Multiloop QCD & CBK identities

based on works of “Karlsruhe-Moscow group” (2001 – 2014 - . . .)

Pavel Baikov (MSU), Johannes Kühn (KIT)

Konstantin Chetyrkin (KIT)

LHCPHENONET, Berlin, 24 November 2014

- **intro: massless propagators in multiloop QCD: main applications**
- **mini-history and current status of the art**
- **the problem of reliabilty of 5-loop calculations**
- **Conformal Symmetry at work:**
  CBK (Crewther-Broadhurst-Kataev relations) and their implications for *very nontrivial* **checks of the five-loop results on the Adler function**
- **a new contribution[*] to the the Bjorken SR for polarized scattering qt $\mathcal{O}({\alpha_s}^4)$, its (ir)relevance to physics[*] and its compliance with the CBK-relation /new result!/**
- **relevance of $\mathcal{O}({\alpha_s}^4)$ term in the Bjorken sum rule: interplay between higher order PT corrections to the Bjorken SR for polarized scattering and higher twist contributions**

---

[*] ignited by: S.A. Larin, *The singlet contribition to the Bjorken SR for polarized DIS*, arxiV:1303.4021

multiloop /in pQCD but not only!/ problems reducible to massless propagators (p-integrals for brevity)

- 2-points correlators at large energies (massless term + $\mathcal{O}(m_q^2)$ corrections) via the optical theorem lead to:

  $R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$

  semi-leptonic τ-decays

  $\Gamma(Z \to$ hadrons$)$

  $\Gamma(H \to \bar{b}b)$ and $\Gamma(H \to gg)$ /via a top quark loop/

- beta-functions and anomalous dimensions

- coefficient functions in OPE of 2 local operators (DIS, SVZ sum rules,. . .)

- massless QCD propagators (e.g. gluon self-energy in the Landau gauge, useful for lattice)

Anom. Dim. from p-integrals

IRR (Infrared ReaRrangement)/Vladimirov, (78)/

+

IR R^* -operation /K. Ch., Smirnov (1984)/ lead to

main **THEOREM** of RG-calculations:

---

any (L+1) loop UV counterterm (read: any (L+1) loop $\overline{\text{MS}}$ Anomalous Dimension) can be expressed through pole and constant terms of some L-loop p-integrals

---

Corollary:

absorptive part of any (L+1) massless 2-point correlator can be expressed through pole and constant terms of some L-loop p-integrals

---

THEOREM + Corollary

is our key tool for multiloop RG calculations as it reduces the general task of evaluation of (L+1)-loop UV counterterms (absrptive part of (L+1)-loop 2-point massless correlators) to a well-defined and clearly posed purely mathematical problem: the calculation of L-loop p-integrals (that is massless propagator-type FI's).

In the following we shall refer to the latter as the L-loop Problem.

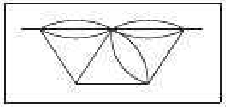
1. 1-loop Problem is trivial.

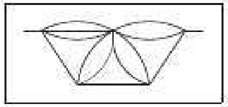
2. the 2-loop Problem was solved after inventing and developing the Gegenbauer polynomial technique in x-space (GPTX) (K.Ch.,F. Tkachov (1980)). Starting idea: : an x-space scalar propagator in $D = 4 - 2\epsilon$ dimensions can be always expanded in spherical harmonics ortogohonal on unit sphere in D-dimensions ($r_1 > r_2, \lambda = 1 - \epsilon$)

$$\frac{1}{(\vec{x}_1 - \vec{x}_2)^\lambda} = \frac{1}{r_1^\lambda} \sum_{n \geq 0} C_n^\lambda(\widehat{x}_1 \widehat{x}_2) \left(\frac{r_1}{r_2}\right)^n \quad (\widehat{x}_1 \widehat{x}_2 \equiv \vec{x_1} \vec{x_2}/(r_1 r_2))$$

GTPX is applicable to analytically compute some quite non-trivial three and even higher loop p-integrals. However, in practice calculations quickly get clumsy, especially for diagrams with numerators. Nevertheless, it proved to be very usefull in cases of scalar diagrams with many multilinear vertexes /appear frequently in supersymmetric theories/

An impressive example of GPTX in action (8 loops!!)

$$\bar{I}_{8d} = $$  $$ = \frac{1}{(4\pi)^8}\left(-\frac{1}{2048\,\varepsilon^4}+\frac{1}{192\,\varepsilon^3}-\frac{1}{64\,\varepsilon^2}-\frac{11}{192\,\varepsilon}\right),$$

$$\bar{I}_{8e} = $$  $$ = \frac{1}{(4\pi)^8}\left(-\frac{1}{6144\,\varepsilon^4}+\frac{1}{256\,\varepsilon^3}-\frac{19}{384\,\varepsilon^2}+\frac{5}{16\,\varepsilon}\right).$$

For their evaluation, we used the *Gegenbauer polynomials x-space technique* (GPXT). We briefly review this technique in Appendix C.

\+ many more similar integrals

copied from the recent work (January of 2013):

"The Leading Order Dressing Phase in ABJM Theory"

Andrea Mauri, Alberto Santambrogio, Stefano Scoleri, arXiv:1301.7732 [hep-th]

The main breakthrough at the three loop level happened with elaborating the method of integration by parts (IBP) of integrals.

Historical references:

At one loop, IBP (for DR integrals) was used in *, a crucial step — an appropriate modification of the *integrand* before differentiation was undertaken in ** (in momentum space, 2 and 3 loops) and in *** (in position space, 2 loops)

---

* G. 't Hooft and M. Veltman (1979)
*** A. Vasiliev, Yu. Pis'mak and Yu. Khonkonen (1981)
** F. Tkachov (1981); K. Ch. and F. Tkachov (1981)

With the use of IBP identities the 3-loop Problem was completely solved and corresponding (manually constructed) algorithm was effectively implemented first in SCHOONSCHIP CAS (Gorishny, Larin, Surguladze, and Tkachov) and then with FORM (Vermaseren, Larin, Tkachov, /1991/ ... Vermaseren 2000–2012).

This achievement resulted to a host of various important 3- and 4-four loop calculations performed by different teams during 80-th and 90-th.

Note that the 4-loop correction to the QCD β function was done only as late as in 1996 and using "massive" way /van Ritbergen, Vermaseren,and Larin/; the reason was too complicated combinatorics of the IR reduction

4-loop Problem has been under study in the Karlsruhe-Moscow group (P. Baikov, K.Ch., J. Kühn . . .) since late 90th. It is essentially solved by now with the help of $1/D$ expansion /reduction to masters, implemented as a FORM program **BAICER**/ and Glue-and-Cut symmetry (analytical evaluation of all necessary masters)

As a result during last 12 years in our group the the results for the Adler function, $R^{VV}(s)$ and a closely related quantity – Z-decay rate into hadrons have been extended by one more loop (that is to order α_s^4, which corresponds 5-loop for the Adler function).

These results $+$some others related to 5 and 4-loop correlators (Higgs decays into hadrons, etc.) can be found in:

Phys.Rev.Lett. 88 (2002) 01200
Phys.Rev.Lett. 95 (2005) 012003
Phys.Rev.Lett. 96 (2006) 012003
Phys.Rev.Lett. 97 (2006) 061803
Phys.Rev.Lett.101:012002,2008
Phys.Rev.Lett. 102 (2009) 212002
Phys.Rev.Lett.104:132004,2010
Phys.Rev.Lett. 108 (2012) 222003
JHEP 1207 (2012) 017
Phys.Lett. B714 (2012) 62-65

Example of Phenomenological Relevance

- With previous α_s^3 calculation* of Γ_Z^h, the theoretical errors were comparable with the experimental ones and, in despair, everybody was using the famous Kataev&Starshenko /1993/ estimation of the α_s^4 term **which (incidentally?) has happened to be quite close to the true number!**

- After our calculations the situation has become significantly better, especially for Γ_Z^h, where the the theoretical error was reduced by a factor of four!

- α_s^4 correction to the τ decay rate has decreased the theoretical error and improved stability the result wrt the renormalization scale (μ) variation

---

* Gorishnii, Kataev, Larin, (1991); Surguladze, Samuel, (1991); (both used Feynman gauge); K. Ch. (1997) (in general covariant gauge)

How reliable are available results at ≤ 4 loops and 5 loops?

A lot of things might go wrong in a multyloop (and usually **multi-month** if not in a sense **years**) calculation: from

- a trivial normalization factor buried somewhere in your programs and not expanded deeply enough in $\epsilon = 2 - D/2$ (this is exactly was happened with the very first calculation of the Adler function in $\mathcal{O}(\alpha_s^3)$ /Gorishny, Kataev and Larin, 1988/, the result was corrected only by three years after)

- ...

- to a **bug** in FORM which shows itself irregularly:
"it affected mainly very big programs that needed the fourth stage of the sorting rather intensively and it showed itself mainly with TFORM with a probability of occurring proportional to at least W^3 if W is the number of workers." (a recent citation of the FORM creator and leading maintainer Jos Vermaseren)

Four loop RG

At 4 loops every calculation was repeated (and confirmed!) by independent computation(s):

4-loop QED β function (in QCD) + R-ratio at α_s^3: an original (Feynman gauge result) /Gorishny, Larin, Kataev (1991)/ was confirmed 5 years later /K. Ch. (1996), (general covariant gauge)/

4-loop QCD β function /T. van Ritbergen, J. Vermaseren, S .Larin, (1997)/ was confirmed 8 years later /M. Czakon, (2004)/ (general covariant gauge in both cases)

4-loop quark anomalous dimension was computed 2 times (general covariant gauge in both cases) once with massless and once with massive setups with identical results

all master integrals apearing in 4-loop calculations (both massless props and massive tadpoles) have been evaluated many times independently, both analytically and numerically

Five loop RG

Here the situation is not so good: since 2002 we have performed many 5-loop RG calculations:

and (almost) no one has yet been confirmed in full by an independent computation. An exception is quark and gluon form factors to three loops in massless QCD: reduction to masters was done in 2 independent ways (with BAICER and FIRE); the pole part was found first by /S. Moch, J.A.M. Vermaseren, A. Vogt (2005)/

All master p-integrals appearing in 5-loop calculations (4-loop massless props) are certainly correct (confirmed by 2 analytical and one numerical — all independent — evaluations).

But how to check two reductions:

1. 4-loop input FI's (typically $\approx 10^8$) to 28 masters (performed via a sophisticated FORM program)

and

2. IR reduction from 5 to 4 loops (human made and also quite complicated)

Exactly at this point CBK (Crewther-Broadhurst-Kataev) relations (conformal symmetry based) enter into the game and provide us with extremely powerful and highly non-trivial test of the 5-loop Adler function /the check has been first suggested by Andrei Kataev/

Bjorken Sum Rule

- the polarized Bjorken sum rule ($a_s \equiv \frac{\alpha_s}{\pi}$)

$$\boldsymbol{B}jp(Q^2) = \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx = \frac{1}{6}|\frac{g_A}{g_V}|C^{Bjp}(a_s)$$

Coefficient function $C^{Bjp}(a_s)$ is fixed by OPE of two EM currents (up to power suppressed corrections)

$$i\int TV_\alpha^E(x)V_\beta^E(0)e^{iqx}dx|_{q^2\to\infty} \approx \frac{q^\sigma}{Q^2}\epsilon_{\alpha\beta\rho\sigma}\times$$

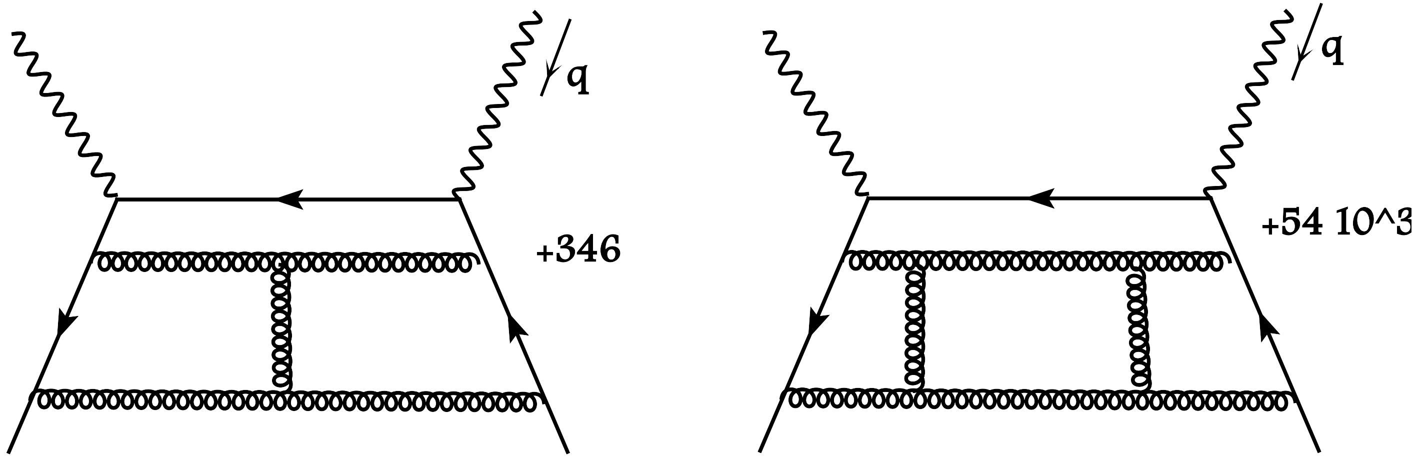
$$\{Tr[E^2 t_c]\ C^{Bjp}(a_s)\ \}\ A_\rho^c(0) + \ldots$$

where $E = diag(Q_i)$,

$V_\alpha^E = \overline{\psi}E\gamma_\rho\psi$ is the EM current,

$A_\rho^c = \overline{\psi}t^c\gamma_\rho\psi$ is (non-singlet!) axial current and $Q^2 = -q^2$

Evaluation of L-loop corrections to a CF of OPE could be done in terms of massless L-loop propagators (S. Gorisny, S. Larin and F. Tkachov (1982)) $\Longrightarrow$ one could use techniques developed for p-integrals



At order α_s^3 the CF was computed in early nineties /Larin, Vermaseren, 91/.The next order is contributed by about 54 thousand of 4-loop diagrams ... (cmp. to $\approx$ 20 thousand of 5-loop diagrams contributing to $R(s)$ at the same order)

We have computed the 4-loop $\mathcal{O}(\alpha_s^4)$ contribution to /PRL, 104 (2010) 1320004/ $C^{Bjp}(a_s)$ for *generic* color group with two aims: to confront to exp. data (here SU(3) would be enough) and to check the Adler function via CBK relation

The Crewther relation states that in the conformal invariant limit $(\beta \equiv 0)$ $C_{Bjp}(a_s)$ is related to the (nonsinglet) Adler function via the following beautiful equality

$$\boldsymbol{C_{Bjp}(a_s) D^{NS}(a_s))|_{c-i} = 1}$$

its generalization for real QCD reads:

$$\boldsymbol{C^{Bjp}(a_s) D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s} \Big[K^{NS} = K_1\, a_s + K_2\, a_s^2 + K_3\, a_s^3 + .}$$

Main ingredients of the derivation: the AVV 3-point function and constraints on it from (approximate) conformal invariance + Adler-Bardeen anomaly theorem

Crewther Relation: (short) bibliography

discovered: R.J. Crewther, *Phys. Rev. Lett.* **28**, 1421 (1972).

generalized for "real" QCD:

D.J. Broadhurst and A.L. Kataev, *Phys. Lett.* B 315, 179 (1993)

further developed:

G.T. Gabadadze and A.L. Kataev,*JETP Lett.* **61**, 448 (1995). S.J. Brodsky, G.T. Gabadadze, A.L. Kataev and H.J. Lu, *Phys. Lett.* B **372**, 133 (1996); . . .

A. Kataev and S. Mikhailov, Archive:1011.5248; recent discussion in A. Kataev, Archive: 1305.4605

"proven" (still with some hand-waving):

R.J. Crewther, *Phys. Lett.* B **397**, 137 (1997).

V. M. Braun, G. P. Korchemsky and D. Müller, Prog. Part. Nucl. Phys. **51**, 311 (2003)

Which exactly constraints come from the Crewther relation?

$$C^{Bjp}(a_s)C_D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K_1\, a_s + K_2\, a_s^2 + K_3\, a_s^3 + \ldots\Big]$$

If it is valid at order a_s^n, then at the next order a_s^{n+1}, we have

$$\left(d_{n+1} - C_{n+1}^{Bjp} + \text{interference terms}\right) a_s^{n+1} = \beta_0\, a_s\Big[K_n\, a_s^n\Big]$$

$$\alpha_s^1 : (d_1 - C_1) : C_F \Longleftrightarrow K_0 \equiv 0 \leftarrow \text{one constraint}$$

$$\alpha_s^2 : (d_2 - C_2) : C_F^2, T\, C_F, C_F\, C_A \Longleftrightarrow K_1 : C_F \leftarrow \text{two constraints}$$

$$\alpha_s^3 : (d_3 - C_3) : C_F^3\,, C_F^2 C_A\,, C_F C_A^2\,, C_F^2 T\,, C_F C_A T\,, C_F T^2$$

$$\Updownarrow$$

$$K_2 : C_F^2, C_F C_A, C_F T \leftarrow \text{three constraints}$$

At last, at $\mathcal{O}(\alpha_s^4)$ there exist exactly 12 color strtuctures:

$$C_F^4\,,\; C_F^3C_A\,,\; C_F^2C_A^2\,,\; C_FC_A^3\,,\; C_F^3T_Fn_f\,,\; C_F^2C_AT_Fn_f\,,$$
$$C_FC_A^2T_Fn_f\,,\; C_F^2T_F^2n_f^2\,,\; C_FC_AT_F^2n_f^2\,,\; C_FT_F^3n_f^3\,,\, d_F^{abcd}d_A^{abcd}\,,\, n_fd_F^{abcd}d_F^{abcd}$$

while the coefficient K_3 is contributed by only **6** color structures:

$$C_FT^2\,, C_F\,C_A^2\,, C_F^2\,T\,, C_F\,C_A\,T\,, C_F^2\,C_A\,, C_F^3$$

Thus, we have 12-6 $=$ **6** constraints on the difference

$$\boldsymbol{d_4 - (C^{Bjp})_4}$$

3 of them are very simple: the above difference cannot contain

$$C_F^4,\; d_F^{abcd}d_A^{abcd} \quad n_fd_F^{abcd}d_F^{abcd}$$

remaining three are a bit more complicated

	d_4	$(1/C^{Bjp})_4$
C_F^4	$\frac{4157}{2048} + \frac{3}{8}\zeta_3$	$\frac{4157}{2048} + \frac{3}{8}\zeta_3$
$n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5$
$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5$	$\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5$
$C_F T_f^3$	$-\frac{6131}{972} + \frac{203}{54}\zeta_3 + \frac{5}{3}\zeta_5$	$-\frac{605}{972}$
$C_F^2 T_f^2$	$\frac{5713}{1728} - \frac{581}{24}\zeta_3 + \frac{125}{6}\zeta_5 + 3\zeta_3^2$	$\frac{869}{576} - \frac{29}{24}\zeta_3$
$C_F T_f^2 C_A$	$\frac{340843}{5184} - \frac{10453}{288}\zeta_3 - \frac{170}{9}\zeta_5 - \frac{1}{2}\zeta_3^2$	$\frac{165283}{20736} + \frac{43}{144}\zeta_3 - \frac{5}{12}\zeta_5 + \frac{1}{6}\zeta_3^2$
$C_F^3 T_f$	$\frac{1001}{384} + \frac{99}{32}\zeta_3 - \frac{125}{4}\zeta_5 + \frac{105}{4}\zeta_7$	$-\frac{473}{2304} - \frac{391}{96}\zeta_3 + \frac{145}{24}\zeta_5$
$C_F^2 T_f C_A$	$\frac{32357}{13824} + \frac{10661}{96}\zeta_3 - \frac{5155}{48}\zeta_5 - \frac{33}{4}\zeta_3^2 - \frac{105}{8}\zeta_7$	$-\frac{17309}{13824} + \frac{1127}{144}\zeta_3 - \frac{95}{144}\zeta_5 - \frac{35}{4}\zeta_7$
$C_F T_f C_A^2$	$-\frac{(\cdots)}{(\cdots)} + \frac{8609}{72}\zeta_3 + \frac{18805}{288}\zeta_5 - \frac{11}{2}\zeta_3^2 + \frac{35}{16}\zeta_7$	$-\frac{(\cdots)}{(\cdots)} - \frac{59}{64}\zeta_3 + \frac{1855}{288}\zeta_5 - \frac{11}{12}\zeta_3^2 + \frac{35}{16}\zeta_7$
$C_F^3 C_A$	$-\frac{253}{32} - \frac{139}{128}\zeta_3 + \frac{2255}{32}\zeta_5 - \frac{1155}{16}\zeta_7$	$-\frac{8701}{4608} + \frac{1103}{96}\zeta_3 - \frac{1045}{48}\zeta_5$
$C_F^2 C_A^2$	$-\frac{592141}{18432} - \frac{43925}{384}\zeta_3 + \frac{6505}{48}\zeta_5 + \frac{1155}{32}\zeta_7$	$-\frac{435425}{55296} - \frac{1591}{144}\zeta_3 + \frac{55}{9}\zeta_5 + \frac{385}{16}\zeta_7$
$C_F C_A^3$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\zeta_3 - \frac{77995}{1152}\zeta_5 + \frac{605}{32}\zeta_3^2 - \frac{385}{64}\zeta_7$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\zeta_3 - \frac{12545}{1152}\zeta_5 + \frac{121}{96}\zeta_3^2 - \frac{385}{64}\zeta_7$

Comments:

The CBK test is highly non-trivial:

- four-loop box-type diagrams (in propagator kinematics) versus five loop propagators

- **No IR-trickery is neccessary** in calculation of $C_{Bjp}(a_s)$

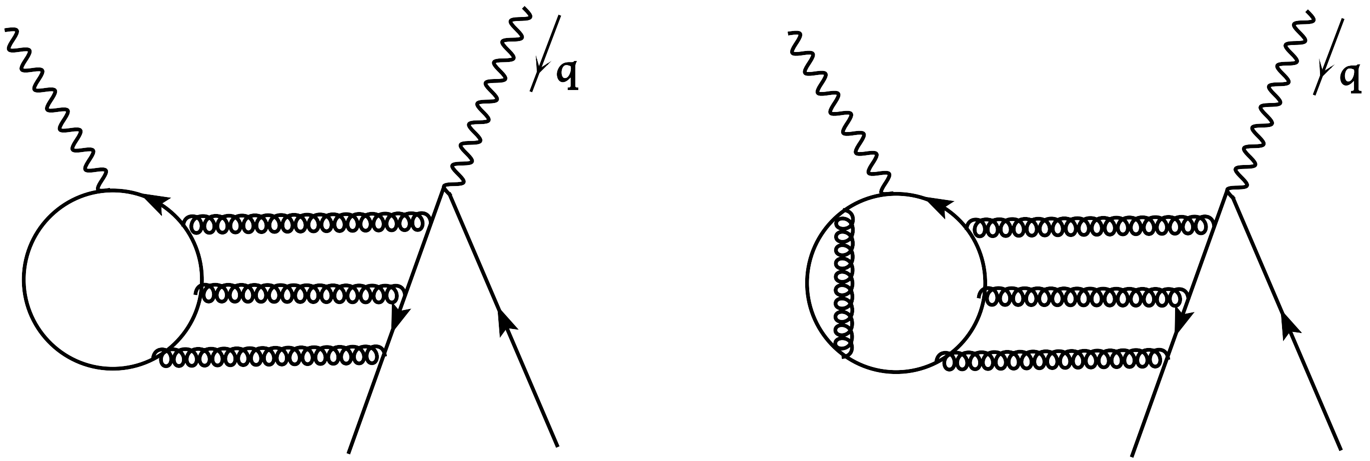
- final 4-loop p-integrals are much simpler for OPE (2 instead of 3 squared propagators inside)

- As a result we have been able to check that $C_{Bjp}(a_s)$ is indeed gauge-independent (the Adler finction was computed in the simplest Feynman gauge only!)

- Technical note: in the course of our calculations we have had to extend the Larin treatment of Hooft-Veltman γ_5 at 4-loop level (a natural object for the dim. reg., which really appears in the course of calculations, is $\gamma^{[\mu\nu\alpha]}$ instead of $\gamma_5\gamma^\mu$ with anticommuting γ_5; the mismatch should be corrected by the Larin factor)

Larin (arxiV:1303.4021) has drawn attention to "missing" contributions to the Bjorken sum rule:

$\mathcal{O}({\alpha_s}^3)$ $\mathcal{O}({\alpha_s}^4)$



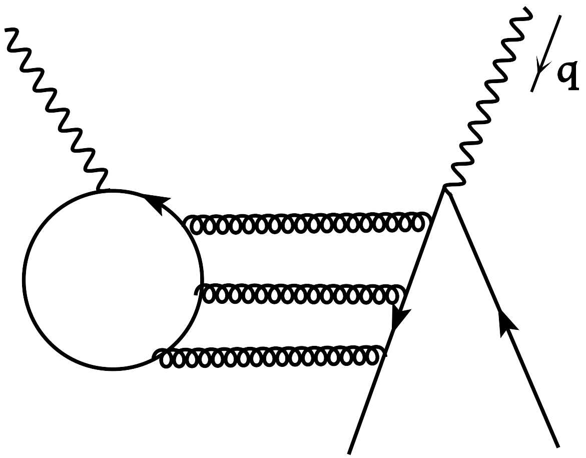
Two important questions:

1. Is CBK relation still valid (no missing contributions to the Adler function as far as we know)?

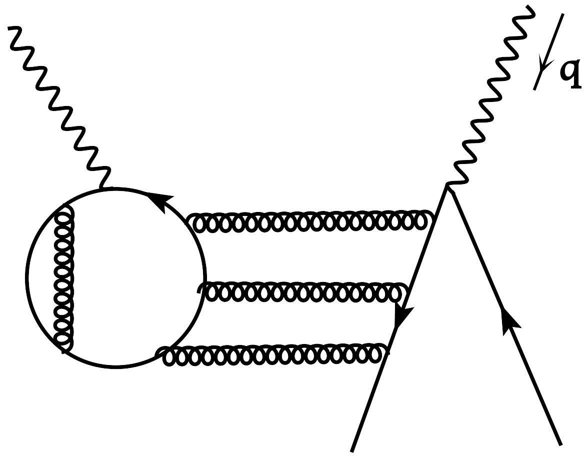
2. Any relevance of the new terms to the phenomenology?

Larin (arxiV:1303.4021) has drawn attention to "missing" contributions to the Bjorken sum rule:

$\mathcal{O}({\alpha_s}^3)$



$\mathcal{O}({\alpha_s}^4)$



color structures: $d^{abc}\,d^{abc} = 40/3$ $\quad$ $(C_f|n_f Tr|C_A) \times d^{abc}\,d^{abc}$

Two important questions:

1. Is CBK relation still valid (no missing contributions to the Adler function as far as we know)? YES

2. Any relevance of the new terms to the phenomenology? /probably/ NO

Bjorken Sum Rule: new term starting from order α_s^3:

- the polarized Bjorken sum rule ($a_s \equiv \frac{\alpha_s}{\pi}$)

$$\boldsymbol{B}jp(Q^2) = \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx = \frac{1}{6}|\frac{g_A}{g_V}|C^{Bjp}(a_s)$$

Coefficient function $C^{Bjp}(a_s)$ is fixed by OPE of two EM currents (up to power suppressed corrections)

$$i\int TV_\alpha^E(x)V_\beta^E(0)e^{iqx}dx|_{q^2\to\infty} \approx \frac{q^\sigma}{Q^2}\epsilon_{\alpha\beta\rho\sigma}\times$$

$$\left\{Tr[E^2t_c]\,C^{Bjp}(a_s) + 3Tr[E]\;C_{SI}^{Bjp}(a_s)\;\right\}A_\rho^c(0) + \dots$$

for c=3 (we neglect sea-quarks in nucleons but assume n_f=4 to have non-zero $Tr(E)$) the content of the figure bracket would read:

$$\frac{1}{6}C^{Bjp}(a_s) + 3\times\frac{2}{3}\,C_{SI}^{Bjp}(a_s) = \frac{1}{6}\left[C^{Bjp}(a_s) + 12\,C_{SI}^{Bjp}(a_s,\boldsymbol{m_c})\right]$$

Let start from the first question: under which conditions CBK relation will survive if Larin's terms are not zero?

$$\Big(C^{Bjp}_{NS}(a_s) + C^{Bjp}_{SI}(a_s)\Big)\, D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K^{NS} = K_1\, a_s + \ldots\Big]$$

with $C^{Bjp}_{SI}(a_s)$ proportional **new** color structure $d^{abc}d^{abc}$

As $\beta_0 = \frac{11}{12}C_A - \frac{T_f\, n_f}{3}$ we conclude that

1. $\mathcal{O}({\alpha_s}^3)$ term in C^{Bjp}_{SI} must be zero (Larin states that it is the case, but in his original work of 1991 there is no discussion at all!)

2. $\mathcal{O}({\alpha_s}^4)$ term in C^{Bjp}_{SI} must have the structure

$$\text{const}\, \beta_0\, d^{abc}d^{abc}$$

Indeed, direct calculation gives (preliminary result at α_s^4, for *massless* c-quark):

$$C^{Bjp}_{SI}(a_s) = 0 \cdot a_s^3 + \frac{1}{9}\beta_0\, d^{abc}d^{abc}\Big(\frac{\alpha_s}{\pi}\Big)^4$$

$$= \Big(\frac{110}{27} - \frac{20}{81}n_f\Big)\, a_s^4 = \Big(4.074 - 0.247 n_f\Big)\, a_s^4 \approx (n_f = 4) 3\, a_s^4$$

Thus, if we assume that c-loop is massless in SI-diagrams and set traditionally $n_f = 3$ for the rest (non-singlet) diagrams we get for full

$$C^{Bjp}_{full} = C^{Bjp}_{NS} + 12\, C^{Bjp}_{SI}$$

the following result:

$$\boldsymbol{C^{Bjp}_{full}(n_f = 3, 4)} =$$

$$\mathbf{1 - a_s - 3.583\, a_s^2 - 20.22\, a_s^3 + (-175.7 + 37.037 = -138.663)\, a_s^4}$$

BUT! we should remember 2 things:

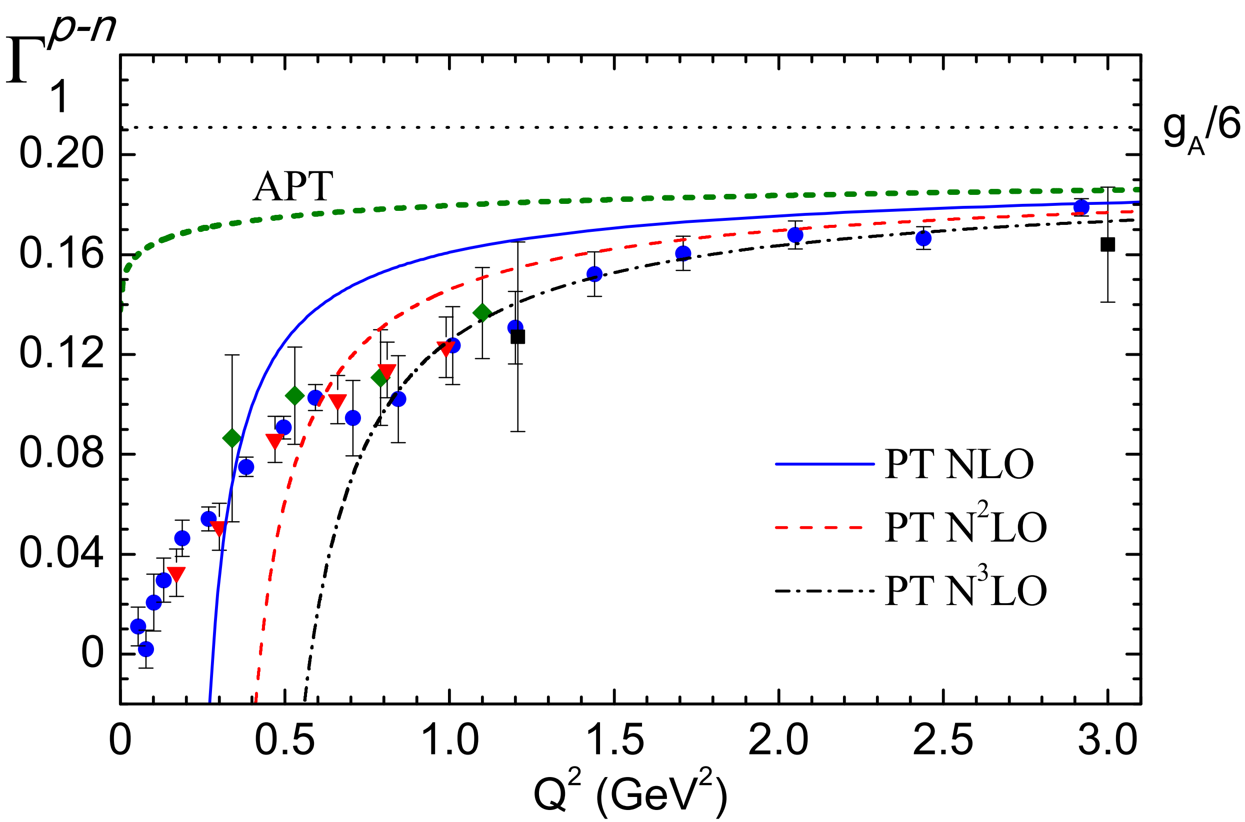
1. that typical momentum scale is $Q^2 = 3\,\mathrm{GeV}^2$, thus we should expect a "decoupling suppression" factor like $\boldsymbol{\frac{Q^2}{4m_c^2} < 1}$

2. with $m_c \neq 0$ the CBK relation stops to work (completely broken conformal invariance!) and one should expect a nonzero contribution already at $\boldsymbol{\mathcal{O}(\alpha_s{}^3)}$ (but again suppressed by a factor $\boldsymbol{\frac{Q^2}{4m_c^2}}$)

in the effective $n_f = 3$ QCD the Bjorken rule is unambiguous

/modulo higher twists!/

predictions of QCD which can be confronted with experimental data:



Perturbative part of the BSR as a function of the momentum transfer squared Q^2 in different orders in both the APT and standard PT approaches against the combined set of the Jefferson Lab (taken from V.L. Khandramai, R.S. Pasechnik, D.V. Shirkov, O.P. Solovtsova, O.V. Teryaev, *Four-loop QCD analysis of the Bjorken sum rule vs data*, Phys.Lett.B706:340-344,2012).

Higher twist contribution to the Borken (polarized) SR

$$\Gamma_1^{p-n}(Q^2) = \frac{|g_A|}{6}\left[1 - \Delta_{\rm Bj}^{\rm PT}(Q^2)\right] + \sum_{i=2}^{\infty} \frac{\mu_{2i}}{Q^{2i-2}} ,$$

Recent analysis of μ_4 from exp. data + PT
/Khandramai, Solovtsova, Teryaev, arXiv:1302.3952v1 (2013)/ has demonstrated a huge sensitivity to the higher order corrections:

B. Higher twist contribution

We expand our consideration, including the HT part which is presented in expression (1). In Table I we show our results for values of the coefficient μ_4 (the errors are statistical only) fitted to the low Q^2 data [10, 11] in different PT orders. One can see that μ_4-values extracted changes rather strongly between different PT orders. The absolute value of μ_4 decreases with the order of PT and just at N^{3}LO becomes compatible to zero. It can be interpreted as a manifestation of duality between higher orders and HT (see Ref. [21]). Note that a value extracted in the

TABLE I: Results of μ_4-extraction from the data on the Bjorken sum rule in different PT orders.

PT order	LO	NLO	N^{2}LO	N^{3}LO
μ_4, GeV^2	-0.037 ± 0.003	-0.025 ± 0.004	-0.012 ± 0.006	0.005 ± 0.008

leading-order (LO) is consistent with a value $\mu_4 = -0.047 \pm 0.025$ GeV^2 presented in Ref. [22] as well as with a value $\mu_4 = -0.028 \pm 0.019$ GeV^2 obtained from the next-to-leading-order (NLO) fit based on the x-dependent structure functions data [23] .

Conclusions

- the new ($\mathcal{O}(\alpha_s{}^4)$) contribution to $C_{pol}^{Bjorken}$ (discussed first by Larin a year ago) is computed (in the massless QCD!), it is a simple, purely rational number proportional to $\beta_0\, d^{abc} d^{abc}$ as is unambiguously dictated by the corresponding CBK relation

- Physically, due to the fact that for $n_f = 3$ the current J_μ^{EM} belongs to the $SU(3)_f$ octet ($Q_u + Q_d + Q_s \equiv 0$) the Larin's term is $completely$ saturated by heavy quarks: its effect on the phenomenology is $presumably$ small. The issue requires further investigation.

- conformal symmetry based CBK relations do provide higly non-trivial and very usefull constraints on VVA^{NS} triangle amplitude and, consequently, on the product

$$C_{Bjp}(a_s) D^{NS}(a_s)$$

  in massless QCD

- these constraints have been successfully tested at five loops (with an account of a new term pointed out by Sergei Larin)

$R(s)$ from p-integrals

Starting object: the polarization operator of EM quark current $j_\mu = e_q \bar{q} \gamma_\mu q$

$$\Pi_{\mu\nu}(q) = i \int \mathrm{d}x e^{iqx} \langle 0 | T[\ j^v_\mu(x) j^v_\nu(0)\ ] | 0 \rangle = (-g_{\mu\nu} q^2 + q_\mu q_\nu) \Pi(q^2)$$

related to $R(s)$ through

$$R(s) \approx \Im \Pi(s - i\delta)$$

Π is not completely physical due to a divergency of $T(j^v_\mu(x) j^v_\nu(0))$ at $x \to 0$, as a result its normalization mode and corresponding evolution equation reads ($(a_s \equiv \alpha_s/\pi)$, massless QCD)

$$\Pi = Z^{\mathrm{em}} + \Pi^B(-Q^2, \alpha_s^B)$$
$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s) a_s \frac{\partial}{\partial a_s} \right) \Pi = \gamma_{\mathrm{em}}(a_s)$$

At first sight, it would be advantageous to avoid this by considering (obviously RG invariant!) Adler function defined as $D = Q^2 \frac{\partial}{\partial Q^2} \Pi_0$ and which is related to $R(s)$ in a unique and simple way

$$R(s) \leftrightarrow D(Q) \quad \Longleftarrow \text{Adler function} \equiv Q^2 \frac{d}{d\,Q^2} \Pi(q^2) = Q^2 \int \frac{R(s)}{(s+Q^2)^2} \, ds$$

BUT, this is not true!

The reason: $\mathcal{O}(\alpha_s{}^L)$ /that is $(L+1)$-loop/ Adler function receives, obviously, contributions from $(L+1)$ loop p-integrals (*including their constant part*).

In fact, only L-loop integrals are enough $\leftarrow$ HUGE simplification. Indeed, let us rewrite the RG equation for Π as follows:

For massless $(L+1)$ loop $\Pi_0(L = \ln \frac{\mu^2}{Q^2}, a_s)$ RG equation amounts to

$$\frac{\partial}{\partial L}\Pi_0 = \gamma_{\rm em}(a_s) - \left(\beta(a_s) a_s \frac{\partial}{\partial a_s}\right) \Pi_0$$

$\nearrow$ **(L+1) loop anom. dim.**

$\nwarrow$ **L-loop integrals only contribute due to the factor of $\beta(a_s)$**

If one knows the rhs to α_s^L, then one could trivially construct the Adler function with the same accuracy!

Anomalous dimensions (as well as β-functions) are simple (no-scale) polynomilas in α_s /at least in MS-like schemes/ $\Longrightarrow$ one loop could be always "undone" with so-called Infrared Rearrangement trick