# $\mathcal{N}=4$ scattering amplitudes and the deformed Graßmannian

Tomasz Łukowski

*Mathematical Institute, University of Oxford*

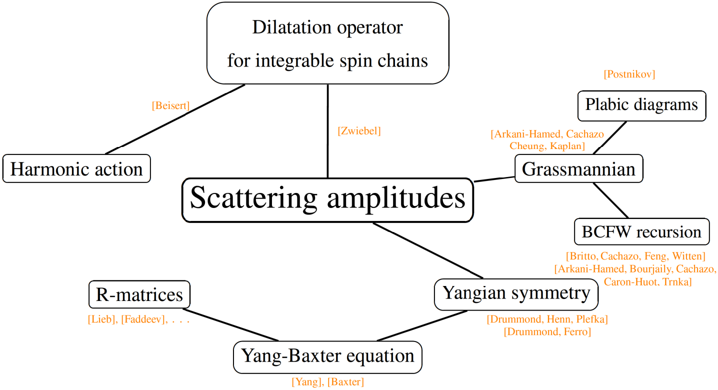
**Integrability in Gauge and String Theory 2014**

Hamburg, 17.07.2014

Livia Ferro, TŁ and Matthias Staudacher – arXiv:1407.xxxx

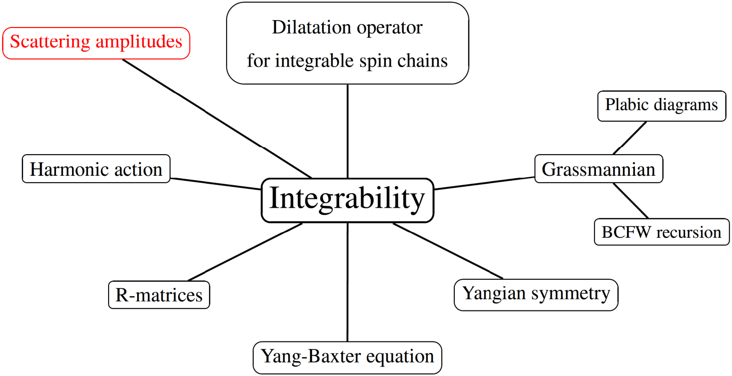
# Outline

## Outline



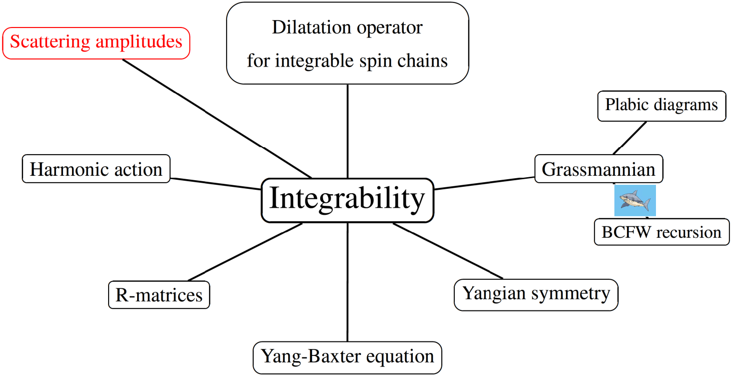
# Outline

## Outline



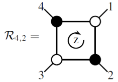
# Outline

## Outline

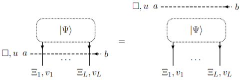


# Zürich → Hamburg

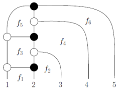
- Original idea
  - Ferro, TŁ, Meneghelli, Plefka, Staudacher – 1212.0850
  - Ferro, TŁ, Meneghelli, Plefka, Staudacher – 1308.3494



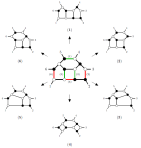

- Algebraic Bethe Ansatz
  - Chicherin, Derkachov, Kirschner – 1309.5748
  - Frassek, Kanning, Ko, Staudacher – 1312.1693




- General construction of Yangian invariants
  - Kanning, TŁ, Staudacher – 1403.3382
  - Brödel, de Leeuw, Rosso – 1403.3670




- Clash with the BCFW recursion relation
  - Beisert, Brödel, Rosso – 1401.7274
  - Brödel, de Leeuw, Rosso – 1406.4024



# Amplitudes in $\mathcal{N} = 4$ SYM

We consider color-ordered scattering amplitudes of superfields

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\,\overline{\Gamma^D} + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD}\, G^-$$

The amplitudes $\mathcal{A}_{n,k}$ are labeled by two numbers:

- number of particles – $n$
- MHV level – $\eta^{4k}$, $\quad k = 2, \ldots n-2\,,$

$$\mathcal{A}_n = \mathcal{A}_{n,2} + \eta^4 \mathcal{A}_{n,3} + \ldots + \eta^{4n-8} \mathcal{A}_{n,n-2}$$

All particles are massless: $p^2 = 0 \Rightarrow p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$.

On-shell superspace – $\Lambda^{\mathcal{A}} = (\lambda^\alpha, \tilde{\lambda}^{\dot{\alpha}}, \eta^A)$

The simplest non-trivial examples are MHV amplitudes – Parke-Taylor formula for the tree-level: [Parke, Taylor]

$$\mathcal{A}_{n,2} = \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}\,, \qquad Q^{\alpha A} = \sum_i \lambda_i^\alpha \eta_i^A\,, P^{\alpha\dot{\alpha}} = \sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}\,, \langle ij\rangle = \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta$$

# Graßmannian integrals

**Remarkable language to express amplitudes $\longrightarrow$ integrals over Graßmannian spaces**

A Graßmannian space $\mathrm{G}(k,n)$:

- set of $k$-planes intersecting the origin of an $n$-dimensional space
- coordinates on $\mathrm{G}(k,n)$ are packaged into a $k \times n$ matrix $C = (c_{ai})$
- matrices $C$ and $A \cdot C$ with $A \in \mathrm{GL}(k)$ correspond to the same point in $\mathrm{G}(k,n)$.

Build super-twistors $\mathcal{W}_j^{\mathcal{A}} = (\tilde{\mu}_j^{\alpha}, \tilde{\lambda}_j^{\dot{\alpha}}, \eta_j^{A})$ with Fourier conjugates $\lambda_j^{\alpha} \to \tilde{\mu}_j^{\alpha}$.

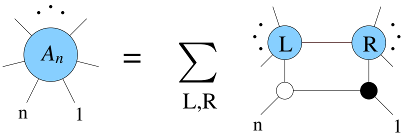
**Graßmannian integral formulation of tree-level amplitudes** [Arkani-Hamed, Cachazo, Cheung, Kaplan]

$$\mathcal{A}_{n,k} = \int \frac{d^{k\cdot n}C}{\mathrm{vol}(\mathrm{GL}(k))} \frac{\delta^{4k|4k}(C\cdot\mathcal{W})}{(1\ldots k)(2\ldots k+1)\ldots(n\ldots n+k-1)}$$

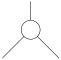
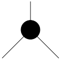
- The $(i\,i+1\ldots i+k-1)$ are the $n$ cyclic $k \times k$ minors of $C$.
- Integration is along „suitable contours".

# On-shell diagrams and the positive Graßmannian

- The integral is a sum over residues, the correct sum is given by the BCFW recursion relation [Britto, Cachazo, Feng, Witten]




- All amplitudes can be written using only two objects:

$A_{3,1}$ =  $A_{3,2}$ = 

- Each BCFW term is an **on-shell diagram** – graph with black and white vertices. [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov Trnka]
- All such graphs are classified by **permutations**. [Postnikov]
- Gives a map to the **positive** part of Graßmannian – broad topic in mathematics.
- New idea for trees and loop integrands – **amplituhedron**. [Arkani-Hamed, Trnka]

# Symmetries

- The tree-level amplitudes enjoy $\mathcal{N} = 4$ superconformal symmetry [Witten]

$$J^{\mathcal{AB}} \cdot \mathcal{A}_{n,k} = 0\,, \qquad \text{with} \qquad J^{\mathcal{AB}} \in \mathfrak{psu}(2,2|4)$$

- However, there is also a non-local dual superconformal symmetry [Drummond, Henn, Korchemsky, Sokatchev]

$$\tilde{J}^{\mathcal{AB}} \cdot \mathcal{A}_{n,k} = 0\,, \qquad \text{with} \qquad \tilde{J}^{\mathcal{AB}} \in \mathfrak{psu}(2,2|4)$$

- Commuting $J$ and $\tilde{J}$, one obtains Yangian symmetry. [Drummond, Henn, Plefka]
- With local generators $J_j^{\mathcal{AB}} = \mathcal{W}_j^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}}$, where $\mathcal{W}_j^{\mathcal{A}}$ are super-twistors, we succinctly express Yangian algebra generators as

$$J^{\mathcal{AB}} = \sum_{j=1}^{n} J_j^{\mathcal{AB}}\,, \qquad \hat{J}^{\mathcal{AB}} = \frac{1}{2} \sum_{i<j} \left( J_i^{\mathcal{AC}} J_j^{\mathcal{CB}} - J_j^{\mathcal{AC}} J_i^{\mathcal{CB}} \right)$$

This is how integrability first appeared in the planar scattering problem!

# Dual Graßmannian integrals

- Planar amplitudes in $\mathcal{N} = 4$ SYM are dual to polygonal light-like Wilson loops. Dual superconformal symmetry is the ordinary superconformal symmetry in the position space of Wilson loops.
- In the dual description one can employ $4|4$ super momentum-twistors $\mathcal{Z}_j$. [Hodges]

With $\hat{k} = k - 2$, there is an equivalent dual description on $\mathrm{G}(\hat{k}, n)$ [Mason, Skinner; Arkani Hamed et. al.]

$$\mathcal{A}_{n,k} = \frac{\delta^4(P_{\alpha\dot{\alpha}})\delta^8(Q^A_\alpha)}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} \int \frac{d^{\hat{k}\cdot n}\hat{C}}{\mathrm{vol}(\mathrm{GL}(\hat{k}))} \frac{\delta^{4\hat{k}|4\hat{k}}(\hat{C}\cdot\mathcal{Z})}{(1\ldots\hat{k})(2\ldots\hat{k}+1)\ldots(n\ldots n+\hat{k}-1)}$$

with the MHV part factored out.

# Deformed symmetries

[Ferro, TŁ, Meneghelli, Plefka, Staudacher]

Of particular interest is the central charge generator of $\mathfrak{gl}(4|4)$

$$C = \sum_{j=1}^{n} c_j \qquad \text{with} \qquad c_j = \lambda_j^\alpha \frac{\partial}{\partial \lambda_j^\alpha} - \tilde{\lambda}_j^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{\alpha}}} - \eta_j^A \frac{\partial}{\partial \eta_j^A} + 2$$

For overall $\mathfrak{psu}(2,2|4)$ we have $C = 0$. Locally we can relax the condition $c_j = 0$.

**Physically**, this deforms the super helicities $h_j = 1 - \frac{1}{2} c_j$.

**Mathematically**, this yields something well-known – the Yangian in evaluation representation. Deforming the $c_j$ switches on the parameters $v_j$

$$J^{\mathcal{AB}} = \sum_{j=1}^{n} J_j^{\mathcal{AB}}, \qquad \hat{J}^{\mathcal{AB}} = \frac{1}{2} \sum_{i<j} \left( J_i^{\mathcal{AC}} J_j^{\mathcal{CB}} - J_j^{\mathcal{AC}} J_i^{\mathcal{CB}} \right) - \sum_{j=1}^{n} v_j J_j^{\mathcal{AB}}$$

# Deformed Graßmannian integrals

[Ferro, TŁ, Staudacher, in preparation]

How are the Graßmannian contour formulas deformed?

The final answer is exceedingly simple and very natural. Define

[Beisert, Brödel, Rosso]

$$v_j^{\pm} = v_j \pm \frac{c_j}{2}$$

Requiring Yangian invariance, we find, with $v_{j+k}^{+} = v_j^{-}$ for $j = 1, \ldots, n$

$$\int \frac{d^{k\cdot n}C}{\text{vol(GL(k))}} \frac{\delta^{4k|4k}(C\cdot\mathcal{W})}{(1,\,\ldots,k)^{1+v_k^{+}-v_1^{-}}\ldots(n,\,\ldots,k-1)^{1+v_{k-1}^{+}-v_n^{-}}}\,.$$

Note that it is not really the Graßmannian space $\text{Gr}(k,n)$ as such that is deformed, but the integration measure on this space. $\text{GL}(k)$ invariance of the integral is preserved!

# Deformed Dual Graßmannian integrals

How are the dual Graßmannian integrals deformed?

Using the parameters $v_j^{\pm}$ we found

$$\frac{\delta^4(P_{\alpha\dot{\alpha}})\delta^8(Q_\alpha^A)}{\langle 12\rangle^{1+v_2^+-v_1^-}\dots\langle n1\rangle^{1+v_1^+-v_n^-}}\times$$

$$\int\frac{d^{\hat{k}\cdot n}\hat{C}}{\mathrm{vol}(\mathrm{GL}(\hat{k}))}\frac{\delta^{4\hat{k}|4\hat{k}}(\hat{C}\cdot\mathcal{Z})}{(1,\,\dots,\hat{k})^{1+v_{\hat{k}+1}^+-v_n^-}\dots(n,\,\dots,\hat{k}-1)^{1+v_{\hat{k}}^+-v_{n-1}^-}}\,.$$

The number of deformation parameters equals $n-1$ since

$$v_{j+k}^+=v_j^-$$

Note that both the MHV-prefactor and the contour integral are deformed.

# Why is it interesting?

**Why should we consider this deformation?** Here are some of the reasons:

- It is very natural from the point of view of integrability.
- In fact, constructing amplitudes by integrability (arguably) requires it.
- Amplitudes are related to the spectral problem, where it is indispensable.
- **Most importantly**: It promises to provide a natural infrared regulator!

The last point was our original motivation. Interestingly, we recently learned that this deformation had been already studied as an infrared regulator in twistor theory in the early seventies by Penrose and Hodges. What is even more interesting, it was already necessary for tree-level amplitudes. This solves some conceptual problem we had with our original deformation.

# Meromorphicity lost and gained

Let us take another look at the deformed Graßmannian contour integral:

$$\int \frac{d^{k\cdot n}C}{\text{vol}(\text{GL(k)})} \frac{\delta^{4k|4k}(C\cdot\mathcal{W})}{(1,\,\ldots,k)^{1+v_k^+-v_1^-}\ldots(n,\,\ldots,k-1)^{1+v_{k-1}^+-v_n^-}}\,.$$

- Choosing the parameters $v_j^\pm$ to be non-integer, we see that the poles in the variables $c_{aj}$ generically turn into branch points.
- **Important point**: We can no longer use the BCFW recursion relations, as they are based on the residue theorem, which does not apply anymore.
- What we can hope to gain is complete meromorphicity in suitable combinations of the deformation parameters $v_j^\pm$. Our ultimate hope is that this will fix the contours uniquely.

# A toy meromorphicity experiment

Consider Euler's first integral, the beta function $B(v_1, v_2)$

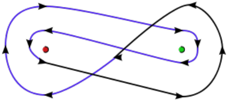
$$\int_0^1 dc \frac{1}{c^{1-v_1}(1-c)^{1-v_2}}$$

- For $v_1, v_2 \in \mathbb{N}$ Euler derived $\frac{(v_1-1)!(v_2-1)!}{(v_1+v_2-1)!}$.
- The analytic continuation for arbitrary $v_1, v_2 \in \mathbb{C}$ is $\frac{\Gamma(v_1)\Gamma(v_2)}{\Gamma(v_1+v_2)}$.
- Meromorphic in both $v_1$ and $v_2$: not obvious from the integral.

This problem was solved by Pochhammer

$$\frac{1}{(1-e^{2\pi i v_1})(1-e^{2\pi i v_2})} \int_{\mathcal{C}} dc \frac{1}{c^{1-v_1}(1-c)^{1-v_2}}$$

where the contour $\mathcal{C}$ goes at least two times through the cut



[Wikipedia]

# A trivial example – $\overline{\text{MHV}}_5 \equiv \text{NMHV}_5$

All $\text{MHV}_n$ and $\overline{\text{MHV}}_n$ amplitudes have been already successfully **deformed**:

[Ferro, TŁ, Meneghelli, Plefka, Staudacher]

$$\mathcal{A}_{n,2} = \frac{\delta^4(P_{\alpha\dot\alpha})\delta^8(Q^A_\alpha)}{\langle 12\rangle^{1+v_2^+-v_1^-}\ldots\langle n1\rangle^{1+v_1^+-v_n^-}}, \qquad \mathcal{A}_{n,n-2} = \frac{\delta^4(P_{\alpha\dot\alpha})\delta^8(Q^A_\alpha)}{[12]^{1-v_2^++v_1^-}\ldots[n1]^{1-v_1^++v_n^-}}$$

In the **momentum twistor space** $\overline{\text{MHV}}_n$ are non-trivial.

Deformation of the $\overline{\text{MHV}}_5$ amplitude:

$$\frac{\delta^{0|4}(\langle 1234\rangle\eta_5 + \langle 5123\rangle\eta_4 + \langle 4512\rangle\eta_3 + \langle 3451\rangle\eta_2 + \langle 2345\rangle\eta_1)}{\langle 1234\rangle^{1+v_1^+-v_4^-}\langle 5123\rangle^{1+v_5^+-v_3^-}\langle 4512\rangle^{1+v_4^+-v_2^-}\langle 3451\rangle^{1+v_3^+-v_1^-}\langle 2345\rangle^{1+v_2^+-v_5^-}}.$$

$$\langle ijkl\rangle = \epsilon^{ijkl}_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D$$

This is a spectral parameter deformation of the **R-invariant**

$$[ijklm] = \frac{\delta^{0|4}(\langle ijkl\rangle\eta_m + \langle jklm\rangle\eta_i + \langle klmi\rangle\eta_j + \langle lmij\rangle\eta_k + \langle mijk\rangle\eta_l)}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}.$$

# A non-trivial example - NMHV$_6$

- Before deformation

$$A_{6,3} = \frac{1}{2}\left([12345] + [23456] + [34561] + [45612] + [56123] + [61234]\right)$$

- Each term of the sum is a **residue** of the Graßmannian integral

$$\int \frac{dc_2 dc_3 dc_4 dc_5 dc_6}{c_2\, c_3\, c_4\, c_5\, c_6}\, \delta^{4|4}(\mathcal{Z}_1 + c_2 \mathcal{Z}_2 + \ldots + c_6 \mathcal{Z}_6)\,,$$

- With deformation

$$\int \frac{dc_2 dc_3 dc_4 dc_5 dc_6}{c_2^{1-\alpha_2} c_3^{1-\alpha_3} c_4^{1-\alpha_4} c_5^{1-\alpha_5} c_6^{1-\alpha_6}}\, \delta^{4|4}(\mathcal{Z}_1 + c_2 \mathcal{Z}_2 + \ldots + c_6 \mathcal{Z}_6)\,,$$

  with $\alpha_i = v_{i-1}^- - v_{i+1}^+$.

- After saturating $\delta$-functions

$$\int d\tau\, \tau^{\alpha_6 - 1}(1-\tau)^{\alpha_5 - 1} \prod_{i=2}^{4} (1 - z_i\, \tau)^{\alpha_i - 1} P(\tau, \eta)$$

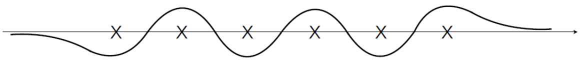
  $P(\tau, \eta)$ is a polynomial in $\tau$ and fermionic variables $\eta$, and $z_2 = \frac{\langle 1234\rangle\langle 6345\rangle}{\langle 6234\rangle\langle 1345\rangle}, \ldots$

# A non-trivial example continued

- This takes the form of the Lauricella $F_D$ hypergeometric function.
- Possible contour in $(2,2)$ signature: $\tau \in (0,1)$

$$\frac{1}{\alpha_6}\frac{\delta^{0|4}(\langle 1234\rangle\eta_5+\langle 5123\rangle\eta_4+\langle 4512\rangle\eta_3+\langle 3451\rangle\eta_2+\langle 2345\rangle\eta_1)}{\langle 2345\rangle^{1-\alpha_1}\langle 3451\rangle^{1-\alpha_2}\langle 4512\rangle^{1-\alpha_3}\langle 5123\rangle^{1-\alpha_4}\langle 1234\rangle^{1-\alpha_5}}$$
$$+\frac{1}{\alpha_5}\frac{\delta^{0|4}(\langle 1234\rangle\eta_6+\langle 6123\rangle\eta_4+\langle 4612\rangle\eta_3+\langle 3461\rangle\eta_2+\langle 2346\rangle\eta_1)}{\langle 2346\rangle^{1-\alpha_1}\langle 3461\rangle^{1-\alpha_2}\langle 4612\rangle^{1-\alpha_3}\langle 6123\rangle^{1-\alpha_4}\langle 1234\rangle^{1-\alpha_6}}+\mathcal{O}(1)$$

  Deformed R-invariants show up as residues in the $\alpha_i$ space!
- How to integrate to find the complete amplitude? In the **non-deformed** case:



- With **deformation** – the same contour?

$$\frac{1}{2\pi i}\left(\frac{e^{i\pi\alpha_6}-1}{\alpha_6}[12345]_{\text{deformed}}+\text{cyclic}\right)+\mathcal{O}(\alpha_i)$$

  Currently under further investigation.

# Further directions

- Gelfand hypergeometric functions – Yangian invariance for the NMHV amplitudes is equivalent to the Gelfand hypergeometric differential equation.
- Yangian invariance can be rewritten as a eigenvector problem of the monodromy matrix

$$M(u)\mathcal{A}_{n,k} = \mathcal{A}_{n,k}$$

  This can be seen as the $p \to 0$ limit of the quantum Knizhnik-Zamolodchikov equation.

# Outlook

**For amplitudes:**

- Work out general deformed tree-level amplitudes explicitly.
- Exciting relations to generalized multi-variate hypergeometric functions.
- Write BCFW recursion relations on the spectral plane.
- Investigate the relation to positivity.
- Establish that the deformed Graßmannian is useful for loop calculations.

**For integrability:**

- Work out all Yangian invariants of $\mathfrak{gl}(N|M)$.
- Yangian invariants are interesting for spin chains and condensed matter

Thank you!