Towards a Numerical Implementation of the Loop-Tree Duality Method

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Outline

1. Introduction
2. A new method for higher order calculations: Loop-Tree Duality
3. Numerical Implementation

Introduction

- When calculating NLO (NNLO) cross-sections one needs to consider the tree- and loop-contributions *separately.* Especially loops with many external legs prove to be challenging.

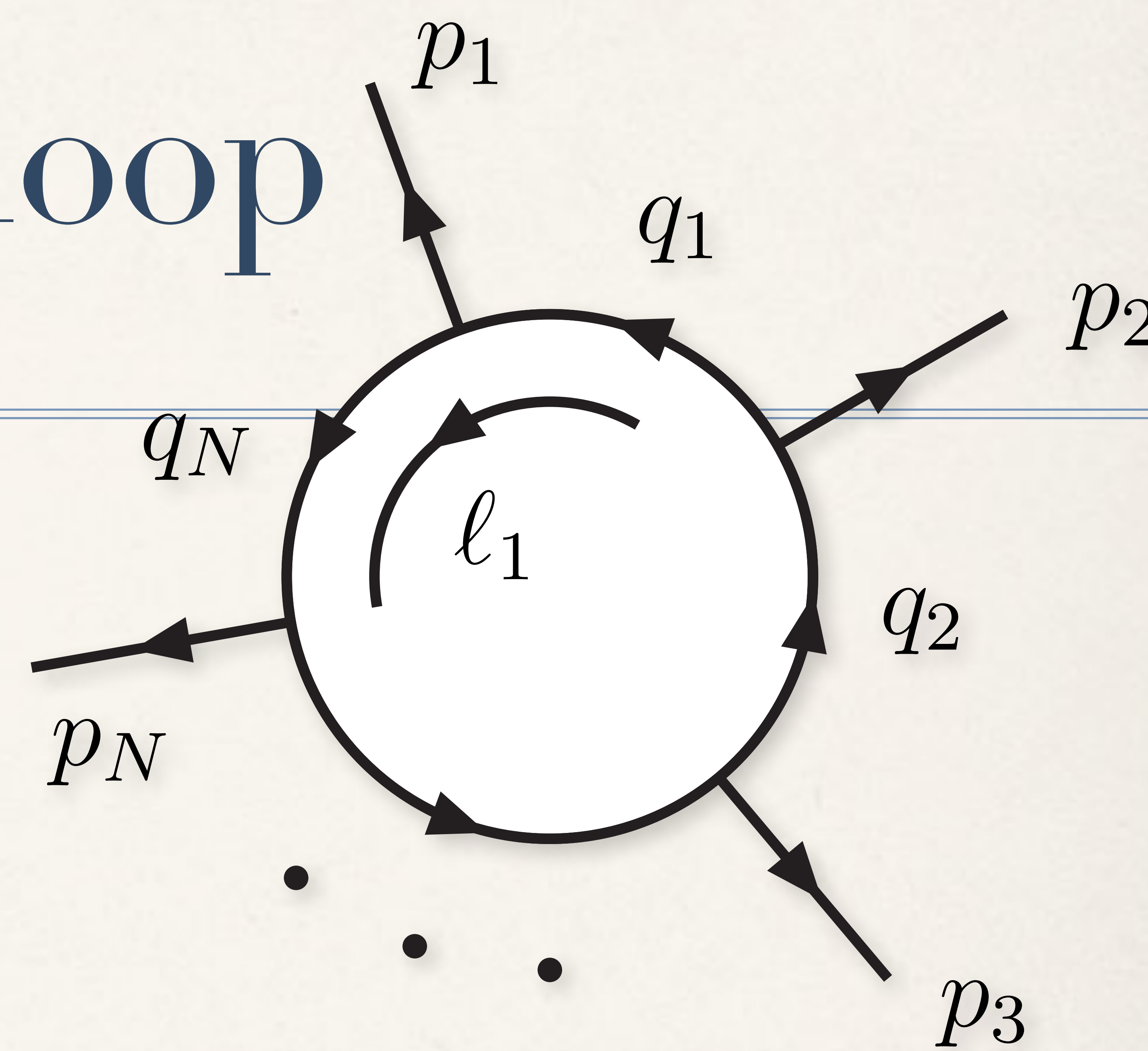
- Considerable progress has already been made in order to attack this problem: OPP-Method, Unitarity Methods, Mellin-Barnes Representation, Sector Decomposition.

- The advantage of these methods is that they made possible what was impossible before, but still a lot of effort has to be put in to cancel **infrared singularities** among real and virtual corrections. Additional difficulties arise from **threshold singularities** that lead to numerical instabilities.

- The Loop-Tree Duality (LTD) method aims towards a **combined** treatment of tree- and loop-contributions. Therefore the Loop-Tree Duality method casts the virtual corrections in a form that closely resembles the real ones.

Loop-Tree Duality at one loop

[Catani, Gleisberg, Krauss, Rodrigo, Winter '08]



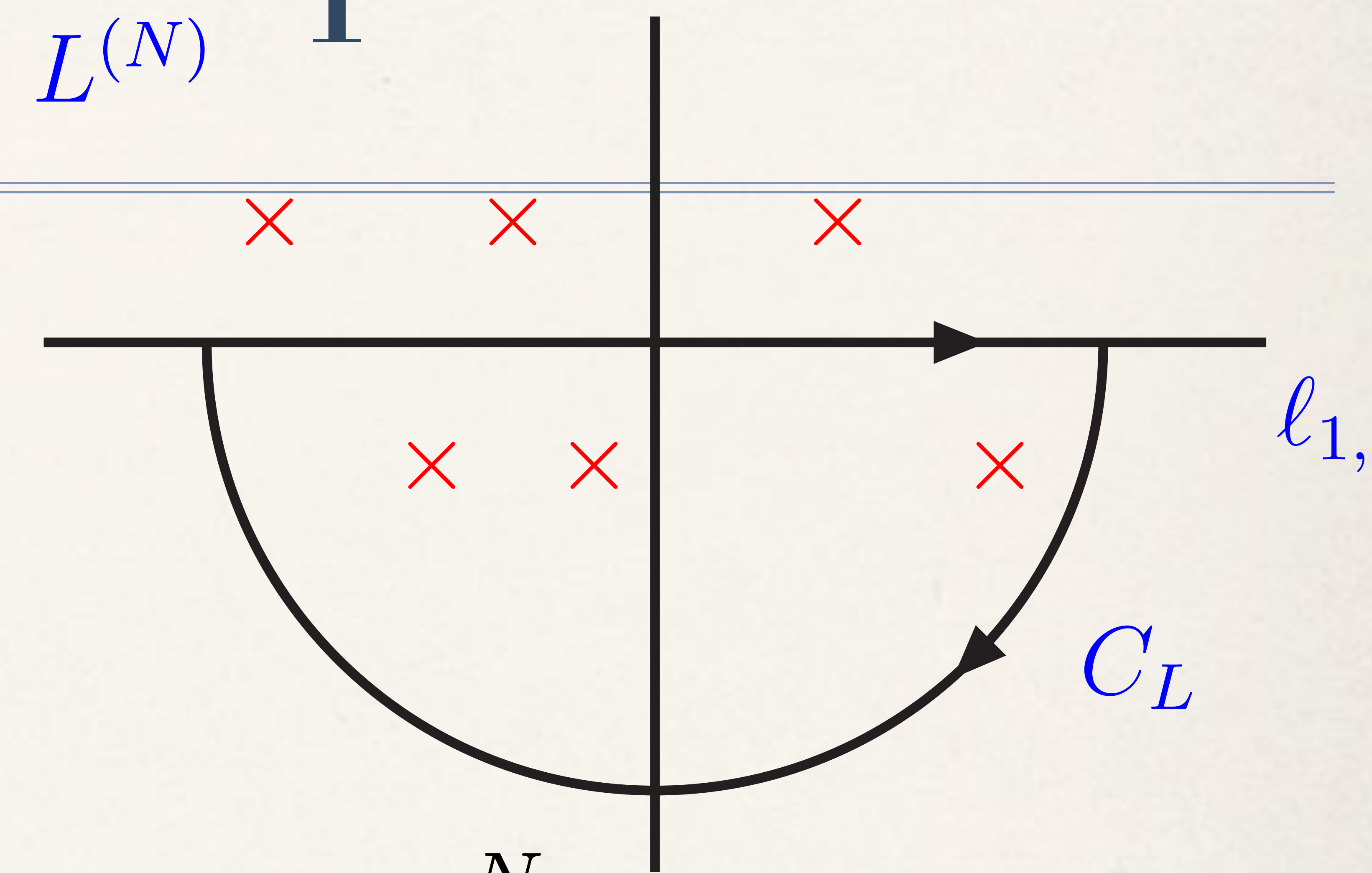
$$L^{(1)}(p_1, p_2, \dots, p_N) = \int_{\ell_1} \prod_{i=1}^{N} G_F(q_i)$$

with $$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0}$$ Feynman propagator

and $$q_i = \ell_1 + p_1 + \cdots + p_i = \ell_1 + k_i$$ Internal momenta

and $$\int_{\ell_1} = -i \int \frac{d^d \ell_1}{(2\pi)^d}$$ Work carried out in dimensional regularization!

Loop-Tree Duality at one loop



Directly apply the residue theorem for complex energy components of the loop momenta!

$$L^{(1)}(p_1, p_2, \ldots, p_N) = -2\pi i \int_{\vec{\ell}_1} \sum \mathrm{Res}_{\mathrm{Im}\{q_{i,0}\}<0} \prod_{j=1}^{N} G_F(q_j)$$

Selects the poles with negative imaginary part and positive energy!

(Duality beyond one-loop: [Bierenbaum, Catani, Draggiotis, Rodrigo ’10],
Duality with higher order poles: [Bierenbaum, Buchta, Draggiotis, Malamos, Rodrigo ’12])

Loop-Tree Duality at one loop

1. $$\text{Res}_{\text{Im}\{q_{i,0}\}<0} \frac{1}{q_i^2 - m_i^2 + i0} = \int d\ell_{1,0} \delta_+(q_i^2 - m_i^2)$$

2. $$\prod_{j\neq i} G_F(q_j)\Big|_{\text{i-th pole}} = \prod_{j\neq i} \frac{1}{q_j^2 - m_j^2 - i0\eta(q_j - q_i)} \equiv \prod_{j\neq i} G_D(q_i; q_j)$$

η future-like vector $\eta^2 \geq 0,\ \eta_0 > 0$

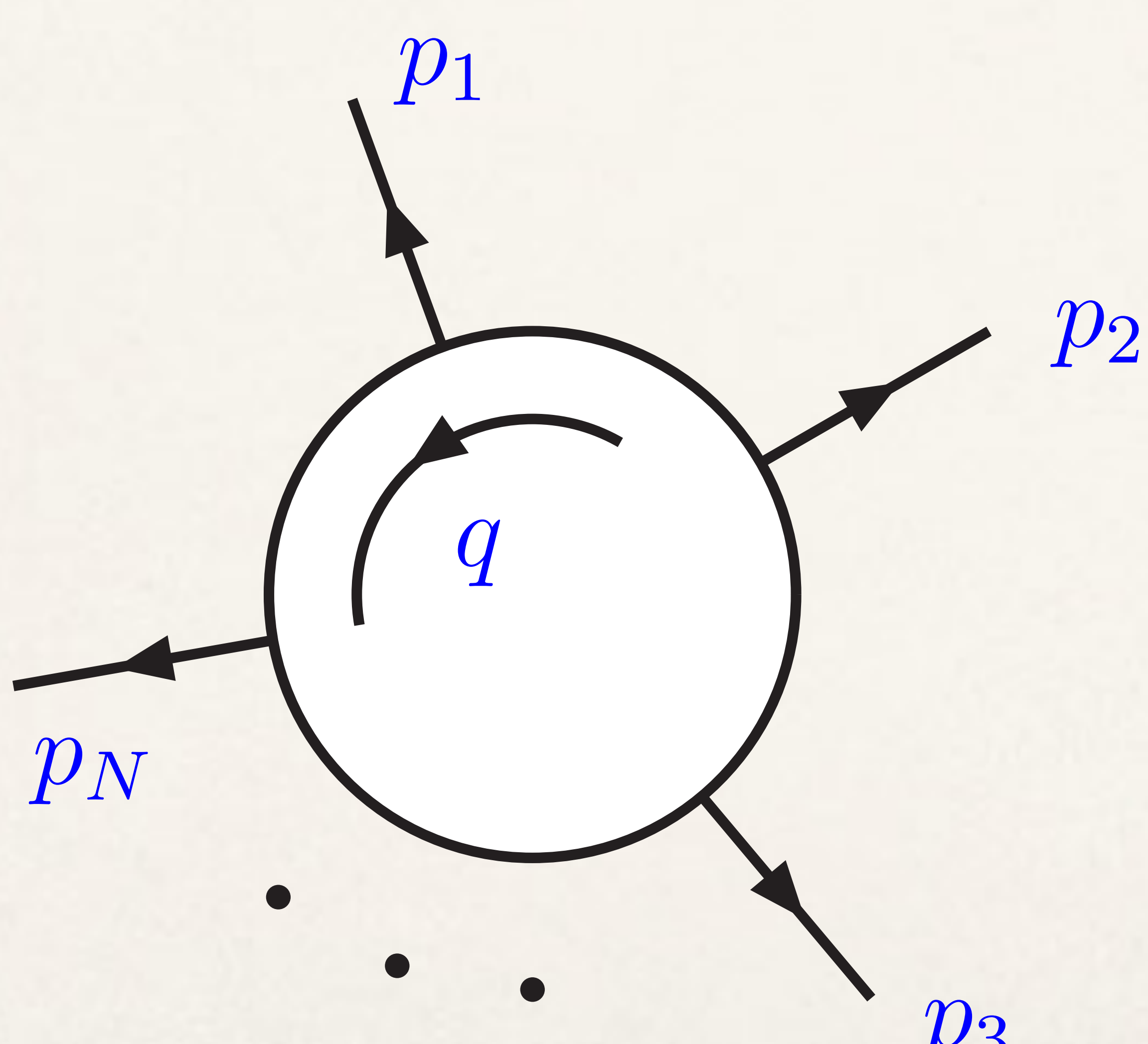
Different choices of η correspond to different coordinate systems.
The sum of all dual contributions is independent of η.

Loop-Tree Duality at one loop

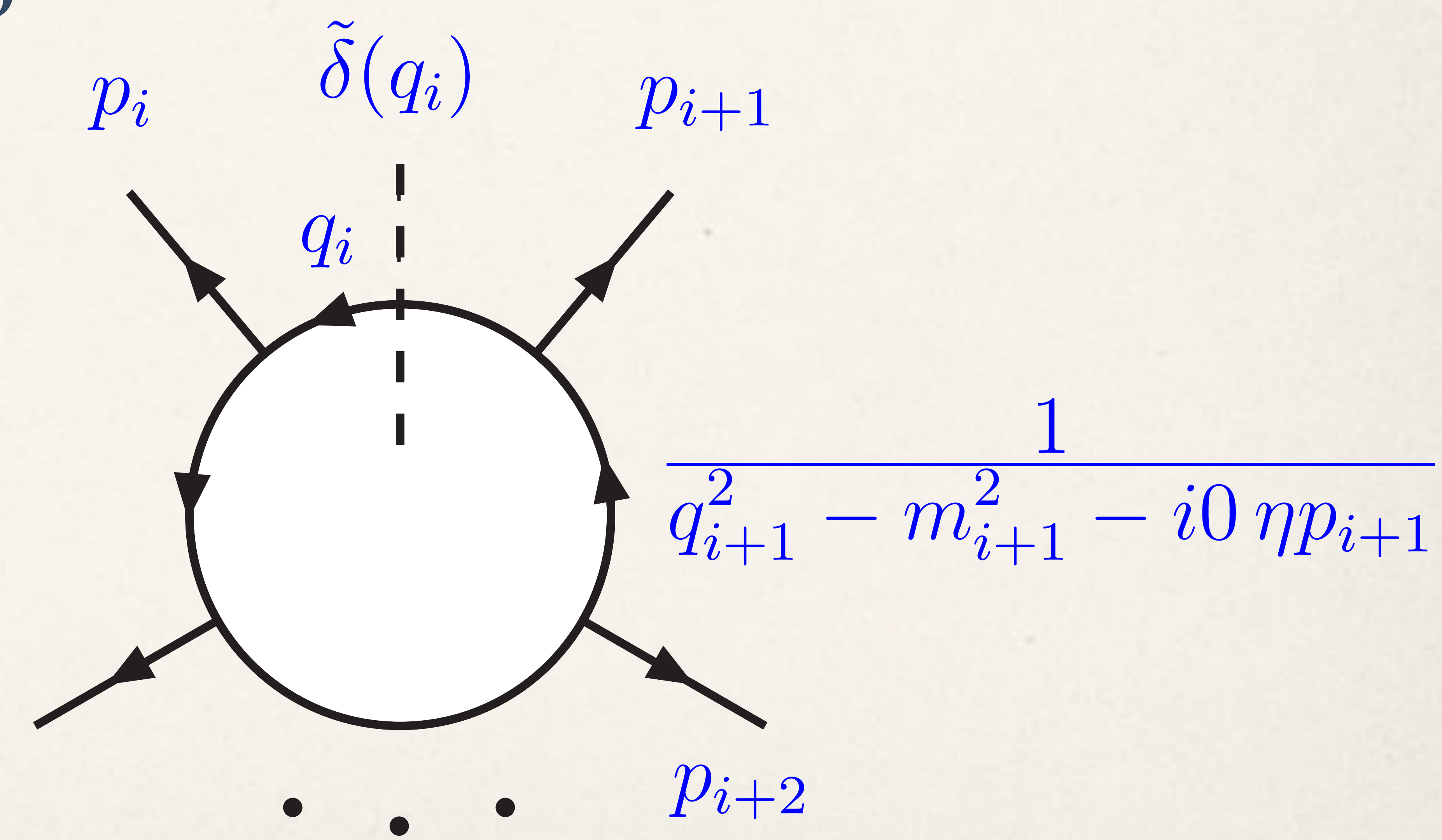
$$\tilde{\delta}(q_i) = 2\pi i \delta_+(q_i^2 - m_i^2)$$

$$L^{(1)}(p_1, p_2, \ldots, p_N) = -\sum \int_{\ell_1} \tilde{\delta}(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} G_D(q_i; q_j)$$

corresponds to



$$= \; - \sum_{i=1}^{N}$$



Loop-Tree Duality at one loop

- The Loop-Tree Duality contains only single cuts while introducing a modified *i0*-prescription, the „dual“ prescription.

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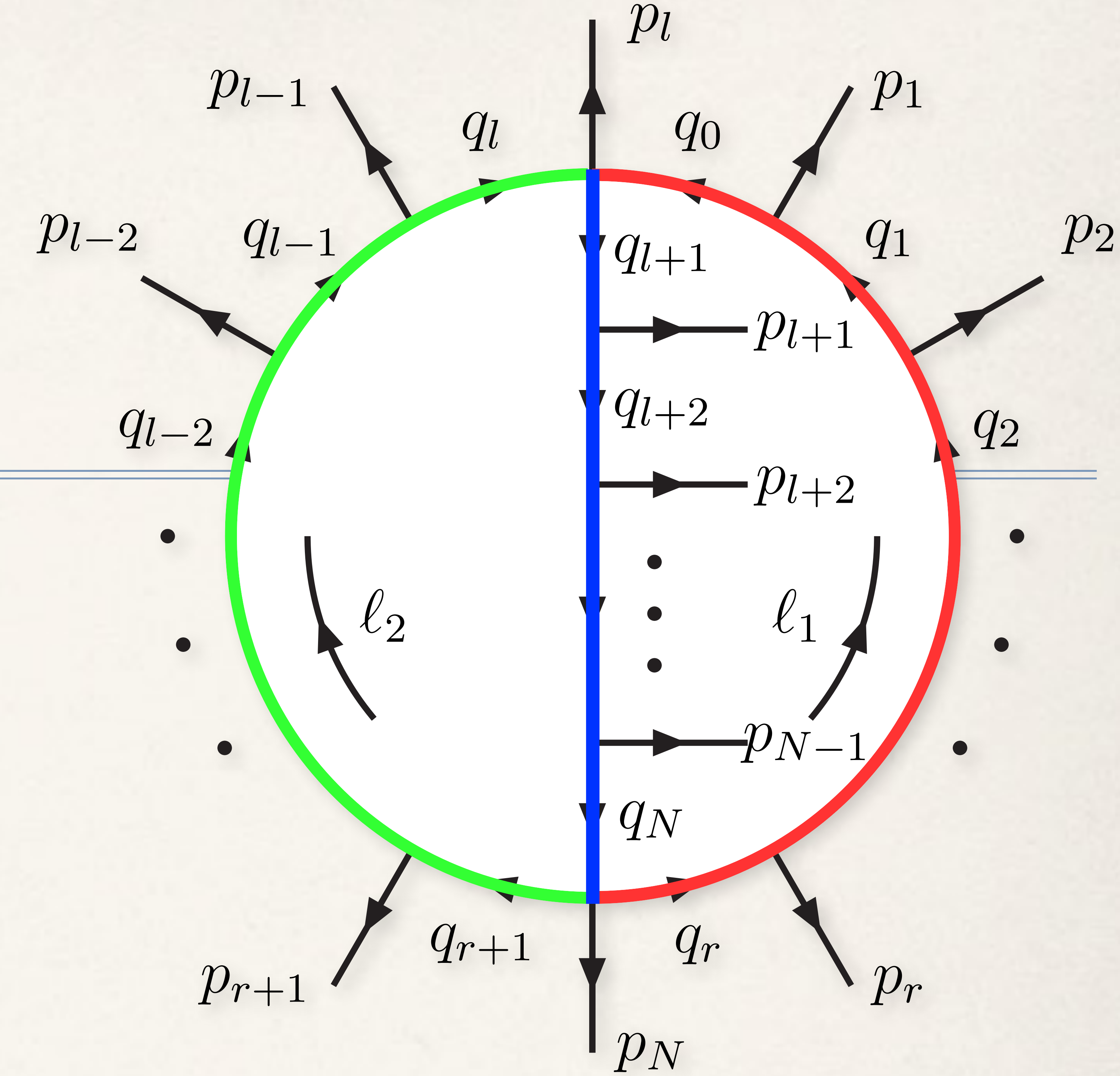
- The Loop-Tree Duality contains only single cuts while introducing a modified *i0*-prescription, the „dual“ prescription.
- Number of single cut Dual Contributions equals the number of legs
- The singularities of the loop diagram appear as singularities of the Dual Integrals.
- Tensor loop integrals and physical scattering amplitudes are treated in the same way since the Loop-Tree Duality works only on propagators.
- Virtual corrections are recast in a form, that closely parallels the contribution of real corrections.

Extension to two loops

[Bierenbaum, Catani, Draggiotis, Rodrigo '10]



Notation:

$$q_i = \begin{cases} \ell_1 + p_{1,i} & i \in \alpha_1 \\ \ell_2 + p_{i,l-1} & i \in \alpha_2 \\ \ell_1 + \ell_2 + p_{i,l-1} & i \in \alpha_3 \end{cases}$$

$$\alpha_1 = \{0, 1, \ldots, r\}, \qquad \alpha_2 = \{r+1, r+2, \ldots, l\}$$

$$\alpha_3 = \{l+1, l+2, \ldots, N\}$$

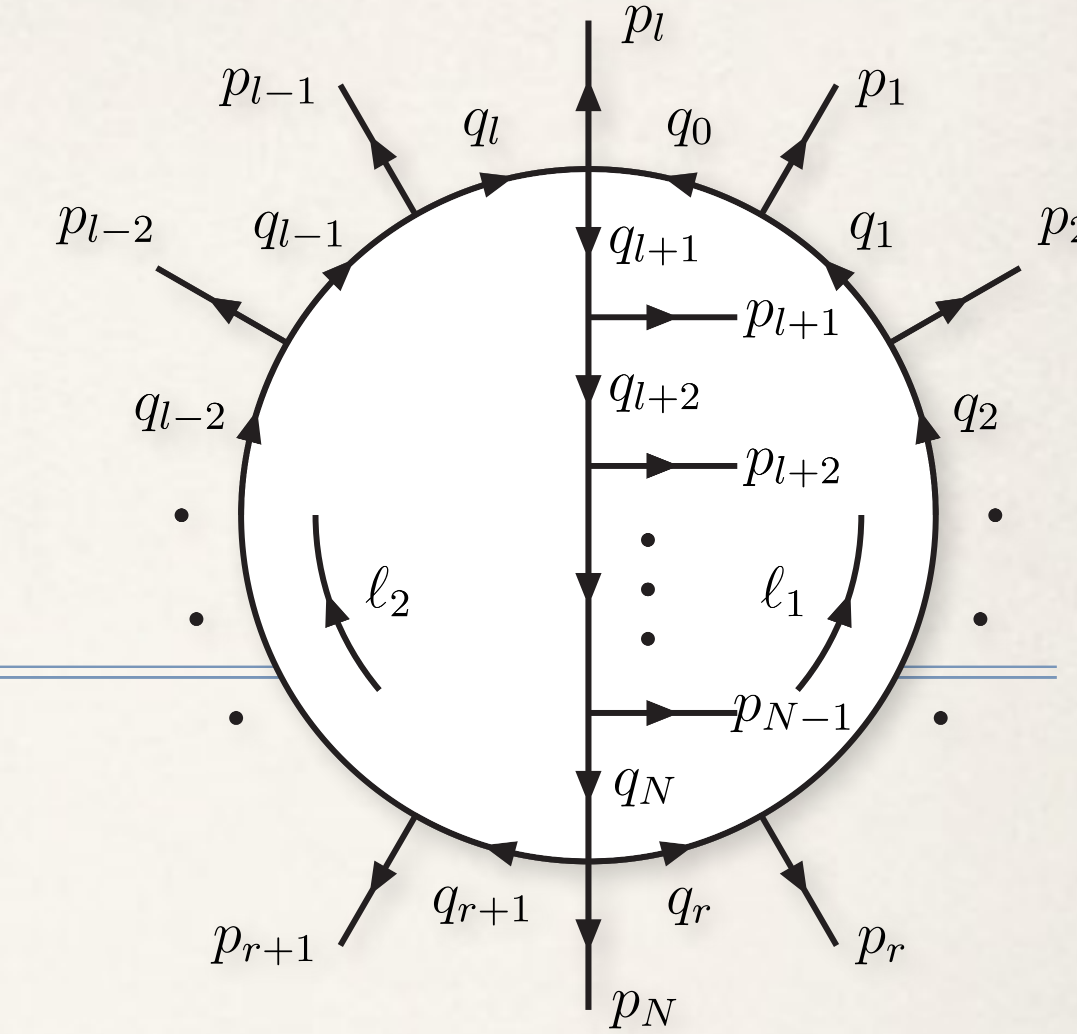
$$G_F(\alpha_k) = \prod_{i \in \alpha_k} G_F(q_i), \qquad G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{\substack{j \in \alpha_k \\ j \neq i}} G_D(q_i; q_j)$$

For a set of looplines belonging to the *same* loop (of a multi loop diagram):

$$\int_{\ell_i} G_F(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n) = - \int_{\ell_i} G_D(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n)$$

Subsequently apply the LTD to the other loops of the diagram.

Extension to two loops



- Each application to a loop introduces an extra single cut.
- Apply it as many times as there are loop: Opening loops to trees
- Every application converts Feynman into Dual Propagators. Since the LTD can only be applied to Feynman P.s, the Dual P.s of the unification of several subsets must be reexpressed in terms of Dual *and* Feynman P.s. before going to the next loop
- One loop line might take part in more that one loop, see „middle“ line in graph

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{\ell_1}\int_{\ell_2} \{-G_D(\alpha_1)G_F(\alpha_2)G_D(\alpha_3) + G_D(\alpha_1)G_D(\alpha_2 \cup \alpha_3) + G_D(\alpha_3)G_D(-\alpha_1 \cup \alpha_2)\}$$

- Reiterate the procedure for higher order loop integrals

Higher order poles

[Bierenbaum, Buchta, Draggiotis, Malamos, Rodrigo '12]

- It is possible to derive the LTD for higher order poles similarly to simple poles
- Yet there is a more practical solution which takes advantage of IBP-relations
- Consider a m-loop scalar integral with n denominators $D_1 \ldots D_n$ raised to exponents $a_1 \ldots a_n$ in d dimensions:

$$\int_{\ell_1} \cdots \int_{\ell_m} \frac{1}{D_1^{a_1} \ldots D_n^{a_n}} = F(a_1, \ldots, a_n)$$

Total derivative:

$$\int_{\ell_1} \cdots \int_{\ell_m} \frac{\partial}{\partial s^\mu} \frac{t^\mu}{D_1^{a_1} \ldots D_n^{a_n}} = 0$$

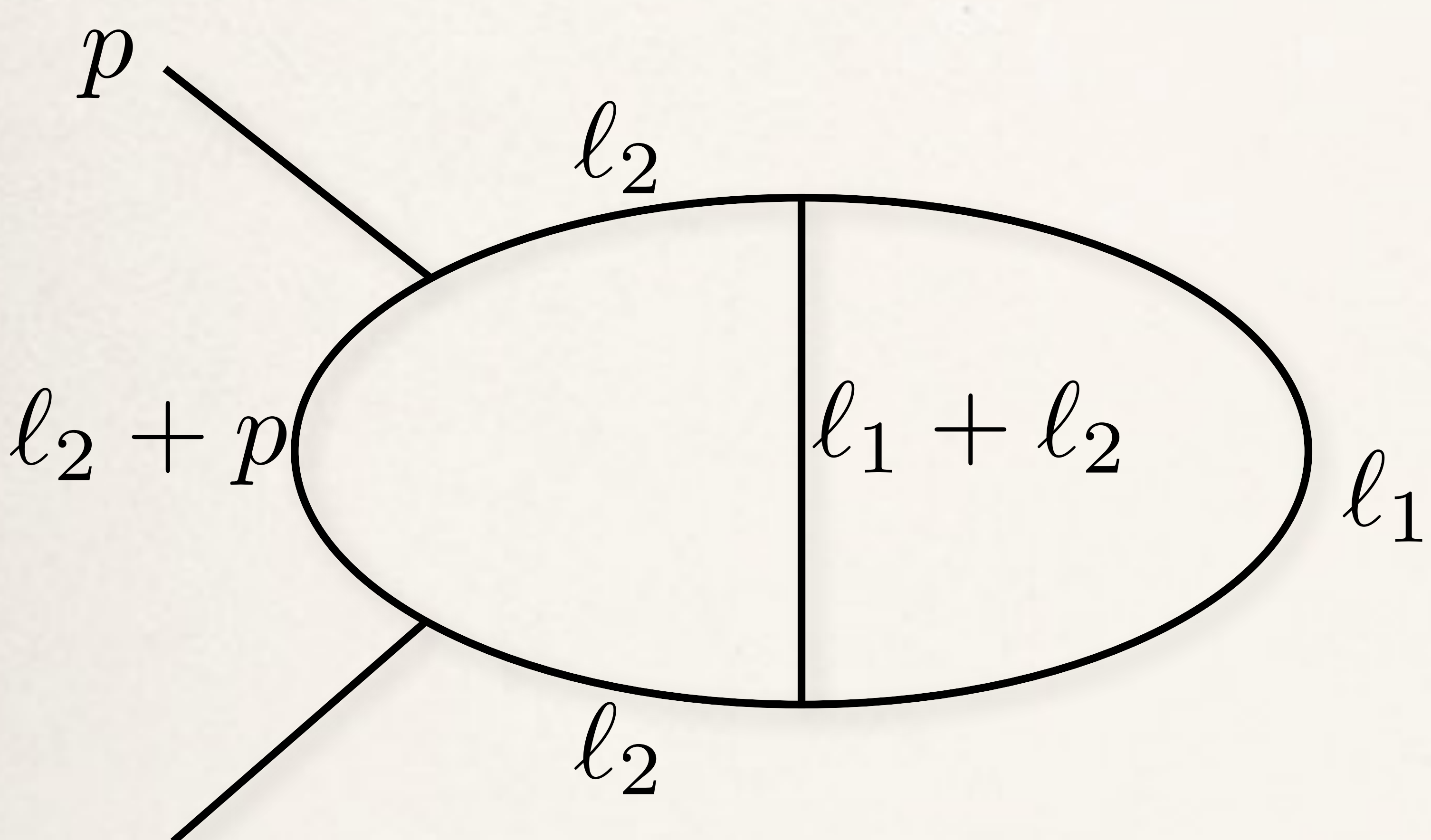
$$s^\mu = \ell_1^\mu, \ldots, \ell_m^\mu$$

$$t^\mu = \ell_1^\mu, \ldots, \ell_m^\mu, \; p_1, \ldots, p_N$$

- Differentiation will raise an exponent or leave it unchanged
- Contractions of loop with external momenta can be expressed as a propagator to lower an exponent
- Sometimes this reexpressing is not possible: Irreducible Scalar Products (ISP). Consider these as extra propagators with negative exponent.

Higher order poles

Example with the simplest graph possible:



$F(12110)$

Higher order poles

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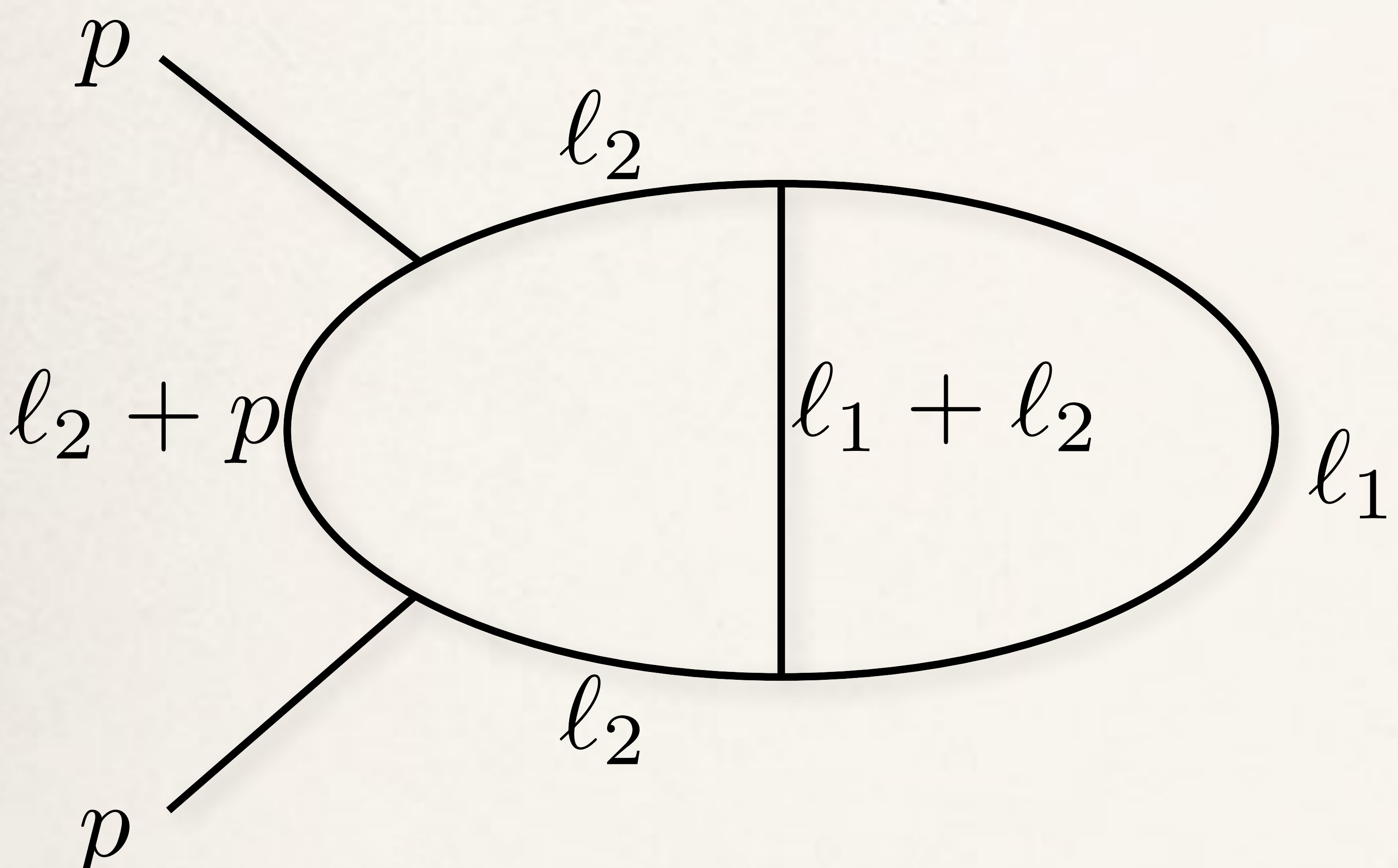

$F(12110)$

Denominators:

$$D_1 = \ell_1^2$$
$$D_2 = \ell_2^2$$
$$D_3 = (\ell_2 + p)^2$$
$$D_4 = (\ell_1 + \ell_2)^2$$
$$D_5 = \ell_1 \cdot p$$

Higher order poles

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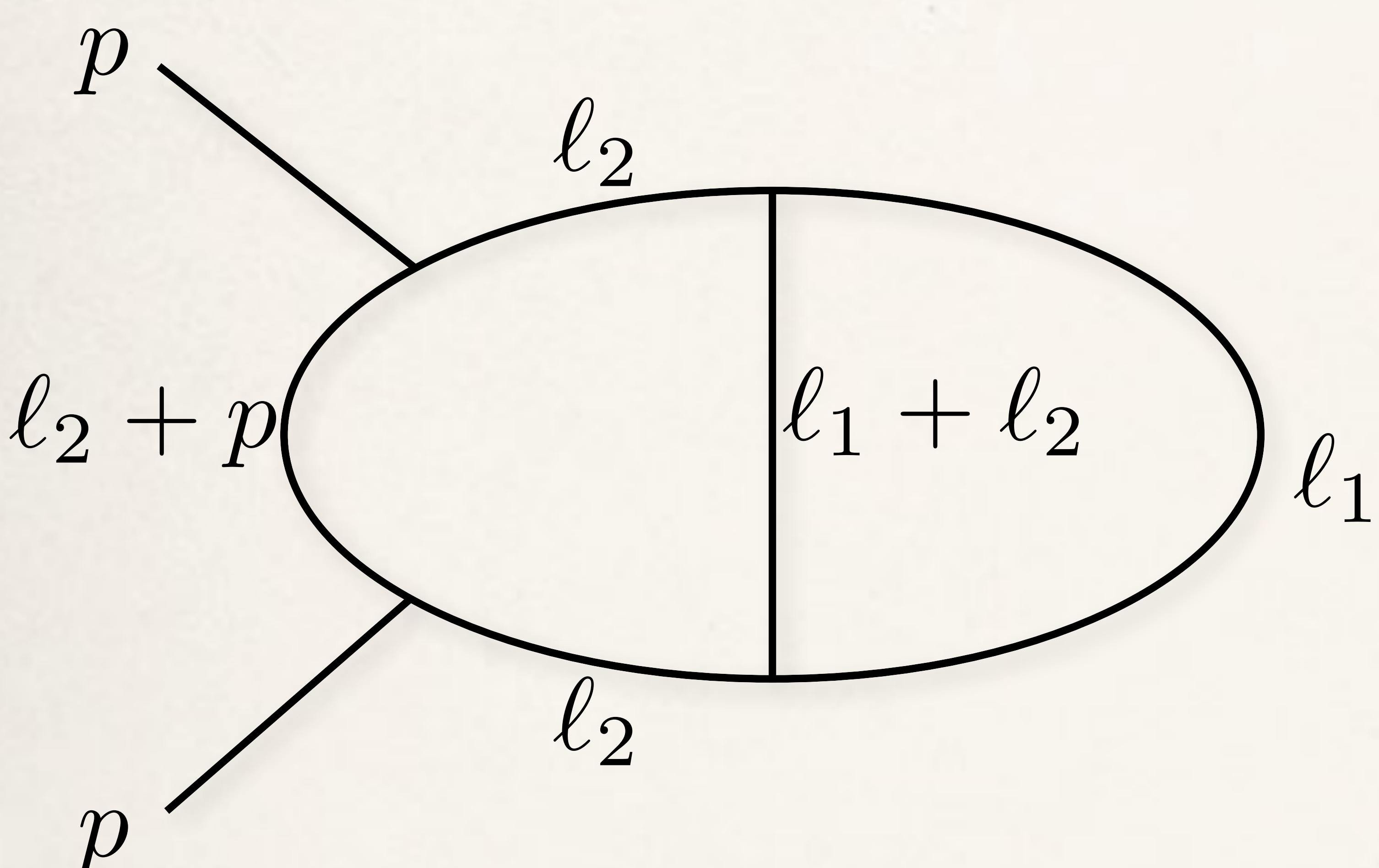

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Total derivatives:

$$\frac{\partial}{\partial \ell_i} \cdot \ell_j$$
$$\frac{\partial}{\partial \ell_i} \cdot p$$
$$i, j = 1, 2$$

Higher order poles

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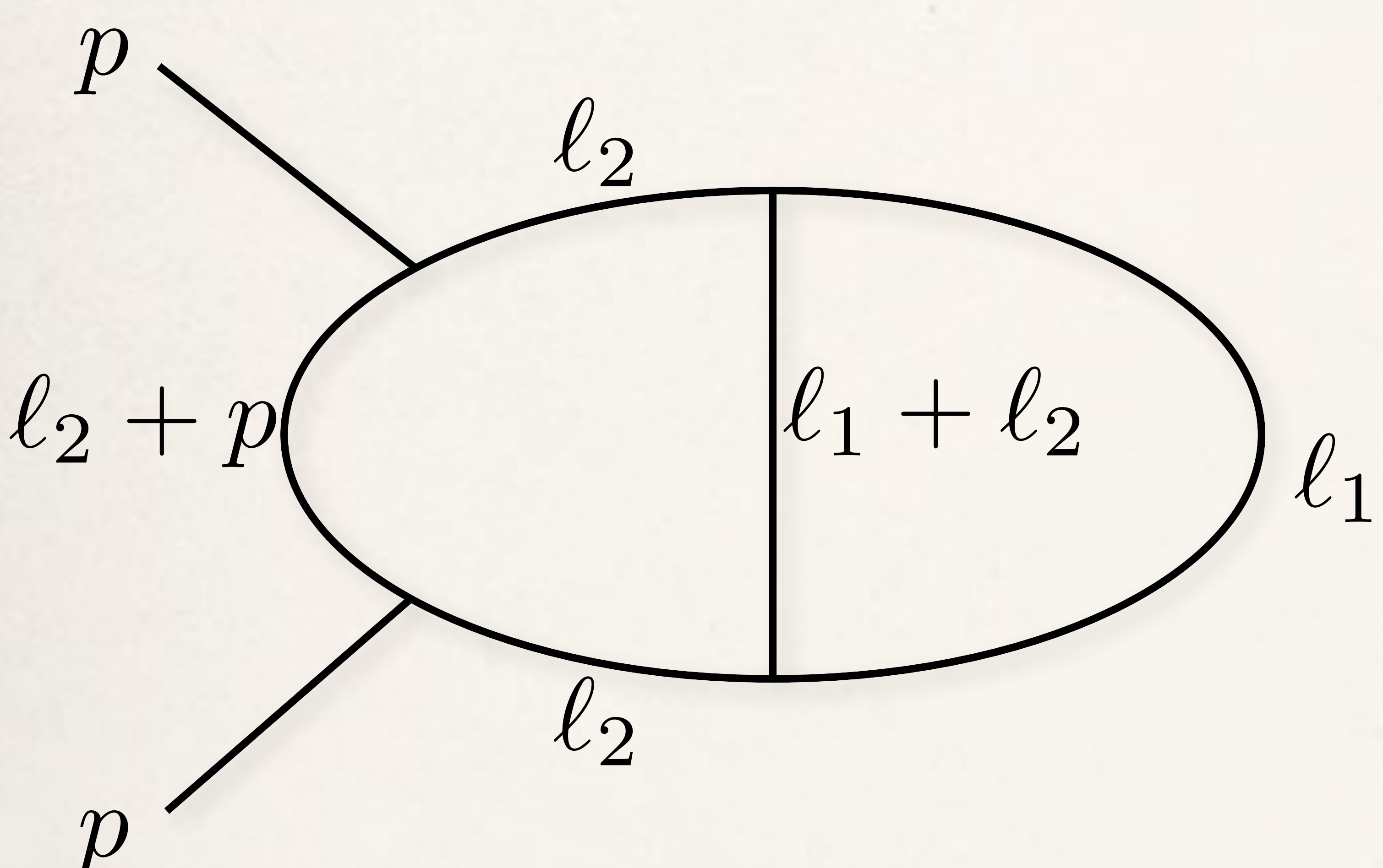

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Solve system of six linear equations: $F(12110) = \dfrac{-1 + 3\epsilon}{(1+\epsilon)s} F(11110), \qquad s = p^2 + i0$

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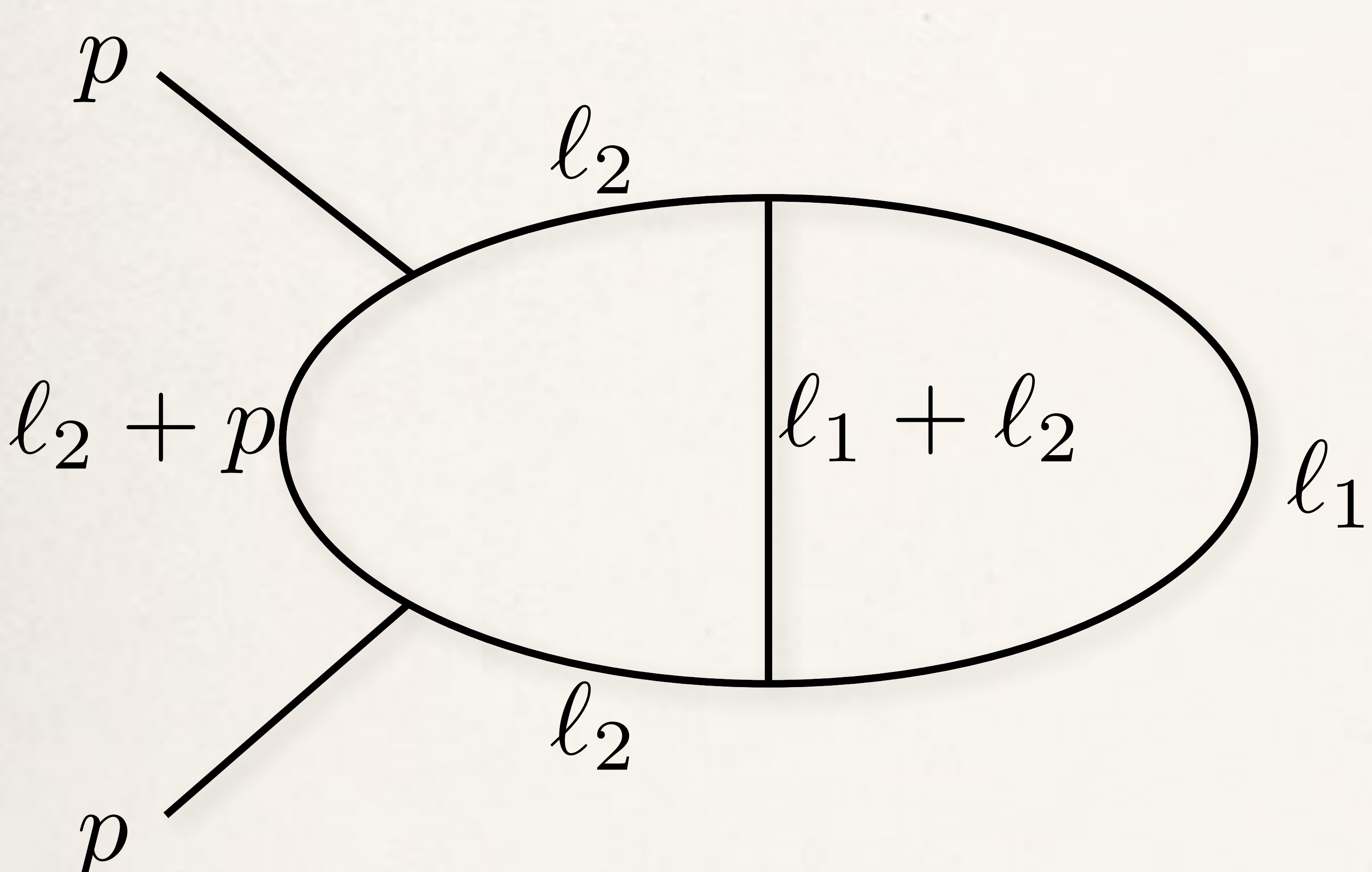

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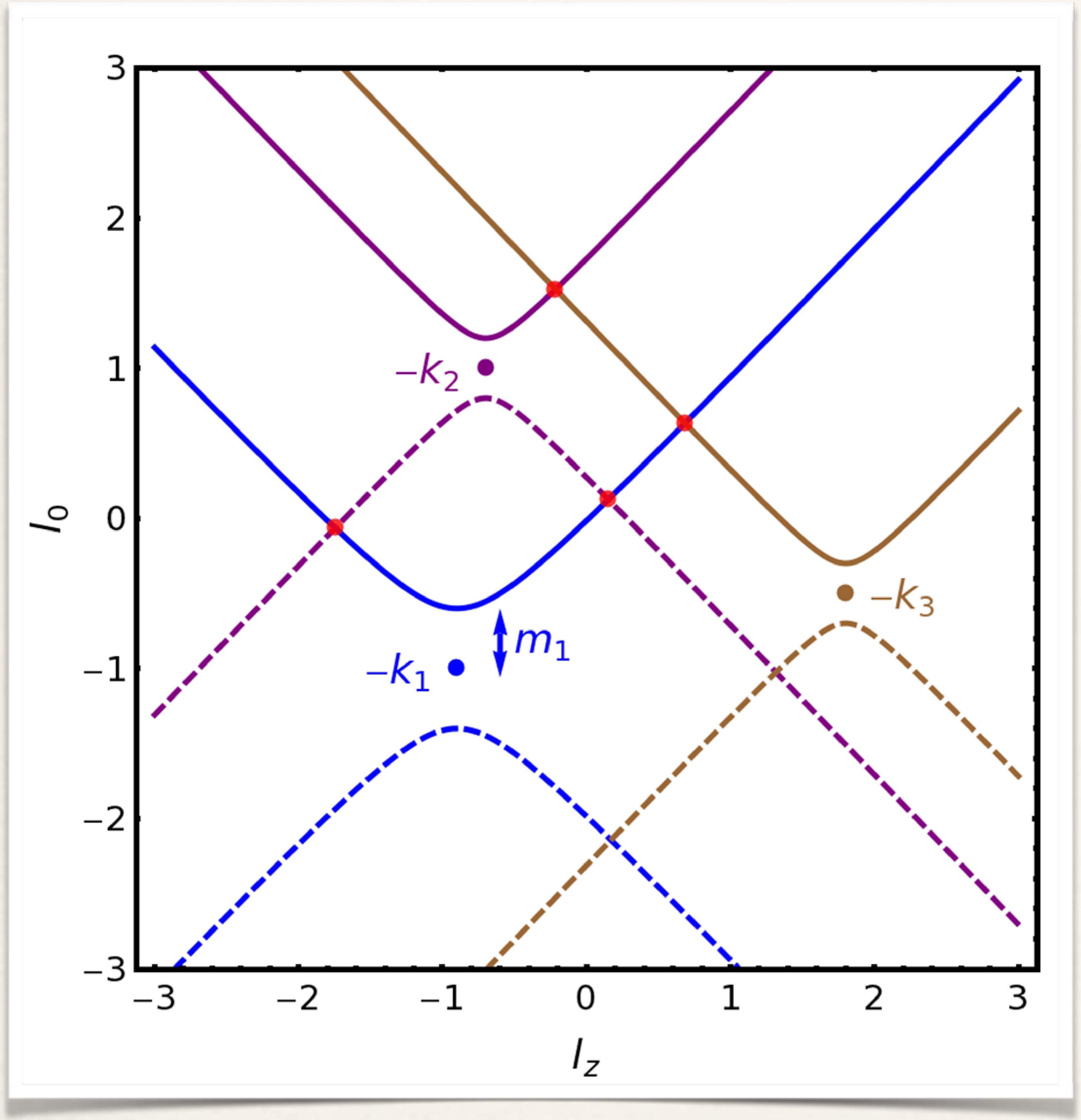
- For more complicated case Mathematica package Fire was used successfully.
- Not necessary to reduce to a certain integral basis. Just get rid of higher order poles.

Singular behavior of the loop integrand

[Buchta, Chachamis, Draggiotis, Malamos, Rodrigo '14]



The loop *integrand* becomes singular at hyperboloids with $q_{i,0}^{(+)} = \sqrt{\mathbf{q}_i^2 + m_i^2 - i0}$ (solid lines) and $q_{i,0}^{(-)} = -\sqrt{\mathbf{q}_i^2 + m_i^2 - i0}$ (dashed lines) and origin in $-k_{i,\mu}$

LT-Duality is equivalent to integrating along the *forward hyperboloids!*

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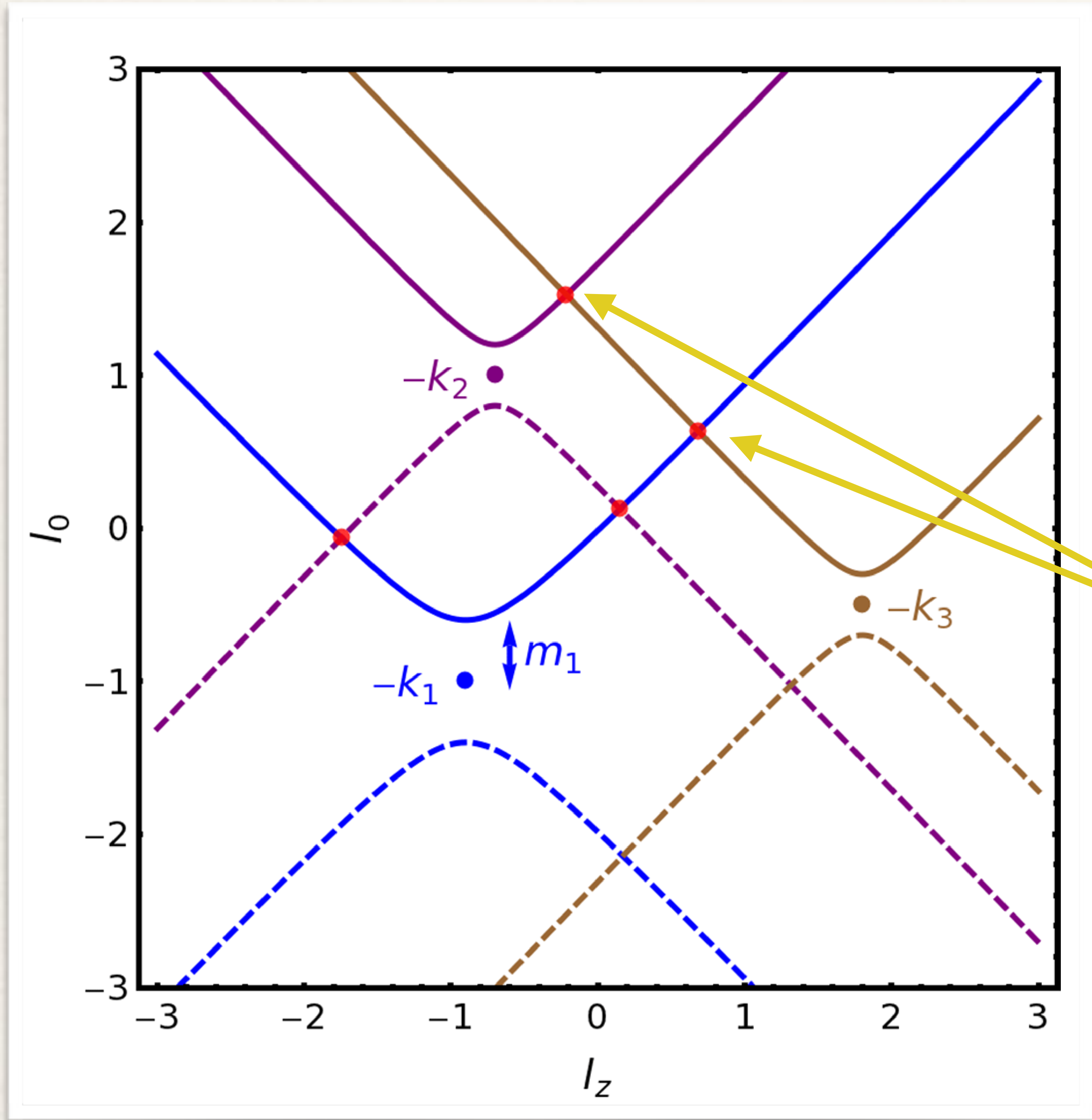

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Forward-forward intersection

These singularities cancel among dual integrals

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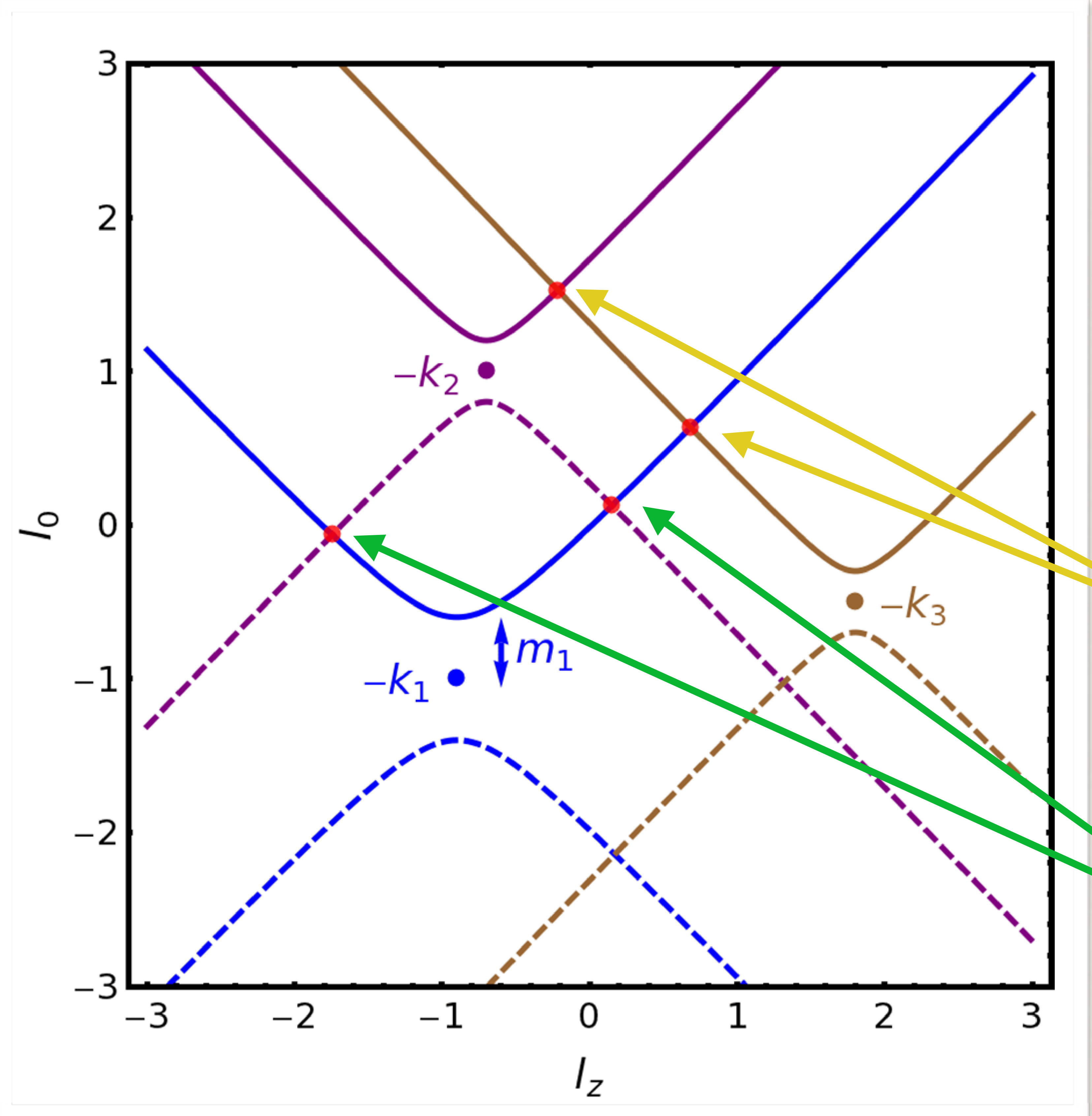

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Forward-forward intersection

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Forward-backward intersection

These singularities remain and require contour deformation

Numerical Implementation: Basics

$$L^{(1)}(p_1, p_2, \ldots, p_N) = -\sum \int_{\ell_1} \tilde{\delta}(q_i) \prod_{\substack{j=1 \\ j \neq i}}^{N} G_D(q_i; q_j)$$

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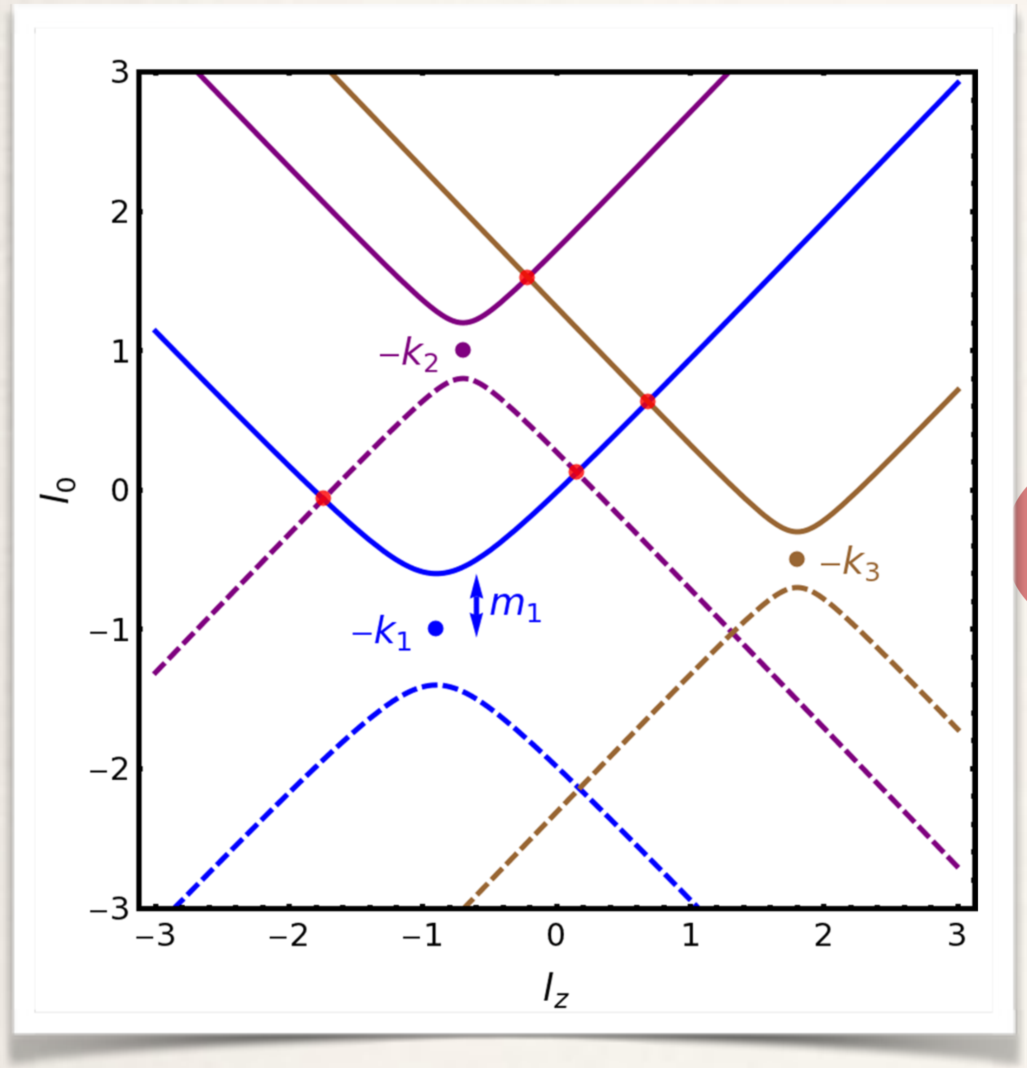
Rewrite dual propagator like this

$$\tilde{\delta}(q_i) G_D(q_i; q_j) = 2\pi i \frac{\delta(q_{i,0} - q_{i,0}^{(+)})}{2q_{i,0}^{(+)}} \frac{1}{(q_{i,0}^{(+)} + k_{ji,0})^2 - (q_{j,0}^{(+)})^2}$$

with $k_{ij} = q_i - q_j$ and $q_{i,0}^{(+)} = \sqrt{\mathbf{q}_i^2 + m_i^2 - i0}$

The resulting N contributions have to be integrated over the loop three-momenta.

Numerical Implementation: Basics



1. $$q_{i,0}^{(+)} + q_{j,0}^{(+)} + k_{ji,0} = 0$$ forward-backward
2. $$q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0} = 0$$ forward-forward

1. The first equation describes an ellipsoid in the loop three-momentum and demands $k_{ji,0} < 0$.

An ellipsoid is the result of the intersection of a forward with a backward hyperboloid.

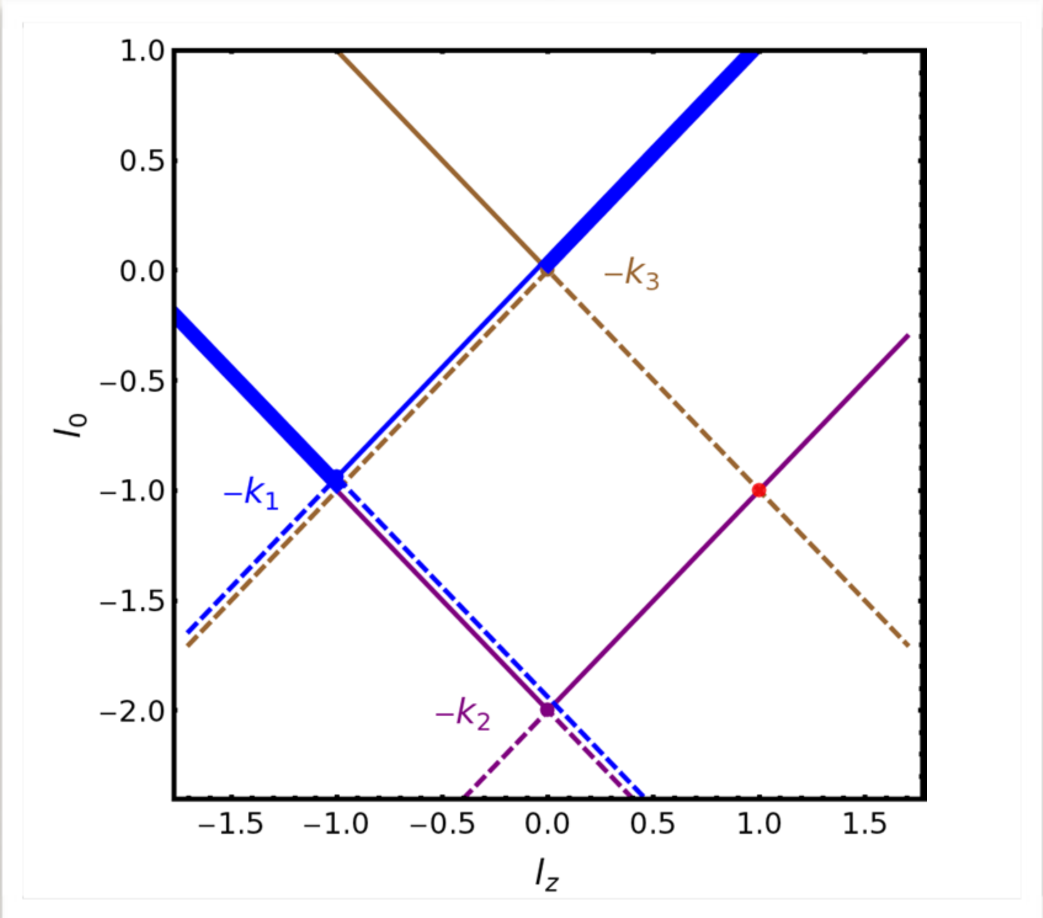
The origins of the hyperboloids are separated in a time-like (light-like) fashion, expressed by the condition:
$$k_{ji}^2 - (m_j + m_i)^2 \geq 0, \quad k_{ji,0} < 0$$

2. The second equation describes a hyperboloid as a result of the intersection of two forward light-cones of space-like (light-l.) separation.
$$k_{ji,0}^2 - (m_j - m_i)^2 \leq 0$$

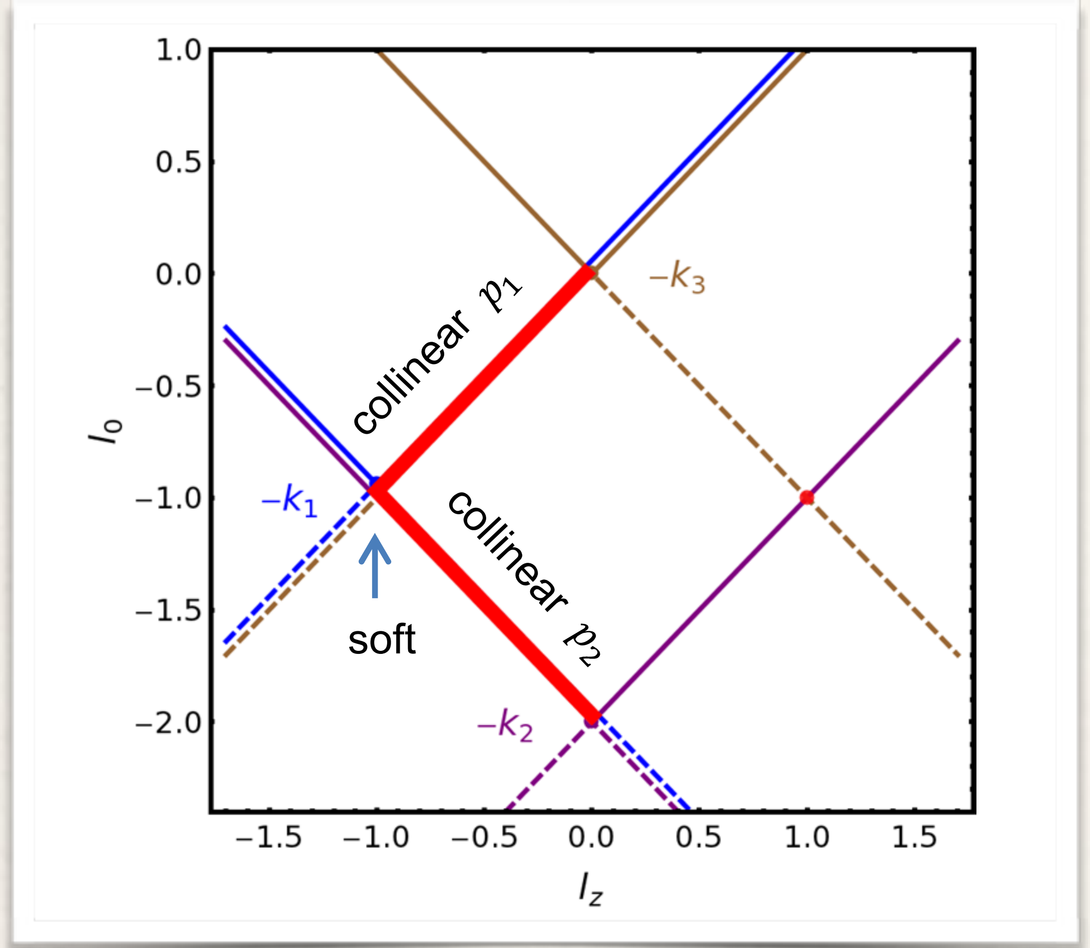
$k_{ji,0}$ may be positive or negative:

Infrared singularities: Massless case

[Buchta, Chachamis, Draggiotis, Malamos, Rodrigo '14]



Forward-forward: Collinear singularities cancel among dual contributions



Forward-backward: Collinear and soft singularities remain. They are restricted to a finite region and can be mapped to the real phase-space emission

Preparation

Feynman-integral

Example:
Box

$G_F\, G_F\, G_F\, G_F$ —LTD→

Positions →

Contributions ↓

δ	G_D	G_D	G_D
G_D	δ	G_D	G_D
G_D	G_D	δ	G_D
G_D	G_D	G_D	δ

Use this scheme to indicate
the position of singularities!

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G_D	δ	G_D	G_D
G_D	G_D	δ	G_D
G_D	G_D	G_D	δ

Use this scheme to indicate the position of singularities!

Example:

0	H	0	H
H	0	H	E
E	H	0	E
H	0	0	0

E: Ellipsoid Sing., H: Hyperboloid Sing.

Preparation: Deformation groups

0	H	0	0	0
H	0	H	E	0
E	H	0	E	0
E	E	E	0	E
0	0	0	0	0

Preparation: Deformation groups

0	H	0	0	0
H	0	H	E	0
E	H	0	E	0
E	E	E	0	E
0	0	0	0	0

Contributions are coupled:

In order to preserve the cancellations of the hyperboloids, every contr. receives all deformations that occur within the coupled contributions

Preparation: Deformation groups

0	H	0	0	0
H	0	H	E	0
E	H	0	E	0
E	E	E	0	E
0	0	0	0	0

Rows 1–3 (}): Contributions are coupled: In order to preserve the cancellations of the hyperboloids, every contr. receives all deformations that occur within the coupled contributions

Row 4 (→): Deform this contribution only with the deformations that itself contains

Preparation: Deformation groups

0	H	0	0	0	} Contributions are coupled: In order to preserve the cancellations of the hyperboloids, every contr. receives all deformations that occur within the coupled contributions
H	0	H	E	0	
E	H	0	E	0	
E	E	E	0	E	→ Deform this contribution only with the deformations that itself contains
0	0	0	0	0	→ No deformation needed here

Preparation: Deformation groups

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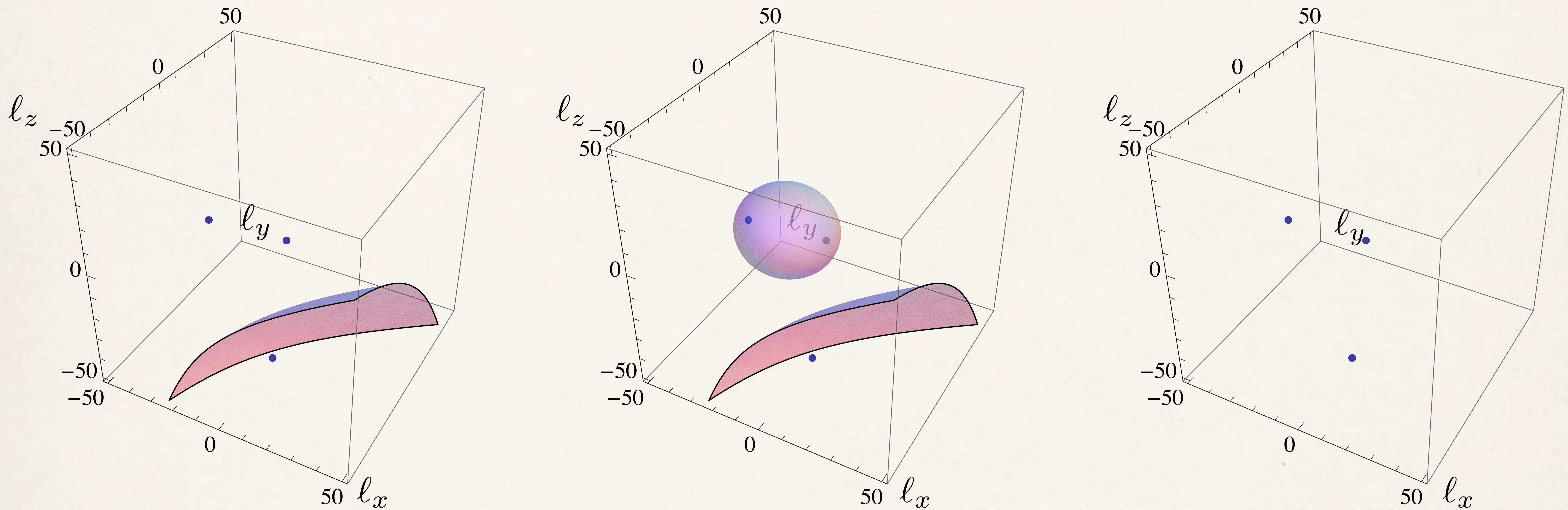
H	0	H	E	0	
E	H	0	E	0	
E	E	E	0	E	→ Deform this contribution only with the deformations that itself contains
0	0	0	0	0	→ No deformation needed here

Organize the contributions into *groups*. Each group is deformed independently. Within a group, every contribution receives the same deformation which accounts for *all* of the ellipsoids of the group.

Singularities in Loop-Momentum Space



Triangle with one ellipsoid and two hyperboloid singularities

0	H	0
H	0	E
0	0	0

Deformation: 1+1 dim

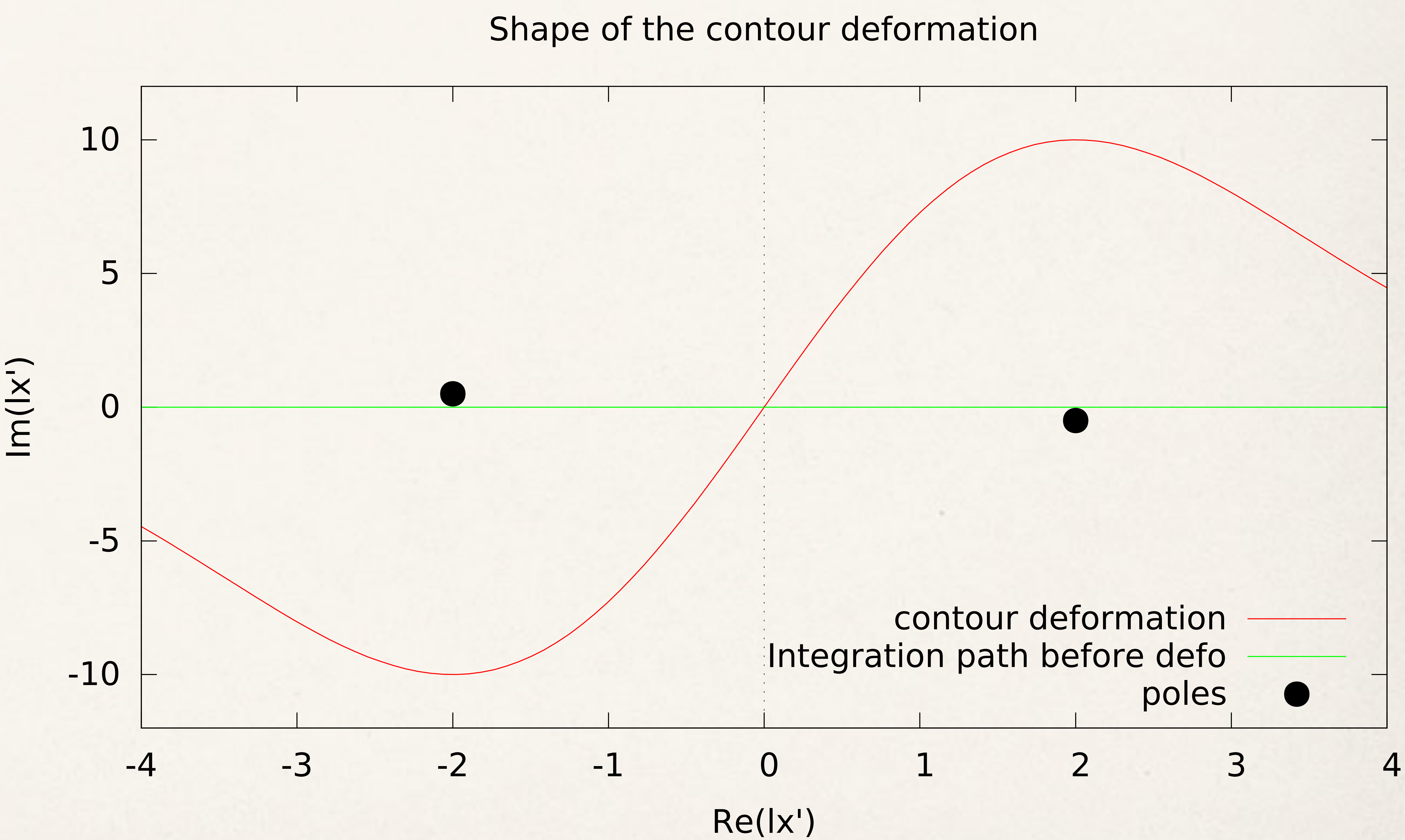
Let's have a short look at an easy case, the one-dimensional integral

$$f(\ell_x) = \frac{1}{\ell_x^2 - E^2 + i0}$$

Corresponding deformation:

$$\ell_x \to \ell_x' = \ell_x + i\lambda\ell_x \exp\left(-\frac{\ell_x^2 - E^2}{2E^2}\right)$$

E: location of the singularity



Deformation: 1+3 dim

For each individual ellipsoid include:

$$\text{def} = i\lambda\vec{\boldsymbol{\ell}}\exp\left(-\frac{G_D^{-2}}{A}\right)$$

λ scaling factor
A width of the deformation

} Can be chosen differently for each individual ellipsoid

Deformation: 1+3 dim

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λ scaling factor
A width of the deformation

} Can be chosen differently for each individual ellipsoid

Sum over the entire group: $$i\vec{\boldsymbol{\kappa}} = \sum_{j\in\text{group}} \text{def}_j$$

Final deformation: $$\vec{\boldsymbol{\ell}} \to \vec{\boldsymbol{\ell}}' = \vec{\boldsymbol{\ell}} + i\vec{\boldsymbol{\kappa}}$$

Numerical Implementation: Results

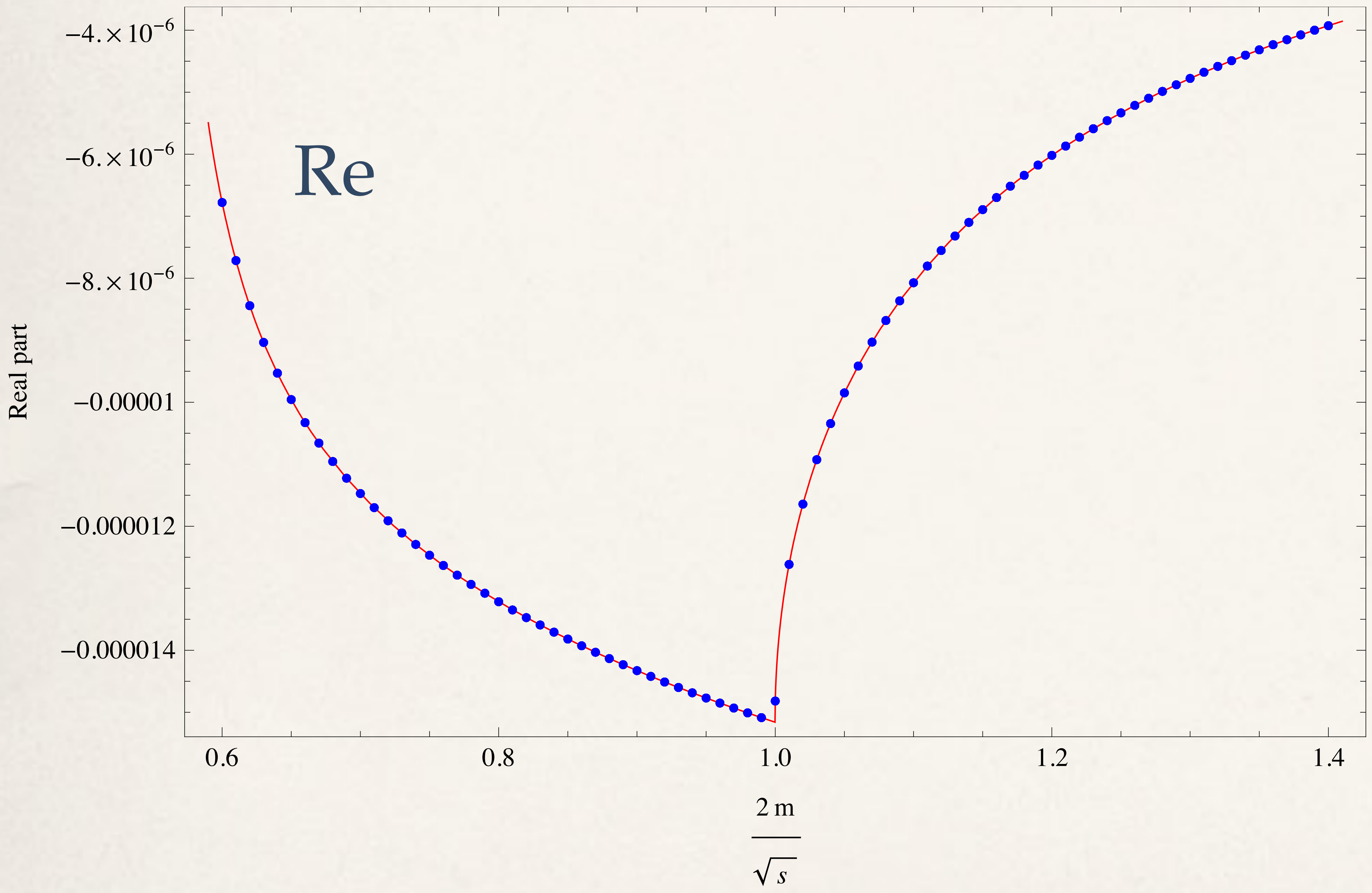
- Implementation in C++. The code runs on an Intel i7 (3.4GHz) desktop machine.
- Triangle, Box, Pentagon with no deformation needed: 4 digits in 0.5s
- Pentagon with deformations: 4 digits in ~25s

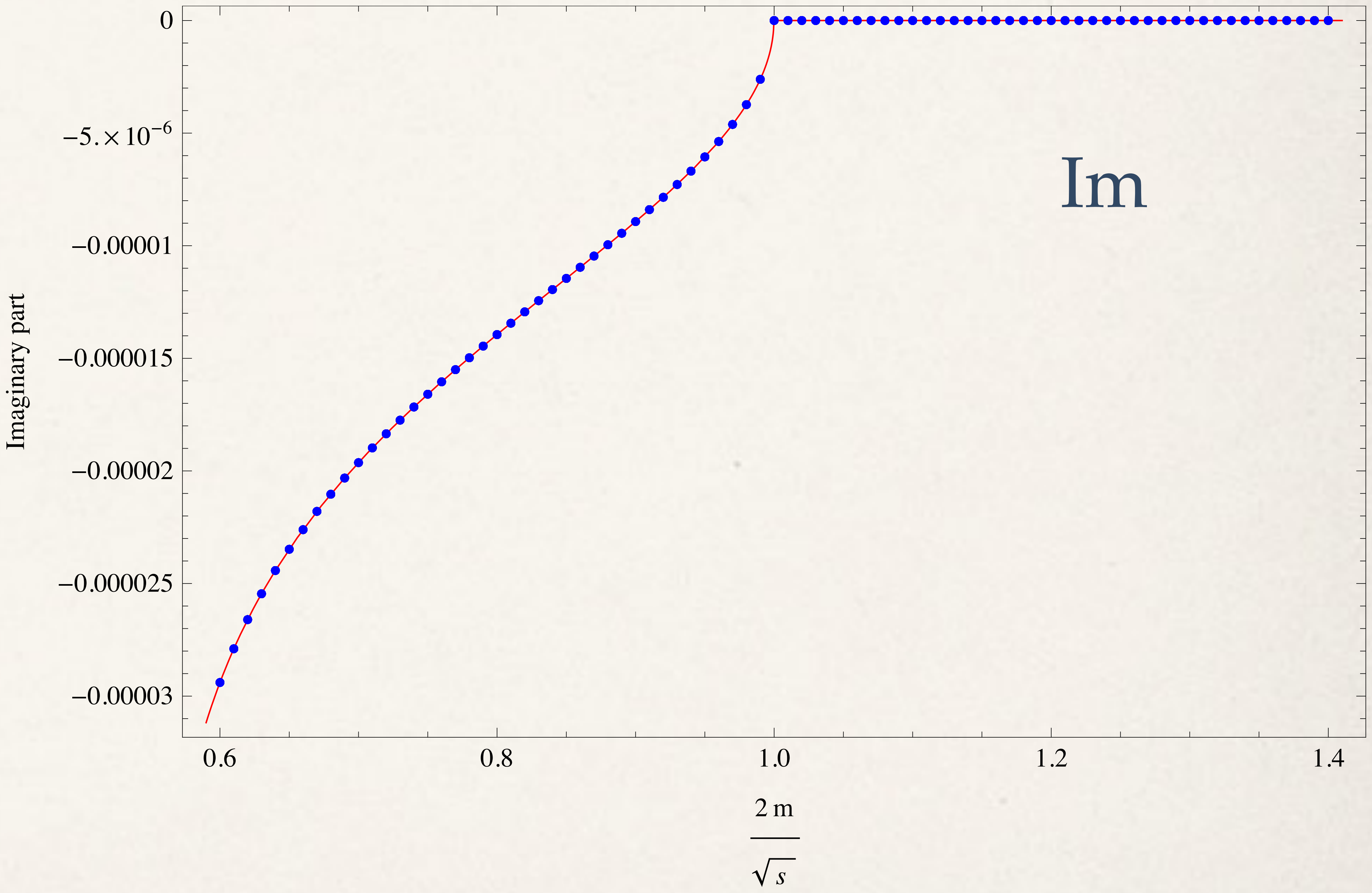
	Real part	Re Error	Imaginary part	Im Error
Analytic value	-1.001066E-10	0	-5.208136E-10	0
LT Duality	-1.001089E-10	9.051720E-16	-5.208556E-10	9.051461E-16

Numerical result produced with Cuhre (Cuba Library) [Hahn ’05], analytic values by LoopTools [Hahn ’99].

Numerical Implementation: Results

Triangle (all internal masses equal), varying the mass around the threshold:
Red curve is LoopTools, blue points is LTD

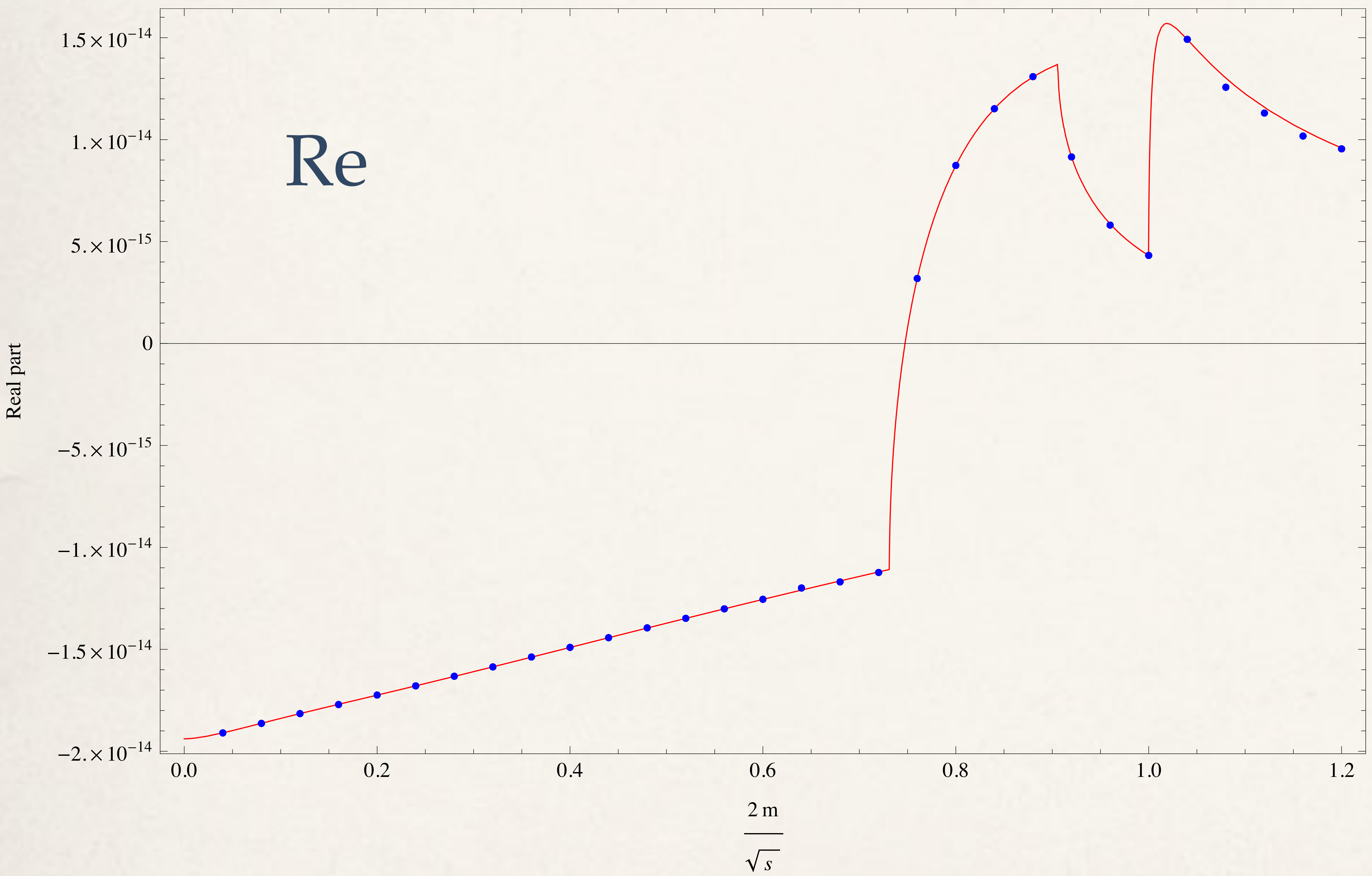


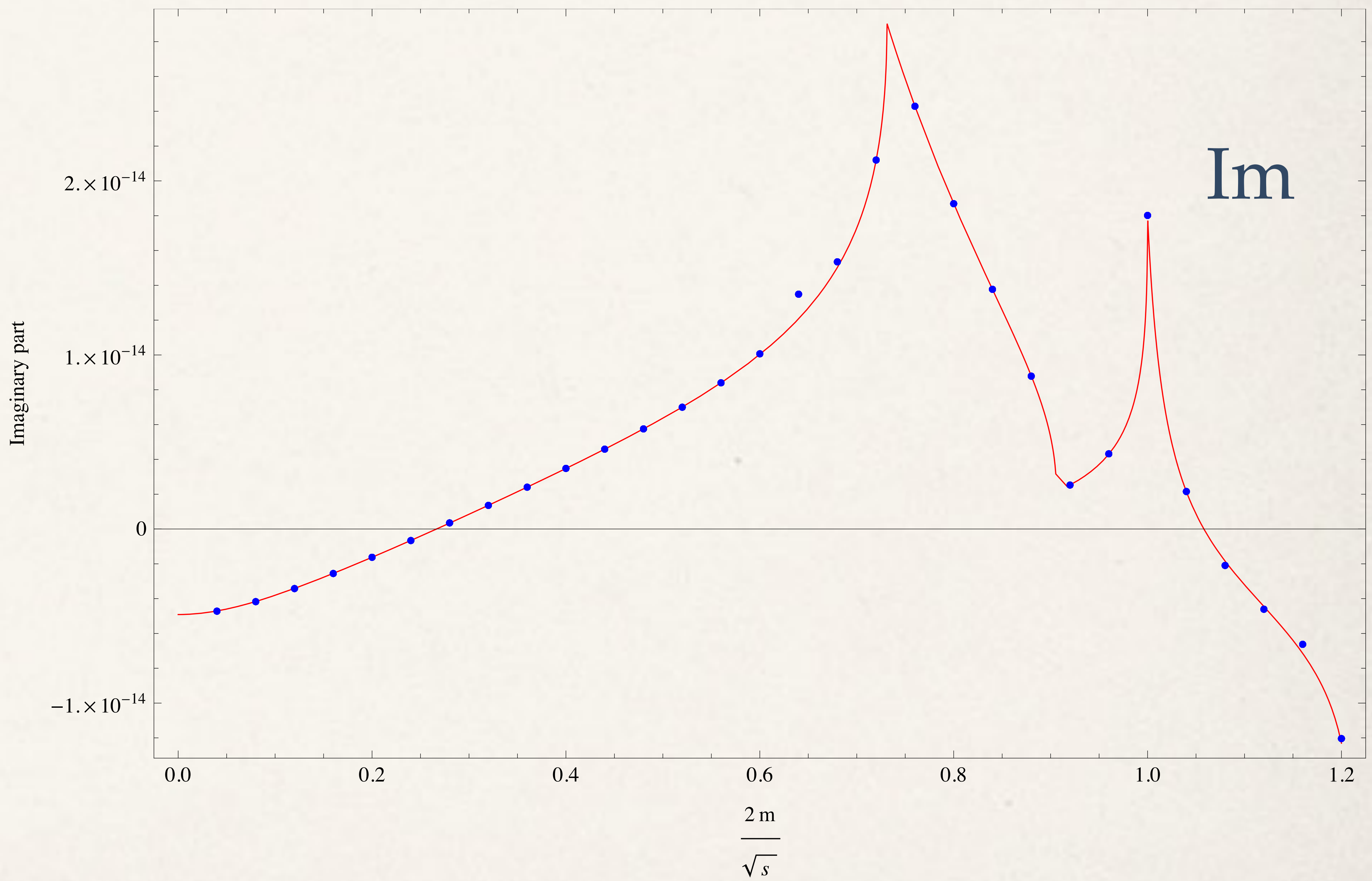


Red and blue are one top of each other! Good precision close to threshold!

Numerical Implementation: Results

Pentagon, varying the mass, all internal masses equal:
Red curve is LoopTools, blue points is LTD





Even complex structures are picked up! Up to 5 deformations!

Conclusions and outlook

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- General purpose numerical implementation soon!