High-energy scattering in strongly coupled $\mathcal{N} = 4$ SYM

Martin Sprenger

IGST 2014

based on 1207.4204, 1311.1512, 1405.3658 and 1407.xxxx
with J. Bartels, J. Kotanski and V. Schomerus

DESY

Particles, Strings, and the Early Universe
Collaborative Research Center SFB 676

Motivation

- planar $\mathcal{N} = 4$ integrable $\leftrightarrow$ can compute observables for any coupling
- scattering amplitudes particularly interesting
  - functions of kinematical invariants
  - techniques for less symmetric theories
- enormous progress on weak coupling side
- how do amplitudes behave at strong coupling?
  - interpolation to intermediate coupling [$\rightarrow$ talk by A.Sever]

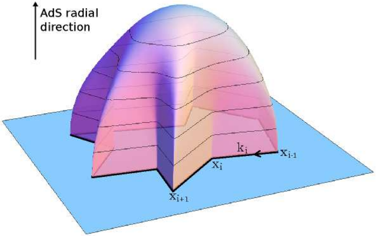
Main results

simple results in high-energy limit at strong coupling:

- MRL corresponds to IR limit of TBA
- $e^{R_6} \sim \left(-(1-u_1)\sqrt{\tilde{u}_2\tilde{u}_3}\right)^{\frac{\sqrt{\lambda}}{2\pi}e_2}$
- 7-point amplitude calculated $\rightarrow$ simple result
- correspondence: Regge cut contributions $\leftrightarrow$ excitations of TBA

Scattering Amplitudes via AdS/CFT

[Alday/Maldacena, A/M/Gaiotto, A/M/Sever/Vieira], [Basso/Sever/Vieira]



[Figure from 1002.2459]

- $A \sim e^{-\frac{\sqrt{\lambda}}{2\pi}\mathrm{Area}\left(+\frac{\sqrt{\lambda}}{48}\frac{(n-4)(n-5)}{n}\right)}$
- $k_i = x_{i-1} - x_i$
  $\rightarrow$ polygon depends only on $u = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$
- $Y_{a,s}(\theta)$ generalized cross ratios
  $\rightarrow$ for n gluons: $3n - 15$ Y-functions

$$\log Y_{a,s}(\theta) = - \; m_s \; \cosh\theta \pm \; C_s + \sum_{a',s'} \int d\theta' \; \mathcal{K}_{s,s'}^{a,a'} \; \left(\theta - \theta' + \; i\varphi_s - i\varphi_{s'} \;\right) \log\left(1 + Y_{a',s'}(\theta')\right)$$

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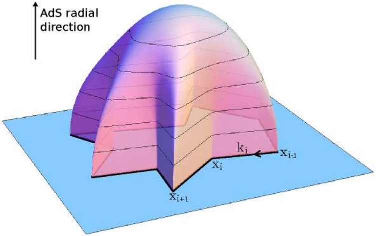

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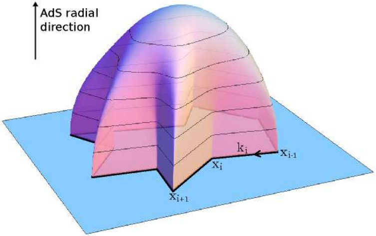

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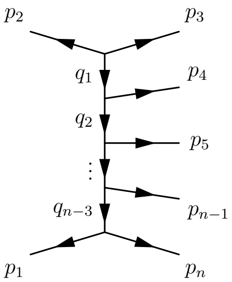
Y-System

$$\text{Area} = \text{A}_{\text{div}}(x_i) + \text{A}_{\text{periods}}(m_s, \varphi_s) + \Delta(u_i)$$
$$+ \sum_s \int \frac{d\theta}{2\pi} |m_s| \cosh\theta \log\left[(1+Y_{1,s})(1+Y_{3,s})(1+Y_{2,s})^{\sqrt{2}}\right](\theta)$$

- non-divergent piece: remainder function e^R
- Y-system easy to solve numerically
- BUT: not solvable analytically for arbitrary kinematics!

Multi-Regge limit



- for $2 \to n-2$ scattering: $3n-10$ Mandelstam invariants
- Multi-Regge limit: rapidities of produced particles strongly ordered
  $\to$ hierarchy for $s_{i\cdots j} := (p_i + \cdots + p_j)^2$

$$s \gg s_{3\cdots n-1}, s_{4\cdots n} \gg s_{3\cdots n-2}, \dots, s_{5\cdots n} \gg \cdots \gg s_{34}, \dots \gg -t_1, \dots, -t_{n-3}$$

- $\mathcal{N}=4$ dual conformal $\to$ choose $3n-15$ cross ratios u_{as}
- kinematical analysis:
  $u_{1s} \to 1,\ u_{2s}, u_{3s} \to 0$ with $\tilde{u}_{2s} = \frac{u_{2s}}{1-u_{1s}} = \mathcal{O}(1), \quad \tilde{u}_{3s} = \frac{u_{3s}}{1-u_{1s}} = \mathcal{O}(1)$

Y-system in the MRL

[1207.4204]

- $u_{as} = \frac{Y_{2s}}{1+Y_{2s}}\Big|_{\theta=i(k\pi/4-\varphi_s)}$
- demand that cross ratios show behavior predicted by MRL

Example: 6-point case

$u_2 \to 0 \Rightarrow Y_2\left(\theta = i\frac{\pi}{4}\right) \overset{!}{\to} 0$

$$\log Y_2\left(\theta = i\frac{\pi}{4}\right) = -\sqrt{2}m\cos\left(\frac{\pi}{4}-\varphi\right) + \sum_{s'}\int d\theta' \mathcal{K}\left(\theta-\theta'\right)\log\underbrace{\left(1+Y_{s'}\left(\theta'\right)\right)}_{\cong 1+e^{-m_{s'}\theta'}}$$

Y-system in the MRL

[1207.4204]

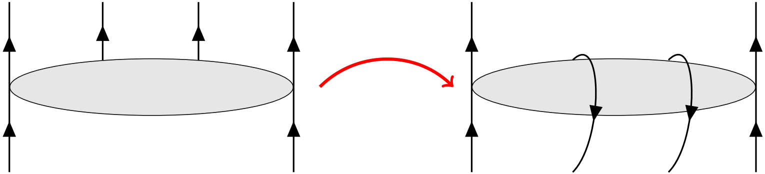
- MRL realized for choice m_s large, $\varphi_s \to -(s-1)\frac{\pi}{4}$, $C_s \to$ const.
- in this limit, integrals in Y-system can be neglected

$$\boxed{\log Y_{a,s}(\theta) \cong -m_s \cosh\theta \pm C_s + \mathcal{O}(e^{-m})}$$

- $\to$ MRL corresponds to IR limit of TBA
- but: in this limit R trivial
- need to generalize Regge limit $\to$ Regge regions

Multi-Regge regions

- 2^{n-4} regions, corresponding to the signs of E_i
- different regions connected by analytic continuation in $s_i \to s_i e^{i\alpha}$



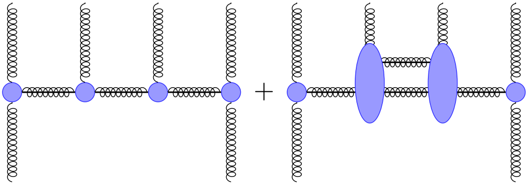
- for the above example $P_{6,--}$:
  $s_{34} \to e^{i\pi} s_{34} \quad s_{56} \to e^{i\pi} s_{56}, \quad s_{345} \to e^{i\pi} s_{345}, \quad s_{456} \to e^{i\pi} s_{456}$
  $$\Rightarrow u_1 \to e^{-2\pi i} u_1, \quad u_2 \to u_2, \quad u_3 \to u_3$$
- probe analytic structure of amplitude

MRL at weak coupling - 6-points

[Bartels/Lipatov/Sabio Vera, Lipatov/Prygarin, Fadin/Lipatov]

- R contains cuts $\Rightarrow$ MRL depends on Regge region:
  - $R_{6,++}$ vanishes in MRL
  - Regge cut appears in Regge region $R_{6,--}$

$$e^{R_6+i\pi\delta}\big|_{\mathrm{MRL}} = \cos\omega_{ab} + i\frac{\lambda}{2}\sum_n \int \frac{d\nu}{\nu^2+\frac{n^2}{4}} |w|^{2i\nu}\, \Phi_{\mathrm{Reg}}(\nu,n) \left(-(1-u_1)\sqrt{\tilde{u}_2\tilde{u}_3}\right)^{-\omega(\nu,n)}$$

$e^{R_6+i\pi\delta}\big|_{\mathrm{MRL}}$ '=' 

- universal building blocks: BFKL eigenvalue $\omega(\nu,n)$, impact factor $\Phi_{\mathrm{Reg}}(\nu,n)$

Multi-Regge regions in the Y-system

- continuation in u_{as} $\sim$ continuation in $m, C, \varphi \to$ [Dorey/Tateo]
- (numerical) inversion of $u_{as} = \frac{Y_{2s}}{1+Y_{2s}}\Big|_{\theta=i(k\pi/4-\varphi_s)}$ to find paths for parameters
- solutions of $Y_{as}(\theta) = -1$ can cross real axis

$$\log Y_{a,s}(\theta) = - m_s \cosh\theta \pm C_s + \sum_{a',s'} \int d\theta' \mathcal{K}_{s,s'}^{a,a'}(\theta - \theta' + i\varphi_s - i\varphi_{s'}) \log(1 + Y_{a',s'}(\theta'))$$

- crossing leads to contributions to R

$$R' = \int \frac{d\theta}{2\pi} |m_s| \cosh\theta \log[(1 + Y_{a,s}(\theta))] \pm i |m_s| \sinh(\theta_0) + \ldots$$

- endpoint explicitly enters R

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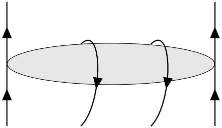
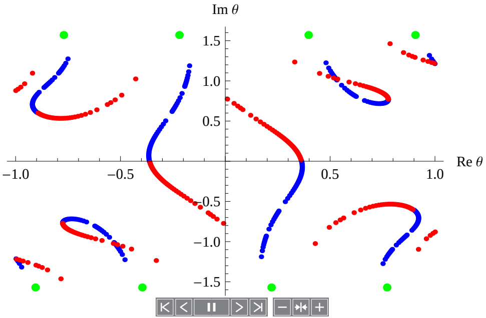
Example: 6-point case

[Bartels/Kotanski/Schomerus, 1311.1512]



$$u_1 \to e^{-2\pi i} u_1,\; u_2 \to u_2,\; u_3 \to u_3$$



Solutions of $Y_3(\theta) = -1$ along continuation

Determination of endpoints

- endpoints of crossed solutions can be determined analytically

Endpoint condition

$$-1 = Y_3'(\theta_+) = e^{-m' \cosh(\theta_+) + C'} \cdot \frac{\mathcal{S}_3(\theta_+, \theta_-)}{\mathcal{S}_3(\theta_+, \theta_+)}$$

- for every Regge region get set of BAE which determines remainder function
- numerical input just provides discrete information on crossing

6-point result

- Remainder function has Regge behavior:

$$e^{R_6+i\pi\delta} \sim \left(-(1-u_1)\sqrt{\tilde{u}_2\tilde{u}_3}\right)^{\frac{\sqrt{\lambda}}{2\pi}e_2}$$

- $e_2 = -\sqrt{2} + \frac{1}{2}\log\left(3+2\sqrt{2}\right) \sim -.533 < 0$
- $\delta = \frac{1}{4}\gamma_K \log\sqrt{\tilde{u}_2\tilde{u}_3}$

- weak coupling:

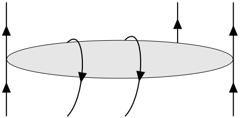
$$e^{R_6+i\pi\delta}\big|_{\mathrm{MRL}} = i\frac{\lambda}{2}\sum_n \int \frac{d\nu}{\nu^2+\frac{n^2}{4}} |w|^{2i\nu}\, \Phi_{\mathrm{Reg}}(\nu,n) \left(-(1-u_1)\sqrt{\tilde{u}_2\tilde{u}_3}\right)^{-\omega(\nu,n)} + \dots$$

- dominant saddle point at strong coupling? [$\rightarrow$ talk by A.Sever]
- cut contribution @ weak coupling $\leftrightarrow$ crossing solution @ strong coupling

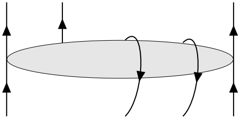
7-point amplitude, predictions from weak coupling

[Bartels/Kormilitzin/Lipatov]

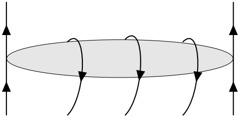
- 7-point cut from two reggeized gluons, as in 6-point case
- predictions obtained from analysis of Regge factorization of BDS ansatz



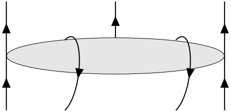
$\rightarrow$ short cut in $1 - u_{11}$



$\rightarrow$ short cut in $1 - u_{12}$

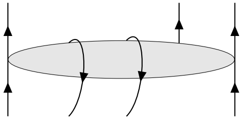


$\rightarrow$ long cut in $(1 - u_{11})(1 - u_{12})$



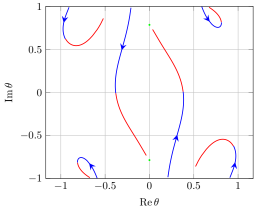
$\rightarrow$ all three cuts contribute

7-point amplitude, paths with short cut



$$u_{11} \to e^{-2i\pi} u_{11},\ u_{21} \to u_{21}, \qquad u_{31} \to u_{31},$$
$$u_{12} \to u_{12}, \qquad u_{22} \to e^{-i\pi} u_{22},\ u_{32} \to e^{i\pi} u_{32}$$

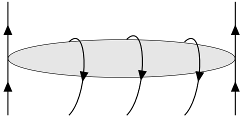
- one pair of crossing solutions
- analogously for mirrored path
- calculation different, we still find:



Crossing solutions of $Y_{32}(\theta) = -1$

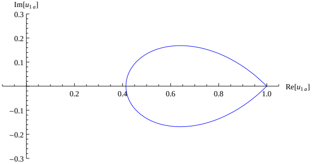
$$R_{7,--+}(u_{as}) = R_6(u_{11}, u_{21}, u_{31})$$
$$R_{7,+--}(u_{as}) = R_6(u_{12}, u_{22}, u_{32})$$

7-point amplitude, paths with long cut



$$u_{11} \to u_{11},\ u_{21} \to u_{21},\ u_{31} \to u_{31},$$
$$u_{12} \to u_{12},\ u_{22} \to u_{22},\ u_{32} \to u_{32}$$

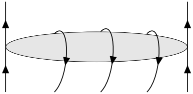
- for 7 points, we get dependent cross ratio $\tilde{u}$
- from kinematics for above path: $\tilde{u} \to e^{-2\pi i}\tilde{u}$
- in conflict with Gram relation: $\tilde{u} \cong \frac{u_{11}+u_{12}-1}{u_{11}u_{12}}$



Deformed path for u_{1a}

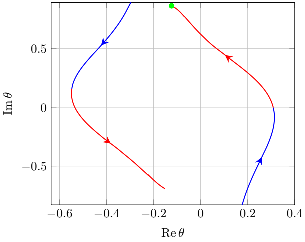
- deform path s.th. winding numbers of cross ratios are preserved
- $u_{11} \to e^{2i\pi}\left(1-\sqrt{1-e^{-2i\pi}}\right)u_{11},\ u_{12} \to e^{2i\pi}\left(1-\sqrt{1-e^{-2i\pi}}\right)u_{12}$

7-point amplitude, paths with long cut



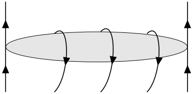
$$u_{11} \to e^{2i\pi}\left(1-\sqrt{1-e^{-2i\pi}}\right)u_{11},\; u_{21} \to u_{21},\; u_{31} \to u_{31},$$
$$u_{12} \to e^{2i\pi}\left(1-\sqrt{1-e^{-2i\pi}}\right)u_{12},\; u_{22} \to u_{22},\; u_{32} \to u_{32}$$

- two pairs of crossing solutions
- both approach same endpoints $\to \theta_\pm = \pm i\frac{\pi}{4}$



Crossing solutions of $Y_{31}(\theta) = -1$

$$\boxed{R_{7,---}(u_{as}) = R_6(u_{11}, u_{21}, u_{31}) + R_6(u_{12}, u_{22}, u_{32})}$$



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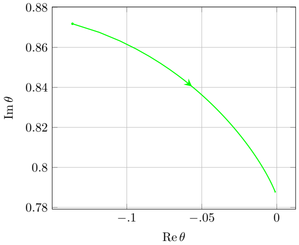
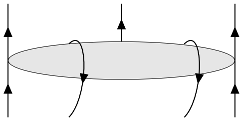


Convergence against endpoint

$$R_{7,---}(u_{as}) = R_6(u_{11}, u_{21}, u_{31}) + R_6(u_{12}, u_{22}, u_{32})$$

7-point amplitude, paths with long cut



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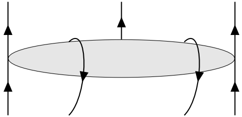
- for this path $\tilde{u} \to e^{-2i\pi} \tilde{u}$, no deformation needed
- no evidence for crossing solutions
  $\Rightarrow R_{7,-+-}(u_{as})$ trivial up to phase
- comparison with weak coupling prediction: possible cancellation?
- path too naive?

7-point amplitude, paths with long cut



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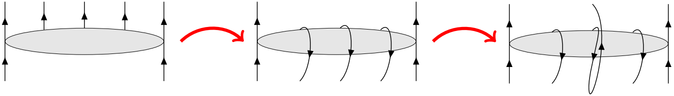

Summary and Outlook

- studied scattering amplitudes in strongly coupled $\mathcal{N} = 4$ SYM
- identified MRL and showed that Y-system equations simplify in MRL
- calculated 6-point and 7-point remainder function
  $\rightarrow$ consistency with weak coupling predictions
- correspondence: Regge cut contributions $\leftrightarrow$ crossing solutions

next steps:

- understand $R_{7,-+-}$: path too naive?
- weak coupling prediction:
  3 Reggeon contribution for 8 points
- make contact with weak coupling results