surface operators in $U(N)$ theory

classified by the partitions of N

$$N = n_1 + \dots + n_k$$

Simplest surface operators in $U(N)$ theory

$$N = 1 + 1 + \dots + 1$$

Target space is the complete flag variety

$$\mathcal{O}_{J_0} = U(N)/U(1)^N$$

Topology: $H_2(\mathcal{O}_{J_0}, \mathbb{Z}) = \mathbb{Z}^{N-1}$

Simplest surface operators in $U(N)$ theory

$$N = 1 + 1 + \dots + 1$$

Gauge theory in four dimensions, instantons

$$k = -\frac{1}{8\pi^2} \int \text{Tr} F_A \wedge F_A$$

Sigma model in two dimensions, instantons

$$\vec{d} \in H_2(\mathcal{O}_{J_0}, \mathbb{Z}) = \mathbb{Z}^{N-1}$$

N charges in total

$$k_1, k_2, \dots, k_N \geq 0 \Leftrightarrow (k; d_1, \dots, d_{N-1})$$



Partition function with surface operator

$$\Psi(q_0, \dots, q_{N-1}) = \sum_{\vec{k}} \prod_{\alpha=0}^{N-1} q_{\alpha}^{k_{\alpha}} \int_{\mathcal{M}[\vec{k}]} \text{Euler}(\mathcal{F}_{\vec{k}})$$

$$\mathcal{M}_k^{\mathbb{Z}^N} = \coprod_{\vec{k}, |\vec{k}|=k} \mathcal{M}[\vec{k}]$$

$$|\vec{k}| = k_0 + k_1 + \dots + k_{N-1}$$





Using qq -characters we prove

For $(1, 1, 1, \dots, 1)$ surface operator

$$\epsilon_1 \epsilon_2 Q \frac{d}{dQ} \Psi(q_0, \dots, q_{N-1}) = \widehat{H}(\epsilon_1 \frac{d}{dx_i}; e^{x_i}) \Psi(q_0, \dots, q_{N-1})$$

$$q_\alpha = Q e^{x_\alpha - x_{\alpha+1}}$$





In the limit $\epsilon_2 \rightarrow 0$ recover

Bethe/gauge correspondence

NN, S.Shatashvili, 2006-9

Together with a formula

for the eigenfunction Ψ of \hat{H} 's





For example, for pure $U(N)$ $\mathcal{N} = 2$ theory

$$\hat{H}(\epsilon_1 \frac{d}{dx_i}; e^{x_i}) = \frac{\epsilon_1^2}{2} \sum_{i=1}^N \frac{d^2}{dx_i^2} + Q \sum_{i=1}^N e^{x_i - x_{i+1}},$$

$x_{N+1} \equiv x_1$ periodic Toda





For example, for $U(N)$ $\mathcal{N} = 2^*$ theory

adjoint hypermultiplet of mass m

microscopic coupling τ , $Q = \exp 2\pi i \tau$

$$\widehat{H}(\epsilon_1 \frac{d}{dx_i}; e^{x_i}) = \frac{\epsilon_1^2}{2} \sum_{i=1}^N \frac{d^2}{dx_i^2} + m(m + \epsilon_1 + \epsilon_2) \sum_{i < j}^N \wp(e^{x_i - x_j}; Q)$$

$+o(\epsilon_1, \epsilon_2)$, elliptic Calogero – Moser





CONCLUSIONS/SPECULATIONS

THE G_γ SYMMETRY CAN BE SEEN

IN THE II STRING REALIZATION OF GAUGE THEORY

YANGIAN $Y(\mathfrak{g}_\gamma)$ AND $U_q(\widehat{\mathfrak{g}}_\gamma)$ IN STRING THEORY?

SH^c degenerate DAHA of S. Kanno, Y. Matsuo, H. Zhang?





CONCLUSION/SPECULATION

GRAVITATIONAL

NON-PERTURBATIVE DS EQUATIONS?

NON-LINEAR WHEELER-DE WIT EQUATION?

SYMMETRY OF THE LANDSCAPE?





CONCLUSION/SPECULATION

TOPOLOGICAL STRING(M-THEORY)

NON-PERTURBATIVE DS EQUATIONS?

qq -characters vs. q, t -characters
Nakajima, Frenkel-Hernandez?





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