Adjoint BFKL at finite coupling

a short-cut
from the collinear limit

with **Benjamin Basso** & **Simon Caron-Huot**

IGST 2014

The **Collinear** and the high energy **Regge** limits

- **Different** expansions of scattering amplitudes

Leading term in one expansion $\Leftrightarrow$ Resummation of **infinite many** terms in the other

- **Different** physics
- Valid at **any coupling**

Related works
[Bartels,Lipatov,Prygarin], [Hatsuda]

In this talk

Fix the Regge (BFKL) one from the collinear (OPE) one at finite coupling

Comment

Color adjoint BFKL (open string) $\neq$ Color singlet BFKL (closed string)

[Lipatov talk] **[Gromov & Lipatov talk]**

Some common physics

Summary of these expansions for n=6

OPE

$$\mathcal{W}_{\text{hex}} = 1 + \sum_{m\neq 0} (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, \hat{\mu}_m(p)\, e^{ip\sigma - \tau E_m(p)} + \dots$$

Leading term dominate at large τ

a,AS,Vieira]

$1/2$ talk explaining $\updownarrow$ $1/2$ talk relating $\updownarrow$

BFKL

$$\mathcal{W}_{\text{hex}}^{\circlearrowleft} e^{-i\pi\delta'} = \sum_{m=-\infty}^{\infty} (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi}\, \hat{\mu}_{\text{BFKL}}(\nu, m)\, e^{i(\sigma-\tau)\nu + (\sigma+\tau)\omega(\nu,m)} + \dots$$

Leading term dominate at large $\tau + \sigma$

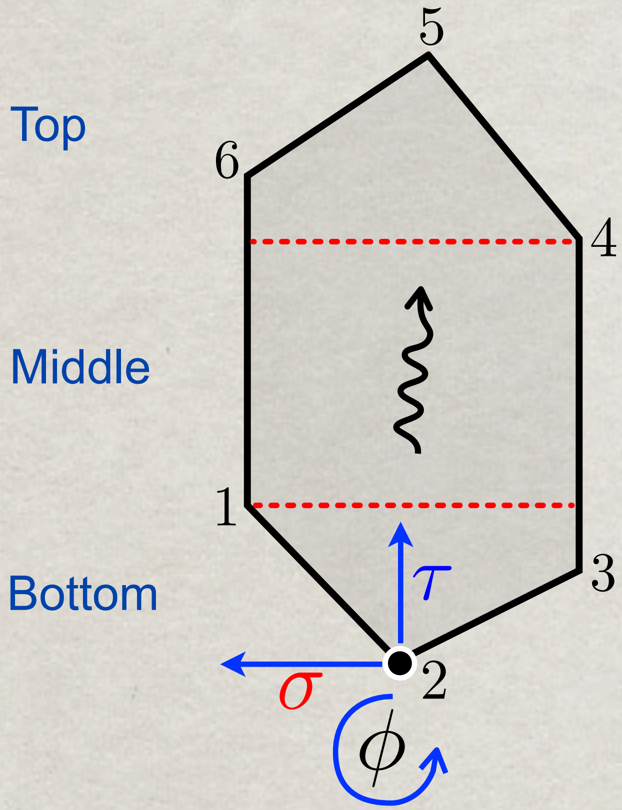
[Bartels,Lipatov,Sabio Vera]

Note - Different kinematical regimes!

Planar $\mathcal{N}=4$ SYM

The collinear expansion (OPE)

Geometry



$$u_i \equiv \frac{x^2_{i-1,i+1}\, x^2_{i-2,i+2}}{x^2_{i-1,i+2}\, x^2_{i+1,i-2}}$$

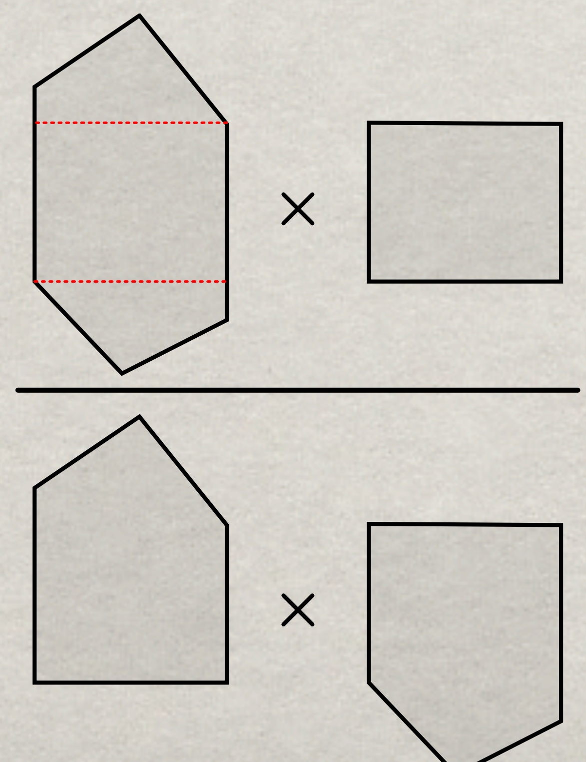
$$\frac{1}{u_2} = 1 + e^{2\tau}$$

$$\frac{u_1}{u_2 u_3} = e^{2\sigma+2\tau}$$

$$\frac{1}{u_3} = 1 + (e^{-\tau} + e^{\sigma+i\phi})(e^{-\tau} + e^{\sigma-i\phi})$$

Expansion

$\mathcal{W}_{\text{hex}} =$



angular momentum

momentum

energy

$$= 1 + \sum_{m\neq 0} (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, \hat{\mu}_m(p)\, e^{ip\sigma - \tau E_m(p)} + \dots$$

multi-particles

single particle

The collinear expansion (OPE)

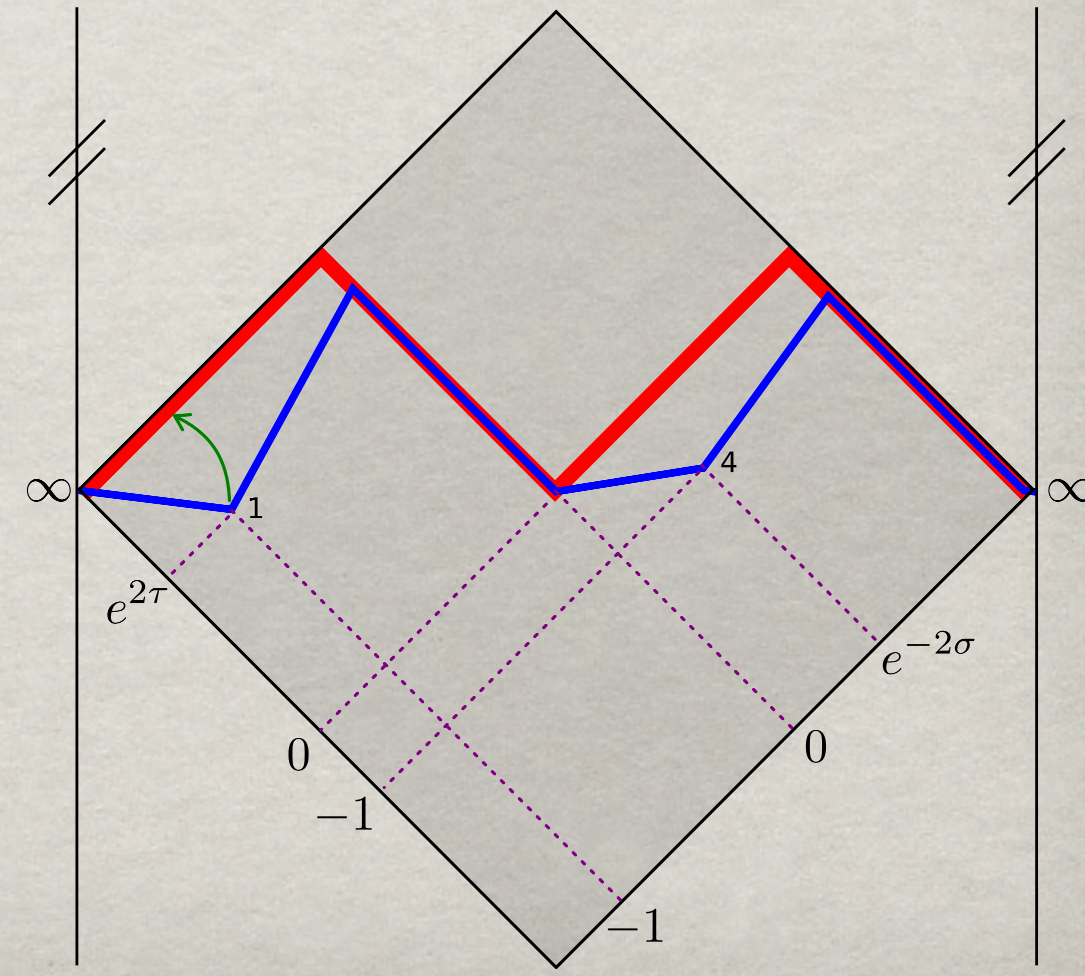
$$\mathcal{W}_{\rm hex} = 1 + \sum_{m\neq 0} (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, \hat{\mu}_m(p)\, e^{ip\sigma - \tau E_m(p)} + \dots$$

- $\mu_m(u),\ E_m(u),\ p_m(u)$ are **known** at finite coupling [Basso], [Basso, AS, Vieira]

  rapidity

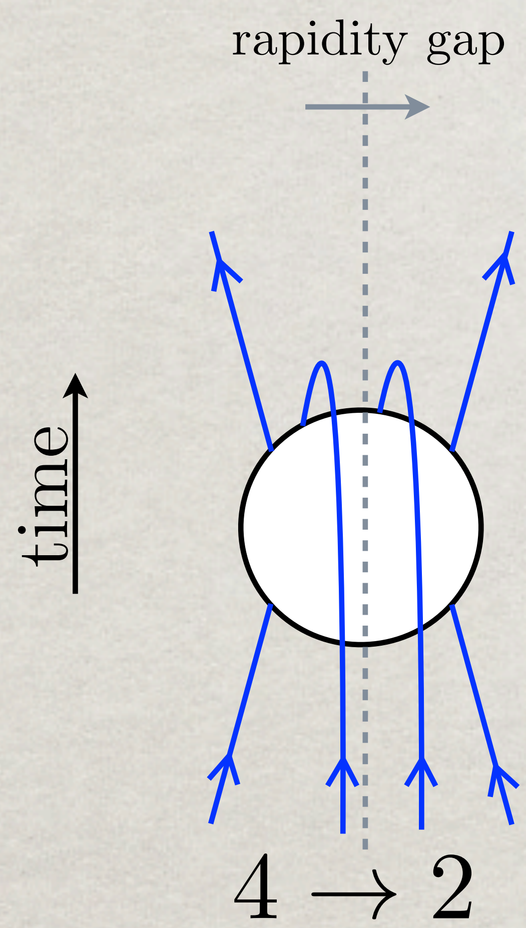
- The **vacuum energy** (twist) is infinite $E_{\rm vac} = \Gamma_{\rm cusp} \log(S)$ but it is subtracted in the ratio $\mathcal{W}_{\rm hex}$

- In **Euclidian** kinematics

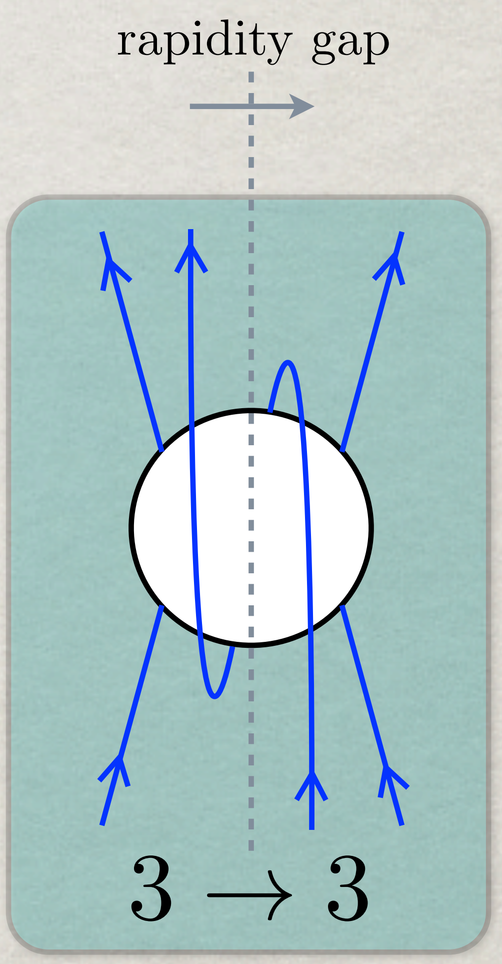


The Regge high energy expansion (BFKL)

Geometry



[Bartels,Lipatov,Sabio Vera]

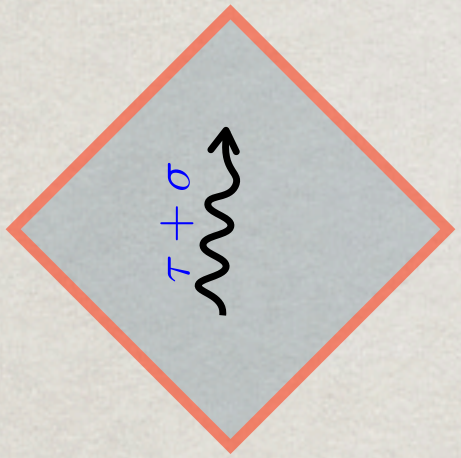


[Bartels,Lipatov,Prygarin]

“Mandelstam region”

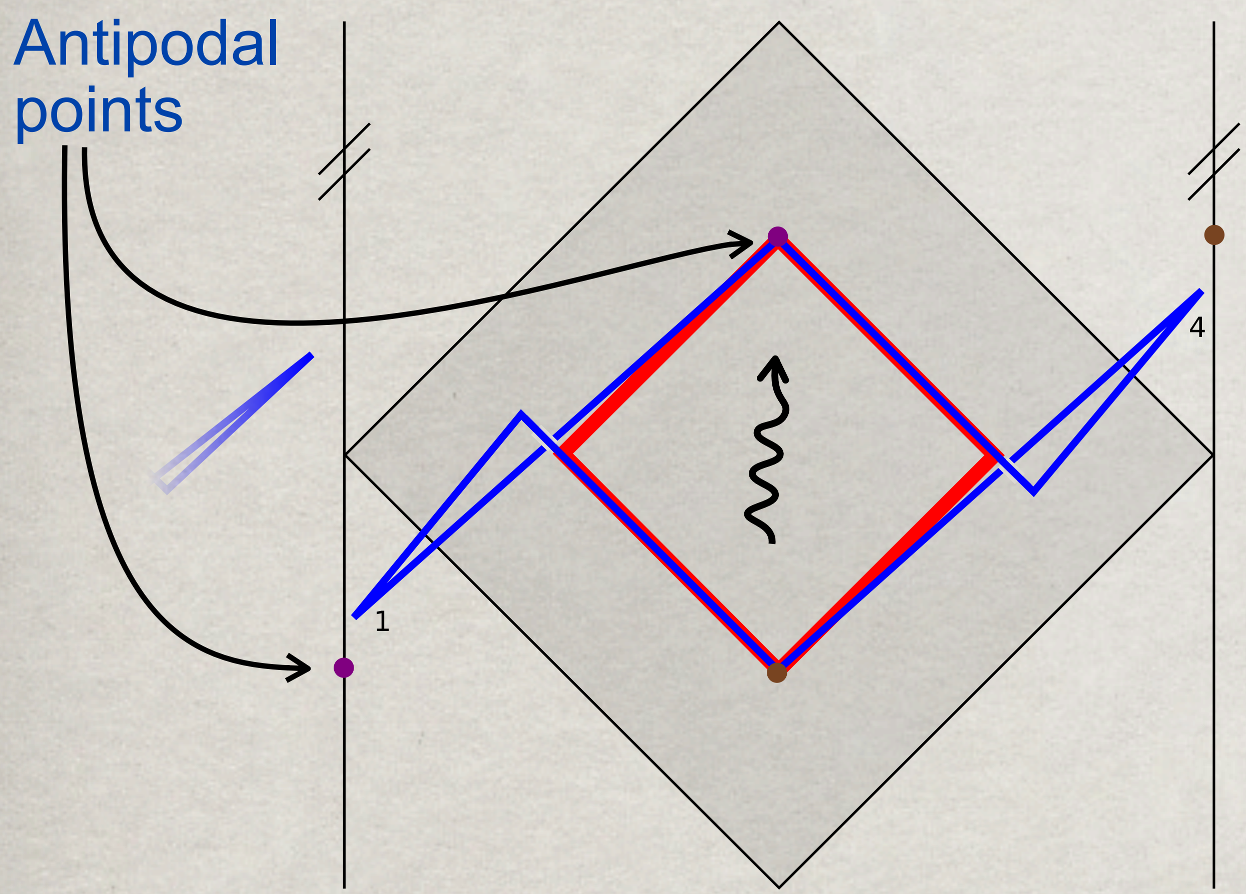
The Regge high energy expansion (BFKL)

Wilson loop picture



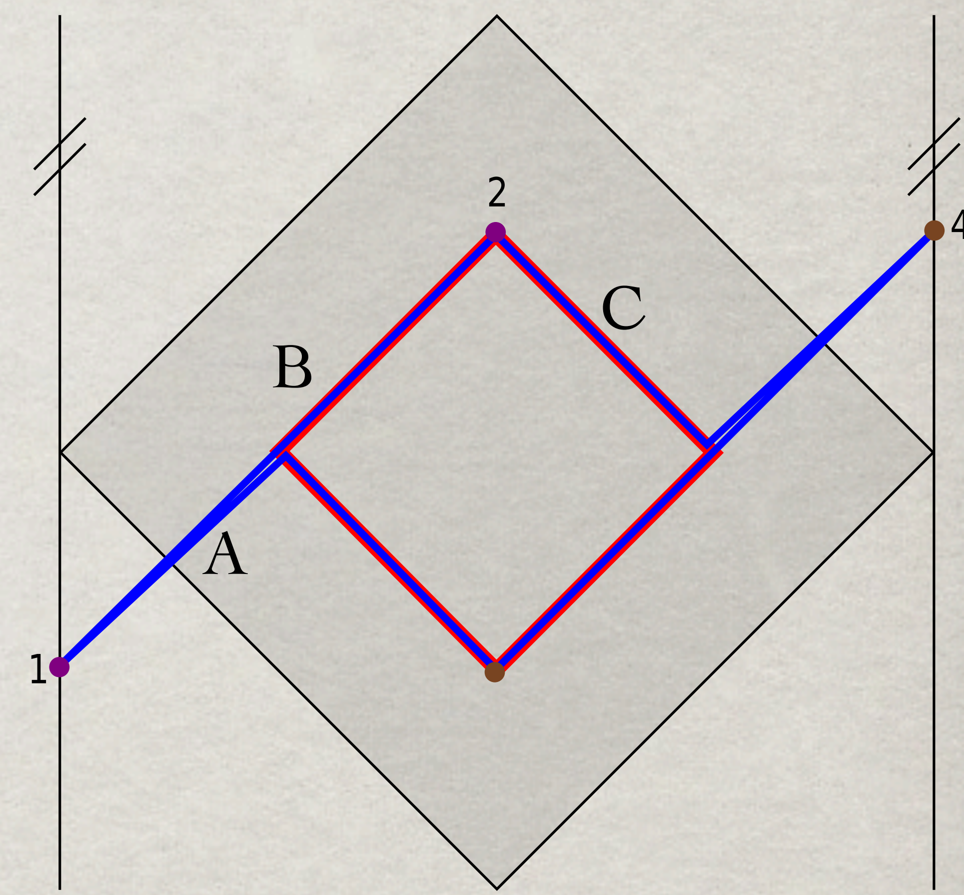
Propagation in 45°

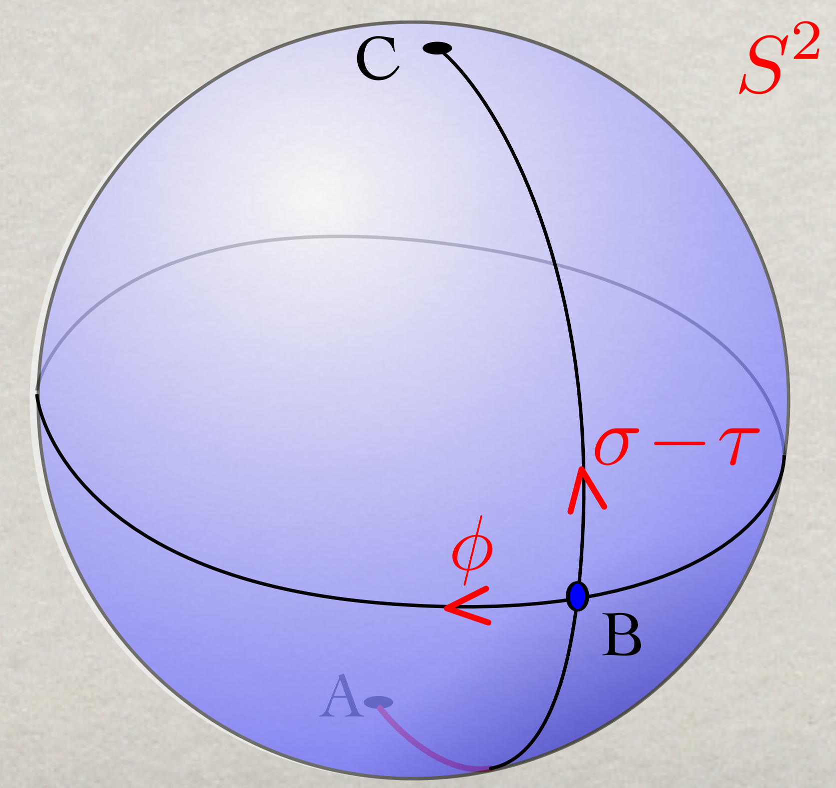
The Regge high energy expansion (BFKL)

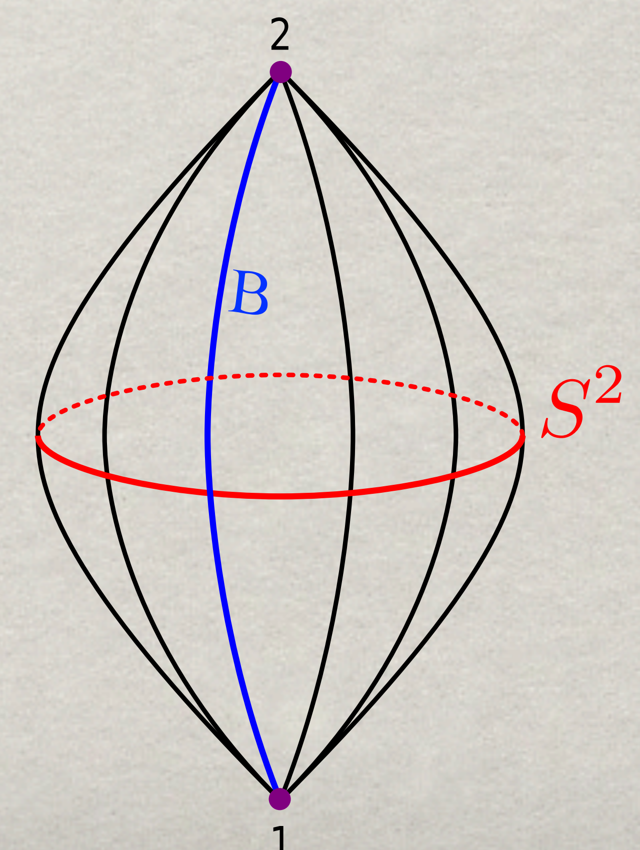


BFKL limit

$(\tau + \sigma) \to \infty$







The Regge high energy expansion (BFKL)

angular momentum

energy

momentum

$$\mathcal{W}^{\circlearrowleft}_{\text{hex}} e^{-i\pi\delta'} = \sum_{m=-\infty}^{\infty} (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi}\, \hat{\mu}_{\text{BFKL}}(\nu, m)\, e^{i(\sigma-\tau)\nu+(\sigma+\tau)\omega(\nu,m)} + \dots$$

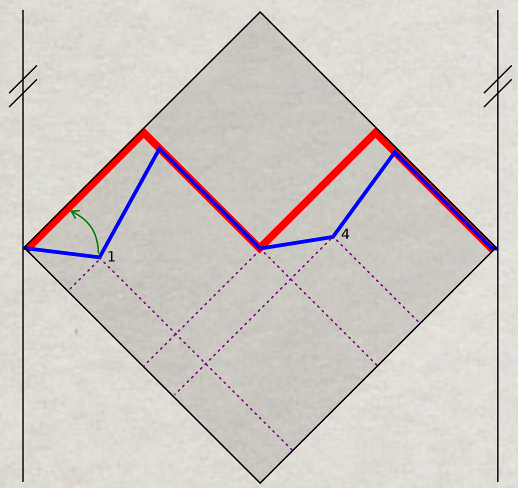
$\delta' = \frac{1}{2}(\sigma - \tau)\Gamma_{\text{cusp}}$ overall momentum shift

The **discontinuity** of the analytic continuation from Euclidian kinematic

- Controlled by $\omega(\nu, m), \quad \hat{\mu}_{\text{BFKL}}(\nu, m)$
- No 1 = trivial vacuum, grows at large $\tau + \sigma$
- The **vacuum energy** (spin) is infinite $-\Gamma_{\text{cusp}} \log(\Lambda_{\text{cut-off}})$ but it is subtracted in the ratio

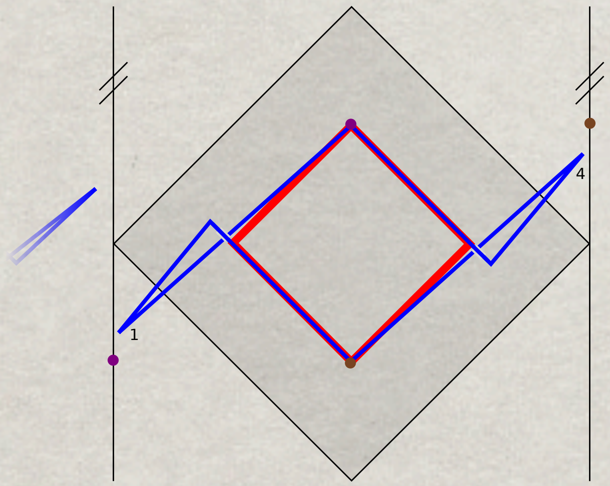
Summary

OPE



$$= 1 + \sum_{m \neq 0} (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \, \hat{\mu}_m(p) \, e^{ip\sigma - \tau E_m(p)} + \ldots$$

BFKL



$$= -2\pi i \sum_{m=-\infty}^{\infty} (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} \, \hat{\mu}_{\text{BFKL}}(\nu, m) \, e^{i(\sigma-\tau)\nu + (\sigma+\tau)\omega(\nu,m)} + \ldots$$

How these are related?

Two **analytic continuations**

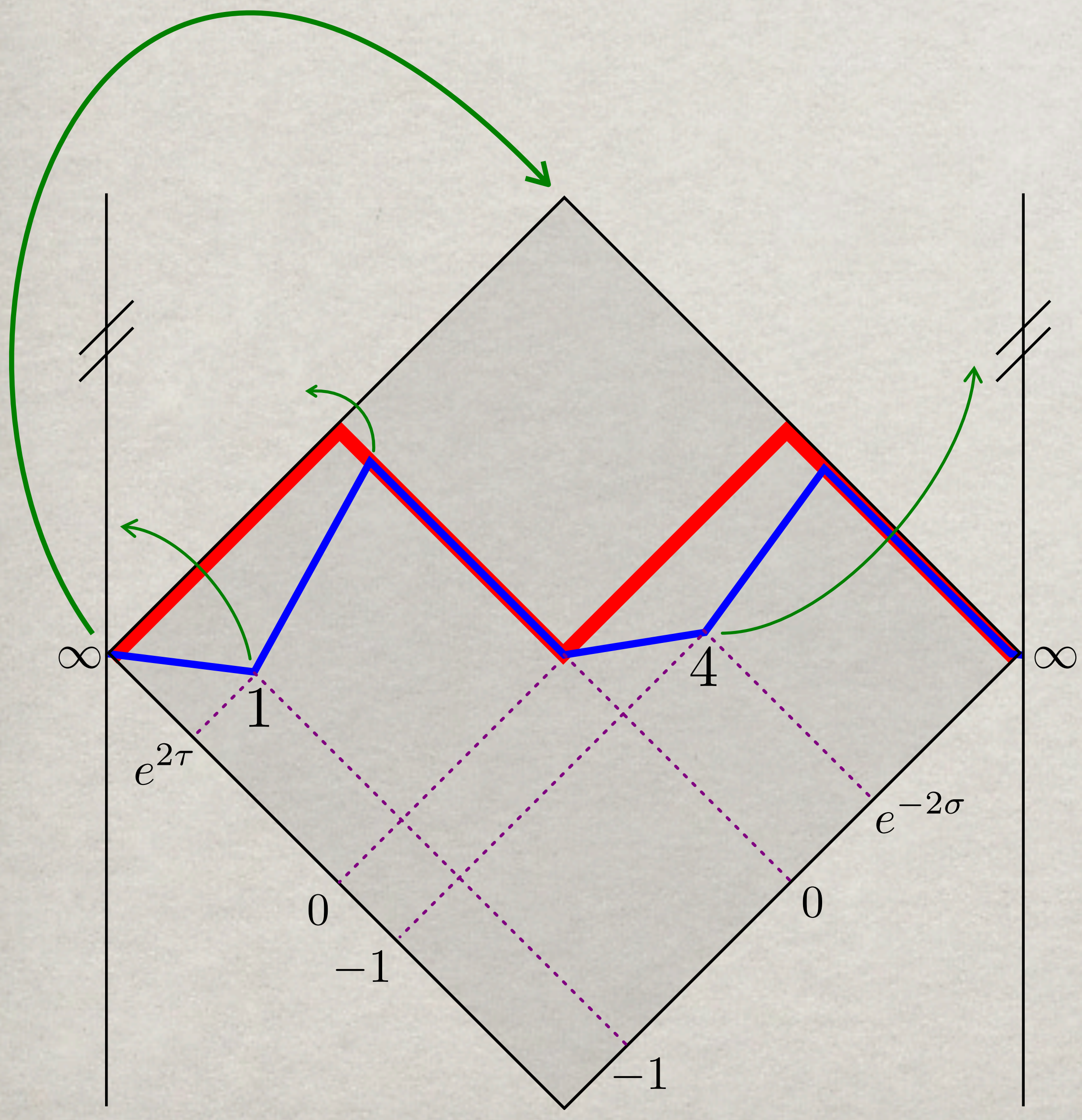
1. In external momenta - $\tau, \ \sigma, \ \phi$ $\quad \Rightarrow \quad$ OPE for $3 \to 3$ discontinuity
2. In flux-tube momenta = p

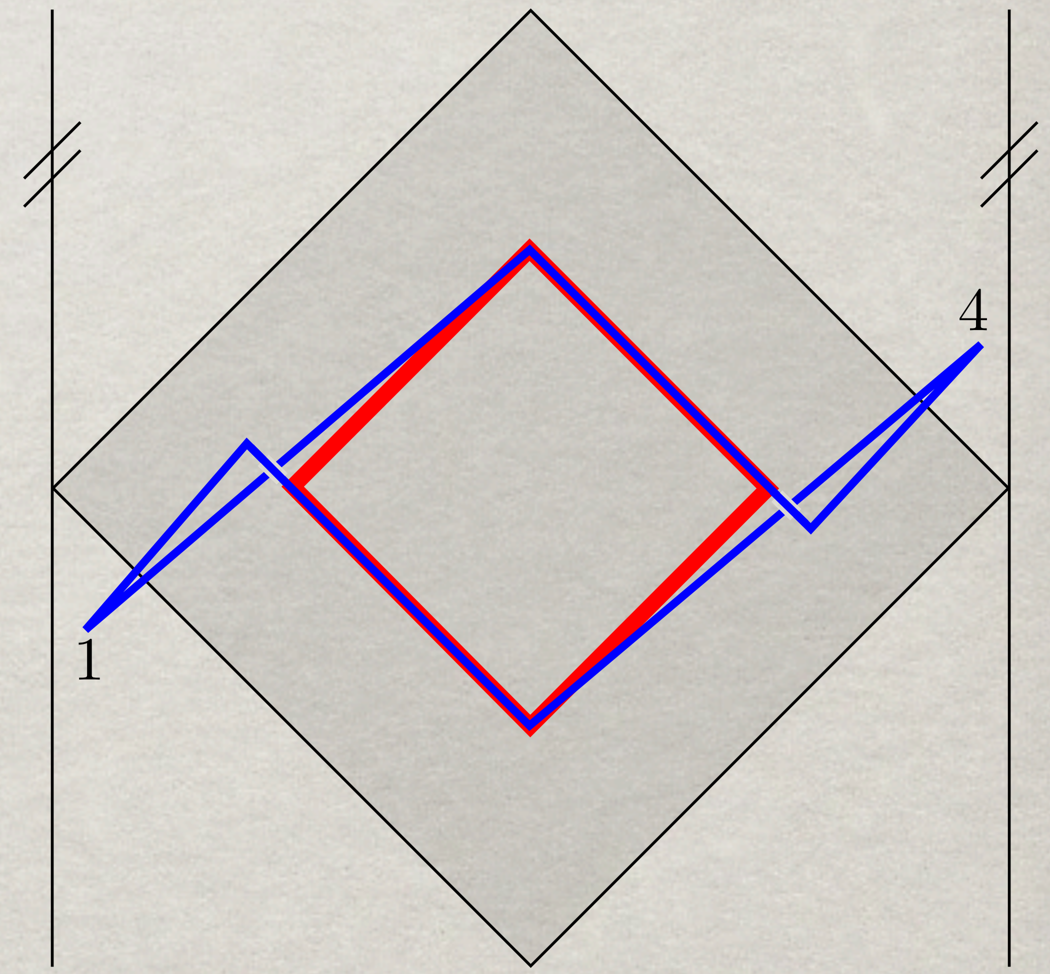
Euclidian to $3 \to 3$ kinematics in the OPE (large τ)

$$\tau \to \tau - i\pi/2 \ , \qquad \sigma \underset{\gamma}{\to} \sigma + i\pi/2 + i0 \ , \qquad \cos\phi \to \cos\phi$$

Euclidian kinematics

$3 \to 3$ kinematics





OPE (large τ) for $3 \to 3$ discontinuity

$$\mathcal{W}_{\rm hex} - 1 = 2\sum_{\ell\geq 1}(-1)^\ell \cos(\ell\phi) \underbrace{\int_{-\infty}^{+\infty} \frac{dp}{2\pi}\hat{\mu}_\ell(p) e^{i\sigma p - \tau E_\ell(p)}} + \dots$$

$$\equiv 2\sum_{\ell\geq 1}(-1)^\ell \cos(\ell\phi) \quad F_\ell(\sigma,\tau) + \dots$$

Example $\ell = 1$ at leading order

$$F_1(\sigma,\tau) = g^2 e^{-\tau}\left[-(e^\sigma + e^{-\sigma})\log(1+e^{2\sigma}) + 2\sigma e^\sigma\right] + O(g^4)$$

Cut at $\mathrm{Im}(\sigma) = \frac{\pi}{2}$ for $\mathrm{Re}(\sigma) > 0$

$$F_\ell^{\downarrow}(\sigma,\tau) \equiv F_\ell\left(\sigma + i\frac{\pi}{2} - i0, \tau - i\frac{\pi}{2}\right)$$

$$F_\ell^{\updownarrow}(\sigma,\tau) \equiv F_\ell\left(\sigma + i\frac{\pi}{2} + i0, \tau - i\frac{\pi}{2}\right) - F_\ell^{\downarrow}(\sigma,\tau) = 2\pi i g^2 e^{-\tau}\left(e^\sigma - e^{-\sigma}\right)\theta(\sigma) + O(g^4)$$

- $F_1^{\updownarrow} \neq 0$ only for $\sigma > 0$
- At $\tau, \sigma \to \infty$ $F_1^{\downarrow} \sim e^{-\sigma-\tau}$ but $F_1^{\updownarrow} \sim e^{\sigma-\tau}$

As seen in Fourier space

Before analytic continuation

$$\hat{\mu}_1(p) = 2\pi g^2 \frac{1}{\cosh(\frac{\pi p}{2})(p^2+1)} + \mathcal{O}(g^4)$$

$$E_1(p) = 1 + 2g^2 \left[\psi\left(\frac{3}{2}+\frac{ip}{2}\right) + \psi\left(\frac{3}{2}-\frac{ip}{2}\right) - 2\psi(1)\right] + \mathcal{O}(g^4)$$

$$\text{leading log} \equiv e^{i\phi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\ \hat{\mu}_1(p)\ e^{i\sigma p - \tau E_1(p)}$$

$1 + O(g^2)$

$O(g^2)$

Right below the cut

$$\hat{\mu}_1(p) \quad \rightarrow \quad \hat{\mu}^{\downarrow}(p) = 2\pi g^2 \frac{ie^{-\pi p/2}\, e^{i\sigma p}}{\cosh(\frac{\pi p}{2})(p^2+1)} + O(g^4)$$

cut $\Rightarrow$ barely converges at large p

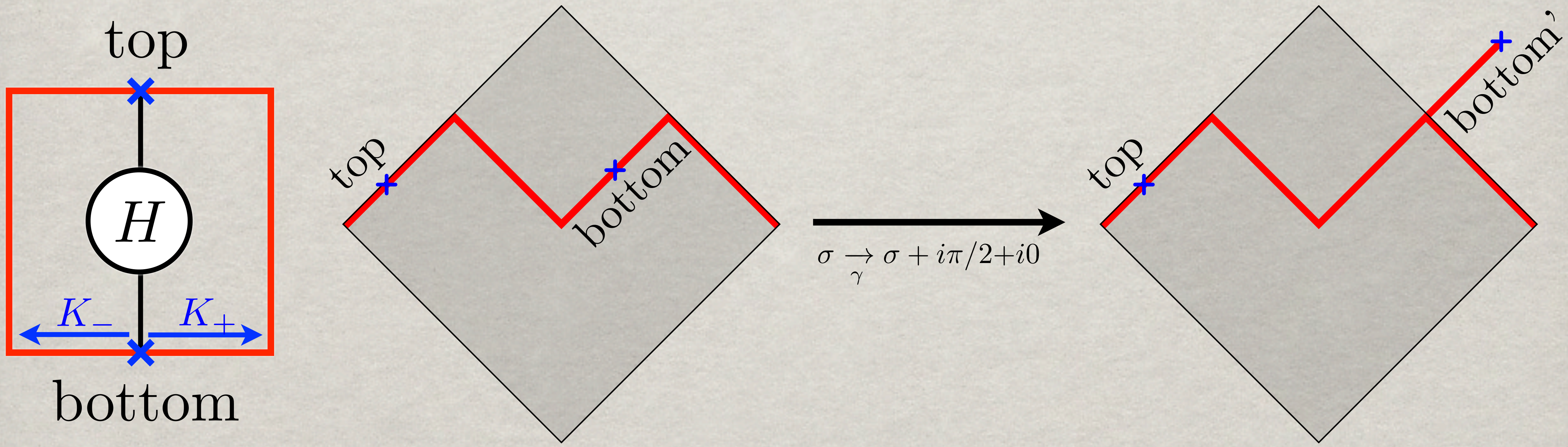
The one loop light-ray hamiltonian

What controls the discontinuity?

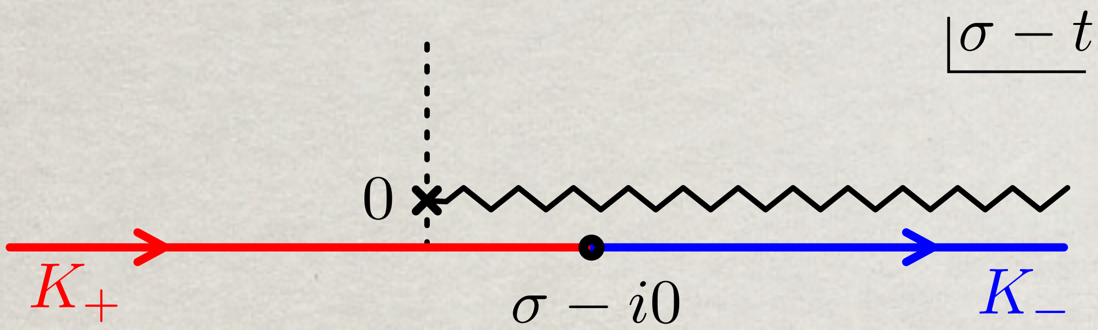
$$-\left(1+\frac{d}{d\tau}\right)F^{\downarrow}(\sigma,\tau) = \int_{-\infty}^{0} dt\, K_{-}(t) F^{\downarrow}(\sigma - t, \tau) + \int_{0}^{\infty} dt\, K_{+}(t) F^{\downarrow}(\sigma - t, \tau)$$

$$K_{\pm}(t) = -4g^2 \frac{e^{-|t|}}{(e^{2|t|}-1)_{+}} + O(g^4)$$

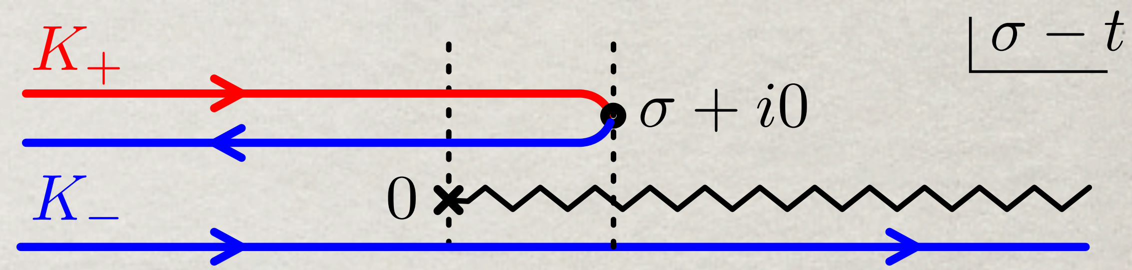
$$\int_0^\infty \frac{dt}{(e^{2t}-1)_+} G(t) \equiv \int_0^\infty \frac{dt}{e^{2t}-1}\left(G(t)-G(0)\right)$$



Analytic continuation of light-ray hamiltonian

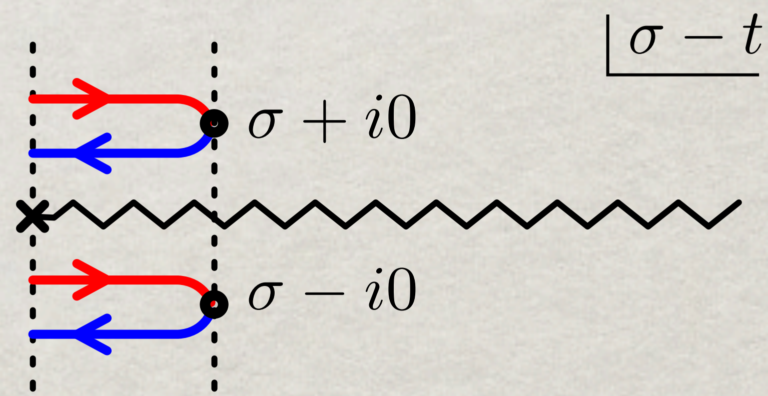


$(\mathbf{a})$



$(\mathbf{b})$

$(\mathbf{b}) - (\mathbf{a}) =$



$$-\left(1 + \frac{d}{d\tau}\right) F^{\updownarrow}(\sigma, \tau) = \int_0^\infty dt\, \big(K_+(t) - K_-(t + i0)\big) F^{\updownarrow}(\sigma - t, \tau)$$

Analytic continuation of light-ray hamiltonian

$$K_+(t) - K_-(t+i0) = -4g^2\left[\frac{e^{-t}+e^{3t}}{(e^{2t}-1)_+} - \frac{i\pi}{2}\delta(t)\right] + O(g^4)$$

$$\Rightarrow \qquad \check{E}_1(p) = 1 + 2g^2\left[\psi\left(\frac{3}{2}+\frac{ip}{2}\right) + \psi\left(-\frac{1}{2}+\frac{ip}{2}\right) - 2\psi(1) + i\pi\right] + O(g^4)$$

"Sister"

Compare with $E_1(p) = 1 + 2g^2\left[\psi\left(\frac{3}{2}+\frac{ip}{2}\right) + \psi\left(\frac{3}{2}-\frac{ip}{2}\right) - 2\psi(1)\right] + \mathcal{O}(g^4)$ before

$$\boxed{-\left(1+\frac{d}{d\tau}\right)F^{\updownarrow}(\sigma,\tau) = \int_0^\infty dt\,\big(K_+(t) - K_-(t+i0)\big)F^{\updownarrow}(\sigma-t,\tau)}$$

Finite coupling - sister dispersion relation

Before the flip

$$E_\ell(u) = \ell + \int_0^\infty \frac{dt}{t} K(t) \left(\cos(ut) e^{-\ell t/2} - 1\right)$$

$$p_\ell(u) = \underline{\underline{2u}} + \int_0^\infty \frac{dt}{t} K(-t) \sin(ut) e^{-\ell t/2}$$

$$K(t) = \frac{2}{1-e^{-t}} \sum_{n\geq 1} (2n)\gamma_{2n} J_{2n}(2gt) - \frac{2}{e^t - 1} \sum_{n \geq 1} (2n-1)\gamma_{2n-1} J_{2n-1}(2gt)$$

$\{\gamma_i\}$ are determined by the BES equation

$$E_0(u) = p_0(u) = 0 \quad \text{for} \quad u \in [-2g, 2g]$$

After the flip

$$\check{E}_\ell(u) = \ell + \frac{i\pi}{2}\Gamma_{\text{cusp}} + \int_0^\infty \frac{dt}{t} \left[K(t) \frac{e^{-iut-\ell t/2} - 2}{2} + K(-t) \frac{e^{-iut+\ell t/2}}{2} \right]$$

$$\check{p}_\ell(u) = 2u + \frac{\pi}{2}\Gamma_{\text{cusp}} - i \int_0^\infty \frac{dt}{t} \left[K(t) \frac{e^{-iut+\ell t/2}}{2} - K(-t) \frac{e^{-iut-\ell t/2}}{2} \right]$$

“sister” $\to$ BFKL, going through the cut

OPE $$\mathcal{W}^{\updownarrow}_{\text{hex}} = \sum_m (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \hat{\mu}^{\updownarrow}_{|m|}(p) e^{i\sigma p - \tau \check{E}_{|m|}(p)} + \dots$$

large τ

BFKL $$\mathcal{W}^{\updownarrow}_{\text{hex}} = e^{\frac{1}{2} i\pi(\sigma-\tau)\Gamma_{\text{cusp}}} \sum_m (-1)^m e^{im\phi} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} \hat{\mu}_{\text{BFKL}}(\nu, m) e^{i(\sigma-\tau)\nu} e^{(\sigma+\tau)\omega(m,\nu)} + \dots$$

large $\tau + \sigma$

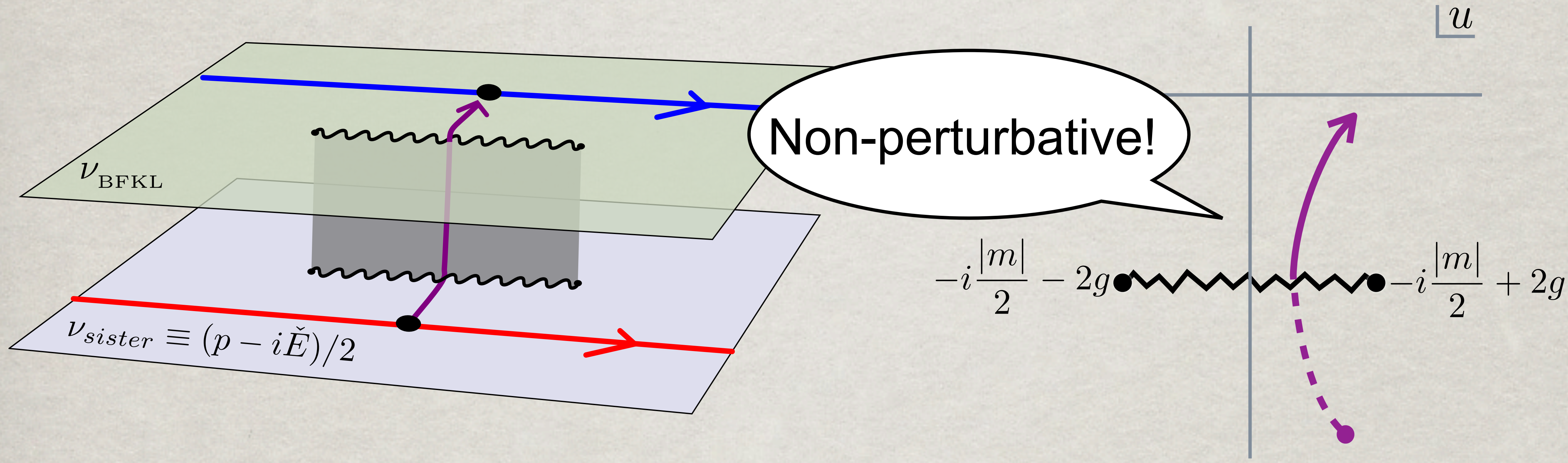
We take both τ **and** $\tau + \sigma$ large $\Longrightarrow$ Both expansions are dominated by the **same** saddle point

$$\frac{\sigma}{\tau} = \frac{1}{i} \frac{d\check{E}}{dp}(p_*) \qquad \qquad \omega(\nu_*) = \frac{i}{2}\left[p_* + i\check{E}(p_*)\right]$$

with

$$\frac{\tau-\sigma}{\tau+\sigma} = \frac{1}{i} \frac{d\omega}{d\nu}(\nu_*) \qquad \qquad \nu_* = \frac{1}{2}\left[p_* - i\check{E}(p_*)\right] - \frac{\pi}{2}\Gamma_{\text{cusp}}$$

“sister” $\to$ BFKL, going through the cut



$$\omega(u,m) = \int_0^\infty \frac{dt}{t}\left(K(t) - \frac{K(-t)+K(t)}{2}\cos(ut)e^{-|m|t/2}\right)$$

$$\nu(u,m) = 2u + \int_0^\infty \frac{dt}{t}\,\frac{K(-t)-K(t)}{2}\sin(ut)e^{-|m|t/2}$$

Also applies to $\mu_m^{\text{OPE}}(u) \quad \to \quad \mu_m^{\updownarrow}(u) \quad \to \quad \mu_m^{\text{BFKL}}(u)$

Checks

Weak coupling

$$-\omega(\nu,m) = 2g^2\left\{-\frac{2|m|}{\nu^2+m^2} + \psi\left(1+\frac{|m|+i\nu}{2}\right) + \psi\left(1+\frac{|m|-i\nu}{2}\right) - 2\psi(1)\right\} + O(g^4)$$

We match with all results in the literature - 3 loops + 4 loops prediction

[Bartels,Lipatov,Sabio Vera], [Fadin,Lipatov], [Dixon,Drummond,Duhr,Pennington]
[Papathanasiou]

Finite coupling

Vanishing of Regge expansion
in the collinear limit

$$\Rightarrow \quad \omega(\nu = \pm\frac{\pi}{2}\Gamma_{\mathrm{cusp}}, m=0) = 0 \quad \checkmark$$

[Caron-Huot]

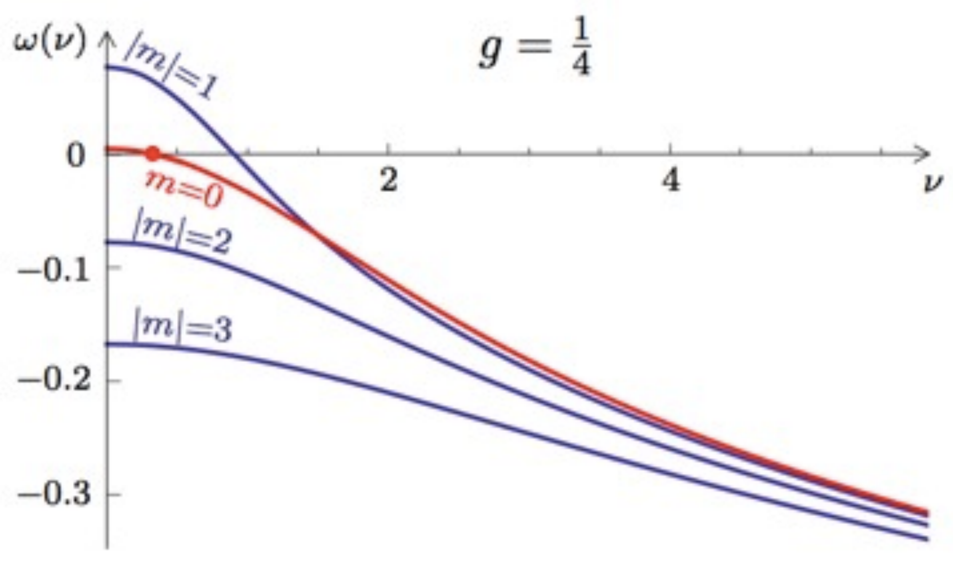
Strong coupling

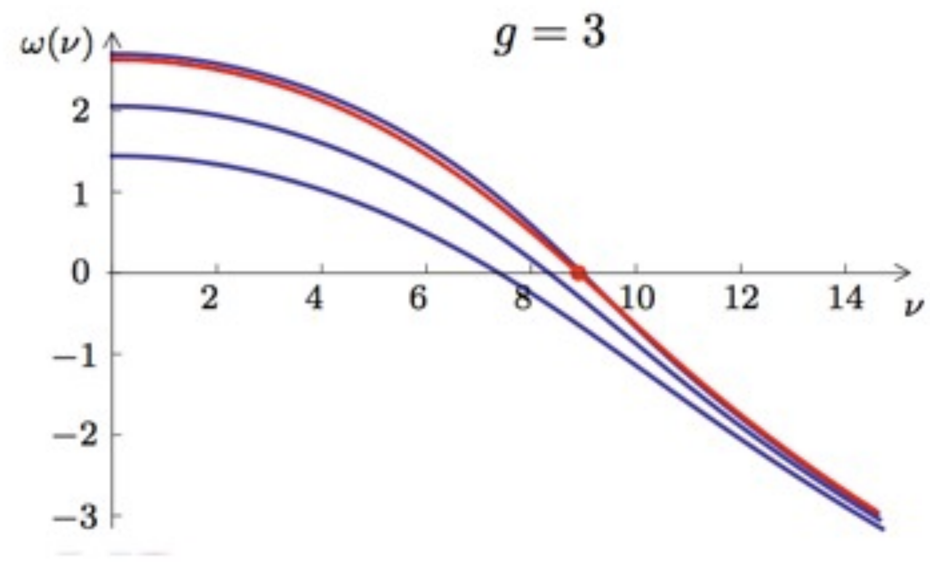
$$\omega(\nu) = \frac{\sqrt{\lambda}}{2\pi}(\sqrt{2} - \log(1+\sqrt{2})) + O(1)$$

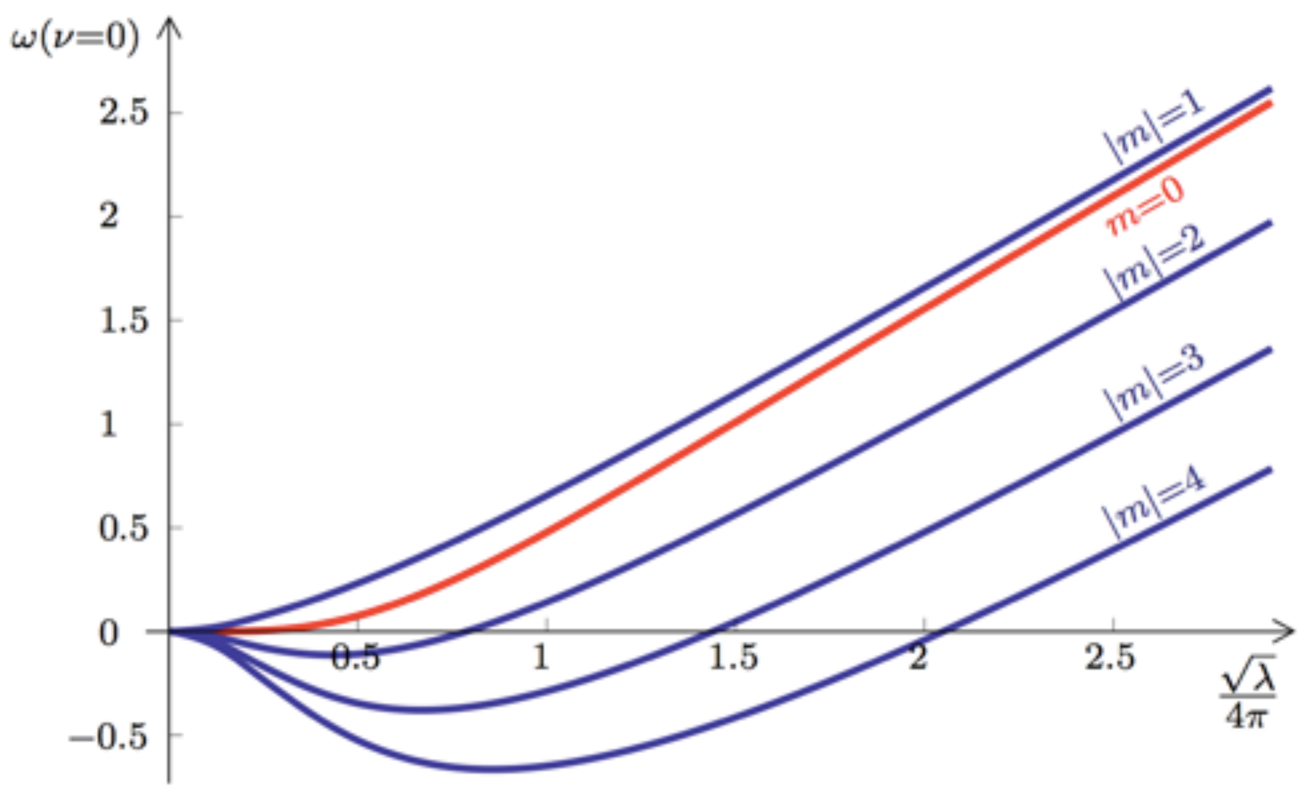
[Bartels,Kotanski,Schomerus,Sprenger]

$$-\frac{|m|-1}{\sqrt{2}} + O(1/\sqrt{\lambda})$$

Finite coupling plots







Scaling limit at strong coupling

Scale both ω and ν with $\sqrt{\lambda}$

$$\omega(\theta) = \frac{\sqrt{\lambda}}{4\pi}\left[\frac{2\sqrt{2}\cosh\theta}{\cosh(2\theta)} - \log\left(\frac{\sqrt{2}\cosh\theta+1}{\sqrt{2}\cosh\theta-1}\right)\right]$$

$$\nu(\theta) = \frac{\sqrt{\lambda}}{4\pi}\left[\frac{2\sqrt{2}\sinh\theta}{\cosh(2\theta)} - i\log\left(\frac{1+i\sqrt{2}\sinh\theta}{1-i\sqrt{2}\sinh\theta}\right)\right] - \frac{\sqrt{\lambda}}{4}$$

$\Rightarrow$ Classical solution!

Relation to giant hole

[Dorey,Losi]

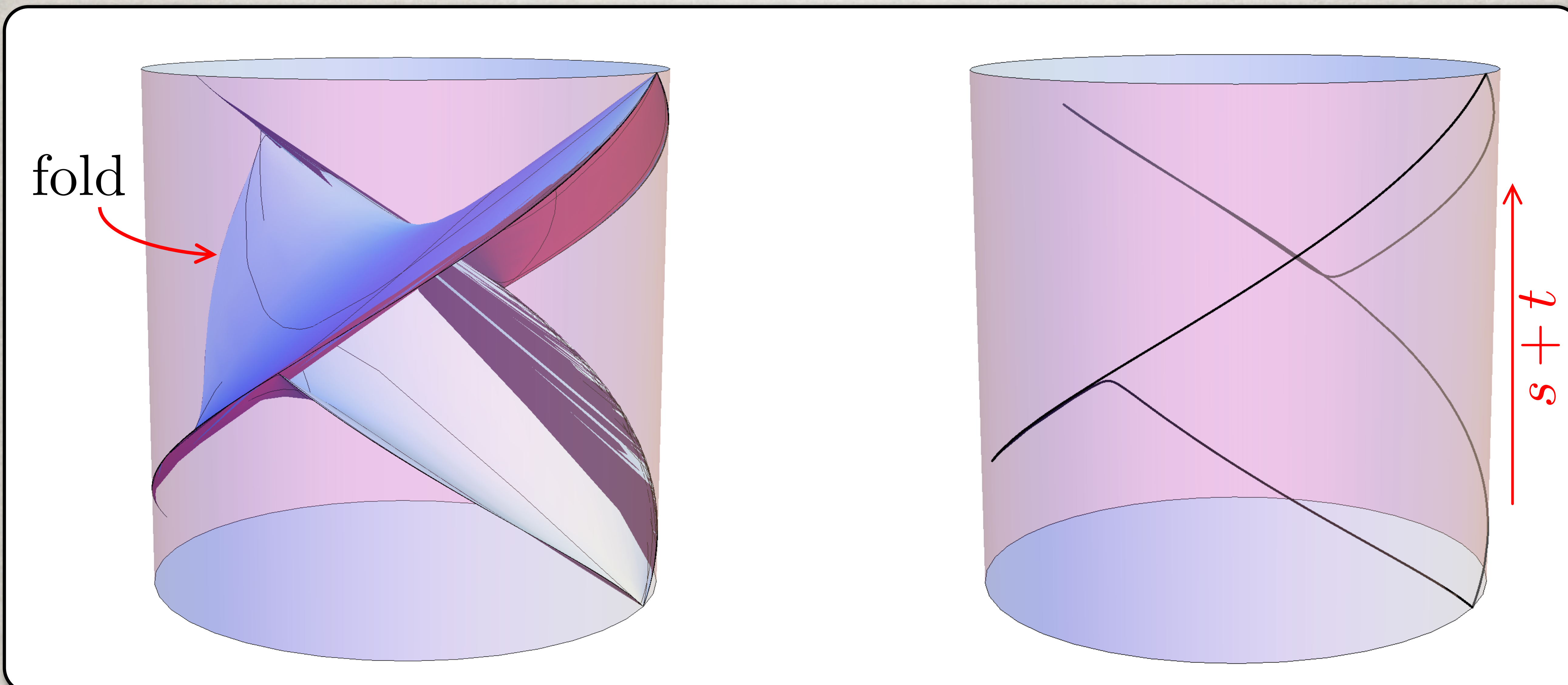
$$\omega(\theta) = -\frac{1}{2}\left[E_{\mathrm{giant}}(\theta+i\pi/4) - iP_{\mathrm{giant}}(\theta+i\pi/4)\right]$$

$$\nu(\theta) = -\frac{i}{2}\left[E_{\mathrm{giant}}(\theta+i\pi/4) + iP_{\mathrm{giant}}(\theta+i\pi/4)\right]$$

Half mirror shift

BFKL / sister solution is an analytic continuation of the giant hole!

Strong coupling classical solution



$$X_{-1} \pm X_2 = \pm e^{\pm t} \frac{e^{\Sigma}(\sqrt{1+v^2}\sinh s - \cosh s) + e^{-\Sigma}(v \mp \sqrt{1+v^2})\cosh s}{v\, e^{\Sigma} + e^{-\Sigma}}$$

[Dorey,Losi],[Sakai,Satoh]

$$X_1 \pm X_0 = \pm e^{\pm t} \frac{e^{\Sigma}(\sqrt{1+v^2}\cosh s - \sinh s) + e^{-\Sigma}(v \mp \sqrt{1+v^2})\sinh s}{v\, e^{\Sigma} + e^{-\Sigma}}$$

$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 = 1 \qquad \Sigma = \frac{s - v\,t}{\sqrt{1+v^2}} \qquad v = -i\tanh\theta$$

Open questions

- Extend to the power (energy) suppressed terms in the Regge limit

  $\Rightarrow$ full amplitude?

- Higher points (n>6)?

  Heptagon should follow from the pentagon transition.

  New states might be appearing for n=8?

  Can we get them from the OPE?

  Make connection with Bethe equations at strong coupling?

  [Bartels,Kotanski,Schomerus,Sprenger]

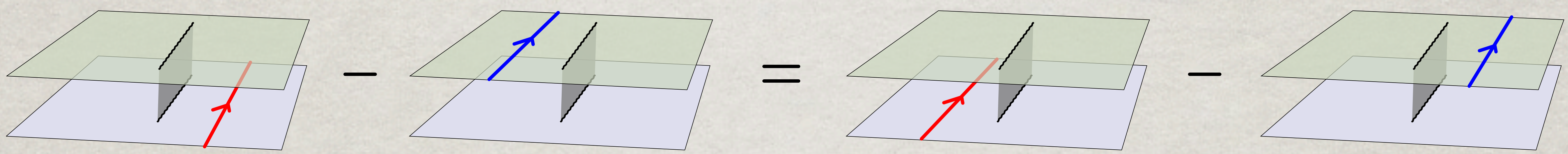
- What can we learn about color-singlet BKP/BFKL spectrum?

  Hint - highly excited BKP states are controlled by color-adjoint eigenvalues (analog of large spin for the spectrum)...

  Can it be used to write all-loop asymptotic Baxter equation for BKP spin chain?

Thank you!

Why we succeeded?



OPE $-$ BFKL $=$ energy suppressed $-$ twist suppressed