Nested sums and integrals for massive Feynman diagrams

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joint work with Jakob Ablinger, Johannes Blümlein, Carsten Schneider

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Outline

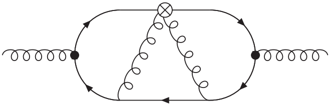
Introduction

Deep-inelastic scattering

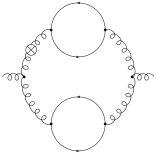
Heavy flavor Wilson coefficients

3-loop contributions in the asymptotic region by massive operator matrix elements at general N

2-loop: [Buza, Matiounine, Smith, Migneron, Neerven hep-ph/9601302]



[Ablinger, Blümlein, CGR, Schneider, Wißbrock 1403.1137]



[Ablinger, Blümlein, De Freitas, Hasselhuhn, Klein, Schneider, Wißbrock 1212.5950]

Main steps

1. Dimensional regularization, . . .
2. ε-expansion via symbolic summation tools [Ablinger, Blümlein, Round, Schneider 1210.1685]
3. Analytic continuation to complex N, . . .

Expressions in the ε-expansion

Nested sums

$$\sum_{i_1=1}^{N} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

Summand structure

Harmonic sums: [Vermaseren hep-ph/9806280]

$$a(i) = \frac{(\pm 1)^i}{i^m}$$

Expressions in the ε-expansion

Nested sums

$$\sum_{i_1=1}^{N} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

Summand structure

Generalized harmonic sums (S-sums): [Moch, Uwer, Weinzierl hep-ph/0110083]

$$a(i) = \frac{c^i}{i^m}$$

Expressions in the ε-expansion

Nested sums

$$\sum_{i_1=1}^{N} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

Summand structure

Generalized (inverse) binomial sums: [Ablinger, Blümlein, CGR, Schneider 1407.1822]

$$a(i) = \frac{c^i}{i^m}\binom{2i}{i}^{\pm 1}, \qquad a(i) = \frac{c^i}{i^m}, \qquad a(i) = \frac{c^i}{i^m(2i+1)}\binom{2i}{i}^{-1}$$

Nested sums and their analytic continuation

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2 \binom{2i}{i}} =$$

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} =$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{(-1)^j \binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} =$$

Nested sums and their analytic continuation

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2 \binom{2i}{i}} = 2 \int_0^1 \frac{x^N - 1}{x - 1} \operatorname{arctanh}\left(\sqrt{1-x}\right) dx$$

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = -\frac{1}{\pi} \int_0^1 \frac{(4x)^N - 1}{x - \frac{1}{4}} \left(\frac{\arccos(2x-1)^3}{6} - \zeta_2 \arccos(2x-1) \right) dx$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{(-1)^j \binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} = \frac{1}{2} \int_0^1 \frac{(-x)^N - 1}{x+1} \frac{x \, \mathrm{H}^*_{(0,\{-\frac{1}{4}\}),0,0}(x)}{\sqrt{x + \frac{1}{4}}} dx - {}_5F_4(const) \int_0^1 \frac{\left(\frac{x}{4}\right)^N - 1}{x - 4} \frac{x}{\sqrt{1-x}} dx$$

Nested sums and their analytic continuation

Examples

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Nested sums and their asymptotic expansion

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2\binom{2i}{i}} = 3\zeta_2 + \sqrt{\frac{\pi}{N}}\left(-2 + \frac{5}{12N} - \frac{21}{320N^2} + \dots\right)$$

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \frac{4^N}{\sqrt{\pi N}}\left(\frac{4\zeta_2}{3N} + \frac{3\zeta_2 - 8}{6N^2} + \frac{363\zeta_2 - 80}{288N^3} + \dots\right) - \frac{2}{3}\zeta_3$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{(-1)^j\binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} = const + (-1)^N\left(\frac{1}{5N^4} - \frac{16}{25N^5} + \frac{67}{125N^6} + \dots\right) - \frac{{}_5F_4(const)}{4^N}\sqrt{\frac{\pi}{N}}\left(-\frac{1}{3} + \frac{25}{72N} - \frac{683}{1152N^2} + \dots\right)$$

Integral representations

Integral representations

General form

$$\sum_{i=1}^{N} a(i) = const + \sum_{j=1}^{k} c_j^N \int_0^1 x^N f_j(x)\, dx$$

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$$\sum_{i=1}^{N} a(i) = const + \sum_{j=1}^{k} c_j^N \int_0^1 x^N f_j(x)\,dx$$

Mellin transform

$$\mathbf{M}\left(f(x)\right)(N) := \int_0^1 x^N f(x)\,dx$$

Iterated integrals

General form

$$\mathrm{H}_{a_1,\dots,a_k}(x) := \int_0^x f_{a_1}(t_1) \int_0^{t_1} f_{a_2}(t_2) \dots \int_0^{t_{k-1}} f_{a_k}(t_k)\, dt_k \dots dt_1$$

Harmonic polylogarithms

[Remiddi, Vermaseren hep-ph/9905237]

$$f_1(x) := \frac{1}{1-x}, \qquad f_0(x) := \frac{1}{x}, \qquad f_{-1}(x) := \frac{1}{1+x}$$

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Multiple polylogarithms

[Kummer 1840, Lappo-Danilevsky 1953, Goncharov 1998]

$$f_a(x) := \frac{1}{|a| - \mathrm{sign}(a)x}, \qquad f_0(x) := \frac{1}{x}$$

Iterated integrals

General form

$$\mathrm{H}^*_{\mathbf{a_1},\dots,\mathbf{a_k}}(x) := \int_x^1 f_{\mathbf{a_1}}(t_1) \int_{t_1}^1 f_{\mathbf{a_2}}(t_2) \dots \int_{t_{k-1}}^1 f_{\mathbf{a_k}}(t_k)\, dt_k \dots dt_1$$

Integrands with square roots

[Hermite 1883, Aglietti, Bonciani hep-ph/0401193, Ablinger, Blümlein, CGR, Schneider 1407.1822]

$$f_a(x) \quad := \quad \frac{\mathrm{sign}(1-a-0)}{x-a},$$

Iterated integrals

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$$f_a(x) := \frac{\mathrm{sign}(1-a-0)}{x-a},$$

$$f_{\{a_1,\ldots,a_k\}}(x) := f_{a_1}(x)^{1/2} \ldots f_{a_k}(x)^{1/2} \qquad k \geq 2,$$

Iterated integrals

General form

$$\mathrm{H}^*_{\mathbf{a_1},\dots,\mathbf{a_k}}(x) := \int_x^1 f_{\mathbf{a_1}}(t_1) \int_{t_1}^1 f_{\mathbf{a_2}}(t_2) \dots \int_{t_{k-1}}^1 f_{\mathbf{a_k}}(t_k)\, dt_k \dots dt_1$$

Integrands with square roots

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$$\begin{aligned} f_a(x) &:= \frac{\mathrm{sign}(1-a-0)}{x-a}, \\ f_{\{a_1,\dots,a_k\}}(x) &:= f_{a_1}(x)^{1/2} \dots f_{a_k}(x)^{1/2} \qquad k \geq 2, \\ f_{(a_0,\{a_1,\dots,a_k\})}(x) &:= f_{a_0}(x) f_{a_1}(x)^{1/2} \dots f_{a_k}(x)^{1/2} \qquad k \geq 1, \end{aligned}$$

Iterated integrals

General form

$$\mathrm{H}^*_{\mathbf{a_1},\dots,\mathbf{a_k}}(x) := \int_x^1 f_{\mathbf{a_1}}(t_1) \int_{t_1}^1 f_{\mathbf{a_2}}(t_2) \dots \int_{t_{k-1}}^1 f_{\mathbf{a_k}}(t_k)\, dt_k \dots dt_1$$

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Examples

Depth 1

- $\mathrm{H}^*_0(x) = \int_x^1 \frac{1}{t}\,dt = -\ln(x)$
- $\mathrm{H}^*_{\{0,1\}}(x) = \int_x^1 \frac{1}{\sqrt{t(1-t)}}\,dt = \arccos(2x-1)$
- $\mathrm{H}^*_{(0,\{1\})}(x) = \int_x^1 \frac{1}{t\sqrt{1-t}}\,dt = 2\,\mathrm{arctanh}(\sqrt{1-x})$

Depth 2

- $\mathrm{H}^*_{1,0}(x) = \int_x^1 \frac{1}{1-t}\,\mathrm{H}^*_0(t)\;dt = \mathrm{Li}_2(1-x)$
- $\mathrm{H}^*_{(2,\{0,8\}),1}(x) = \int_x^1 \frac{\ln(1-t)}{(2-t)\sqrt{t(8-t)}}\,dt$

Algebraic relations

Shuffle relations

$$
\begin{aligned}
\mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1}}(x) \\
\mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2},\mathbf{a_1}}(x) \\
&\vdots
\end{aligned}
$$

Algebraic relations

Shuffle relations

$$\begin{aligned} \mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1}}(x) \\ \mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2},\mathbf{a_1}}(x) \\ &\vdots \end{aligned}$$

Theorem

The functions $\mathrm{H}^*_{\mathbf{a_1},\dots,\mathbf{a_k}}(x)$ do not satisfy any algebraic relations apart from the shuffle relations.

Mellin transforms of iterated integrals

Theorem

Let $h(x) = \frac{rat(x)}{\sqrt{poly(x)}}$ and let $\mathbf{a_1}, \dots, \mathbf{a_k}$ be some generalized letters as above. If for all $i \in \{0, \dots, k\}$ the product

$$h(x) f_{\mathbf{a_1}}(x) \dots f_{\mathbf{a_i}}(x)$$

has at most one square-root singularity different from $x = 0$, then

$$\mathbf{M}\left(h(x)\,\mathrm{H}^*_{\mathbf{a_1},\dots,\mathbf{a_k}}(x)\right)(N)$$

is expressible in terms of generalized (inverse) binomial sums.

Examples

Basic examples

- $$\mathbf{M}\left(\frac{1}{\sqrt{x-a}}\right)(N) = \frac{(4a)^N}{(N+1/2)\binom{2N}{N}}\left(\sqrt{1-a} - \sqrt{-a} + \sqrt{1-a}\sum_{i=1}^{N}\frac{\binom{2i}{i}}{(4a)^i}\right)$$
- $$\mathbf{M}\left(\frac{1}{\sqrt{x(x-a)}}\right)(N) = \left(\frac{a}{4}\right)^N\binom{2N}{N}\left(2\operatorname{arcsinh}\left(\frac{1}{\sqrt{-a}}\right) + \sqrt{1-a}\sum_{i=1}^{N}\frac{(4/a)^i}{i\binom{2i}{i}}\right)$$

Mellin transform

Basic properties

$$\mathbf{M}\left(c_1 f_1(x) + c_2 f_2(x)\right)(N) = c_1 \mathbf{M}\left(f_1(x)\right)(N) + c_2 \mathbf{M}\left(f_2(x)\right)(N)$$

$$\mathbf{M}\left(f(x)\right)(N+k) = \mathbf{M}\left(x^k f(x)\right)(N)$$

$$\mathbf{M}\left(f_1(x)\right)(N)\,\mathbf{M}\left(f_2(x)\right)(N) = \mathbf{M}\left(\int_x^1 \frac{f_1(\frac{x}{t}) f_2(t)}{t}\, dt\right)(N)$$

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Summation

$$\sum_{k=1}^{N} c^k \mathbf{M}\left(f(x)\right)(k) = c^N \mathbf{M}\left(\frac{x}{x-\frac{1}{c}} f(x)\right)(N) - \mathbf{M}\left(\frac{x}{x-\frac{1}{c}} f(x)\right)(0)$$

Mellin transform (cont.)

Basic transforms

$$
\begin{aligned}
\frac{1}{N} &= \mathbf{M}\left(\frac{1}{x}\right)(N) \\
\binom{2N}{N} &= \frac{4^N}{\pi}\mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(N) \\
\frac{1}{N\binom{2N}{N}} &= \frac{1}{4^N}\mathbf{M}\left(\frac{1}{x\sqrt{1-x}}\right)(N)
\end{aligned}
$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)$$

$$= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)$$

Example

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Example

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\begin{aligned}
\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} &= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j) \\
&= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right) \\
&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{1}{x}\right)(i) \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right) \\
&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)
\end{aligned}
$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)$$

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$$= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)$$

$$= \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i) \, \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)$$

Example

$$\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2} = \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)$$

$$= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\left(\zeta_2+\mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)$$

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$$= \sum_{i=1}^{N}\frac{4^i}{\pi}\mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\,\mathbf{M}\left(\frac{\zeta_2-\mathrm{Li}_2(1-x)}{x}\right)(i)$$

$$= \sum_{i=1}^{N}\frac{4^i}{\pi}\mathbf{M}\left(\zeta_2\frac{\arccos(2x-1)}{x}-\frac{\arccos(2x-1)^3}{6x}\right)(i)$$

Example

$$\begin{aligned}
\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2} &= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)\\
&= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\left(\zeta_2+\mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)\\
&= \sum_{i=1}^{N}\binom{2i}{i}\mathbf{M}\left(\tfrac{1}{x}\right)(i)\left(\zeta_2+\mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)\\
&= \sum_{i=1}^{N}\binom{2i}{i}\mathbf{M}\left(\frac{\zeta_2-\mathrm{Li}_2(1-x)}{x}\right)(i)\\
&= \sum_{i=1}^{N}\frac{4^i}{\pi}\mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\,\mathbf{M}\left(\frac{\zeta_2-\mathrm{Li}_2(1-x)}{x}\right)(i)\\
&= \sum_{i=1}^{N}\frac{4^i}{\pi}\mathbf{M}\left(\zeta_2\frac{\arccos(2x-1)}{x}-\frac{\arccos(2x-1)^3}{6x}\right)(i)\\
&= \frac{4^N}{\pi}\mathbf{M}\left(\zeta_2\frac{\arccos(2x-1)}{x-1/4}-\frac{\arccos(2x-1)^3}{6(x-1/4)}\right)(N)-\frac{2}{3}\zeta_3
\end{aligned}$$

Mellin convolution via differential equations

Mellin convolution

Recall

$$\mathbf{M}\left(f_1(x)\right)(N)\,\mathbf{M}\left(f_2(x)\right)(N) = \mathbf{M}\left(\int_x^1 \frac{f_1(\frac{x}{t})f_2(t)}{t}\,dt\right)(N)$$

Approach

Compute

$$F(x) = \int_x^1 \frac{f_1(\frac{x}{t})f_2(t)}{t}\,dt$$

as solution of a differential equation

$$c_m(x)F^{(m)}(x) + \cdots + c_0(x)F(x) = r(x)$$

Construction of the ODEs

Parametric integration

Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

Construction of the ODEs

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Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

1. Find $g(x,t)$ and $c_0(x), \ldots, c_m(x)$ s.t.

$$c_m(x) D_x^m f(x,t) + \cdots + c_0(x) f(x,t) = D_t g(x,t)$$

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$$c_m(x) D_x^m f(x,t) + \cdots + c_0(x) f(x,t) = D_t g(x,t)$$

2. Transfer this to a relation of corresponding integrals

$$c_m(x) \int_{a(x)}^{b(x)} D_x^m f(x,t)\,dt + \cdots + c_0(x) \int_{a(x)}^{b(x)} f(x,t)\,dt =$$

Construction of the ODEs

Parametric integration

Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

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$$c_m(x) D_x^m f(x,t) + \cdots + c_0(x) f(x,t) = D_t g(x,t)$$

2. Transfer this to a relation of corresponding integrals

$$c_m(x) \int_{a(x)}^{b(x)} D_x^m f(x,t)\,dt + \cdots + c_0(x) \int_{a(x)}^{b(x)} f(x,t)\,dt = \\ = g(x, b(x)) - g(x, a(x))$$

Construction of the ODEs

Parametric integration

Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

1. Find $g(x,t)$ and $c_0(x),\ldots,c_m(x)$ s.t.
$$c_m(x)D_x^m f(x,t) + \cdots + c_0(x)f(x,t) = D_t g(x,t)$$
2. Transfer this to a relation of corresponding integrals
$$c_m(x)\int_{a(x)}^{b(x)} D_x^m f(x,t)\,dt + \cdots + c_0(x)\int_{a(x)}^{b(x)} f(x,t)\,dt = \\ = g(x,b(x)) - g(x,a(x))$$
3. Rewrite as ODE for $F(x)$
$$c_m(x)F^{(m)}(x) + \cdots + c_0(x)F(x) = r(x)$$

Computer algebra

Problem

Given: C a field, V a C-vector space, $f(x,t),\dots,D_x^m f(x,t) \in V$

Find: $g(x,t) \in V$ and $c_0(x),\dots,c_m(x) \in C$ s.t.

$$c_m(x)D_x^m f(x,t) + \cdots + c_0(x)f(x,t) = D_t g(x,t)$$

Computer algebra

Problem

Given: C a field, V a C-vector space, $f(x,t),\dots,D_x^m f(x,t) \in V$

Find: $g(x,t) \in V$ and $c_0(x),\dots,c_m(x) \in C$ s.t.

$$c_m(x)D_x^m f(x,t) + \cdots + c_0(x)f(x,t) = D_t g(x,t)$$

Differential fields

[Risch 1969, Singer, Saunders, Caviness 1985, Bronstein 1990/97, CGR 2012]

- (V, D_t) a differential field containing $f(x,t),\dots,D_x^m f(x,t)$
- $C = \mathrm{Const}_{D_t}(V)$ its constant field

Ore algebras

[Almkvist, Zeilberger 1990, Chyzak 2000, Chyzak, Kauers, Salvy 2009, Koutschan 2009, Chen, Kauers, Koutschan 2014]

- V a $C(t)[D_x, D_t]$-module containing $f(x,t)$
- C the field of rational functions in x

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\, \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\, \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)$$

$$F(x) = \int_x^1 \frac{\zeta_2 - \mathrm{Li}_2(1-t)}{t\sqrt{x(t-x)}}\, dt$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\, \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)$$

$$F(x) = \int_x^1 \frac{\zeta_2 - \mathrm{Li}_2(1-t)}{t\sqrt{x(t-x)}}\, dt$$

$$D_x^3 f - \frac{3(4x-3)}{2x(1-x)} D_x^2 f - \frac{7x-3}{x^2(1-x)} D_x f - \frac{1}{x^2(1-x)} f =$$
$$D_t \frac{4(t-x)^2 + 2t(t-x)\ln(t) - (1-t)(t+2x)(\zeta_2 - \mathrm{Li}_2(1-t))}{4(1-x)x^2(t-x)^2\sqrt{x(t-x)}}$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\, \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)$$

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$$D_x^3 f - \frac{3(4x-3)}{2x(1-x)} D_x^2 f - \frac{7x-3}{x^2(1-x)} D_x f - \frac{1}{x^2(1-x)} f = D_t \frac{4(t-x)^2 + 2t(t-x)\ln(t) - (1-t)(t+2x)(\zeta_2 - \mathrm{Li}_2(1-t))}{4(1-x)x^2(t-x)^2\sqrt{x(t-x)}}$$

$$F^{(3)}(x) - \frac{3(4x-3)}{2x(1-x)} F''(x) - \frac{7x-3}{x^2(1-x)} F'(x) - \frac{1}{x^2(1-x)} F(x) = \frac{1}{x^2(1-x)\sqrt{x(1-x)}}$$

Solution of the ODEs

Observations

- ODEs obtained factor completely into first-order ODEs

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Consequence

Solutions can be written as linear combinations of expressions of the form

$$\frac{r(x)}{\sqrt{p(x)}} \mathrm{H}^{*}_{\mathbf{a_1},\ldots,\mathbf{a_k}}(x)$$

where r is a rational function and p is a squarefree polynomial

Example

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k^2}}{j2^j}}{i\binom{2i}{i}} = \sum_{i=1}^{N} \frac{1}{i\binom{2i}{i}} \left(\frac{1}{2^i} \mathbf{M} \left(\frac{\mathrm{H}^*_{1,0}(x) - \zeta_2}{2-x} \right)(i) + \frac{5}{8}\zeta_3 \right)$$

involves convolution $F(x) = \int_x^1 \frac{\mathrm{H}^*_{1,0}(t) - \zeta_2}{x(2-t)\sqrt{1-\frac{x}{t}}} dt$ satisfying

$$F^{(5)}(x) + \cdots + \frac{15}{4x^3(1-x)(2-x)} F(x) = -\frac{3}{4x^4(1-x)^{7/2}(2-x)}$$

Example

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k^2}}{j 2^j}}{i\binom{2i}{i}} = \sum_{i=1}^{N} \frac{1}{i\binom{2i}{i}} \left(\frac{1}{2^i} \mathbf{M}\left(\frac{\mathrm{H}^*_{1,0}(x) - \zeta_2}{2-x}\right)(i) + \frac{5}{8}\zeta_3 \right)$$

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$$F^{(5)}(x) + \cdots + \frac{15}{4x^3(1-x)(2-x)} F(x) = -\frac{3}{4x^4(1-x)^{7/2}(2-x)}$$

Solution is given by

$$F(x) = \frac{1}{x}\left(\mathrm{H}^*_{0,0,(0,\{1\})}(x) - \mathrm{H}^*_{(0,\{1\}),(0,\{1\}),(0,\{1\})}(x) + \zeta_2 \mathrm{H}^*_{(0,\{1\})}(x)\right) - \frac{2}{x\sqrt{2-x}}\left(\mathrm{H}^*_{(0,\{2\}),0,(0,\{1\})}(x) - \mathrm{H}^*_{(0,\{1,2\}),(0,\{1\}),(0,\{1\})}(x) + \zeta_2 \mathrm{H}^*_{(0,\{1,2\})}(x)\right)$$

Rewrite Rules

Alternative method

Absorption of simple prefactors

$$\frac{1}{N}\mathbf{M}\left(f(x)\right)(N) = \mathbf{M}\left(\frac{1}{x}\int_x^1 f(t)\,dt\right)(N)$$

Alternative method

Absorption of simple prefactors

$$\frac{1}{N}\mathbf{M}(f(x))(N) = \mathbf{M}\left(\frac{1}{x}\int_x^1 f(t)\,dt\right)(N)$$

$$\binom{2N}{N}\mathbf{M}(f(x))(N) = \frac{4^N}{\pi}\mathbf{M}\left(\frac{1}{\sqrt{x}}\int_x^1 \frac{f(t)}{\sqrt{t-x}}\,dt\right)(N)$$

$$\frac{1}{N\binom{2N}{N}}\mathbf{M}(f(x))(N) = \frac{1}{4^N}\mathbf{M}\left(\frac{1}{x}\int_x^1 \frac{\sqrt{t}f(t)}{\sqrt{t-x}}\,dt\right)(N)$$

$$\vdots$$

Assume $f(x) = h(x)\,\mathrm{H}^*_{\vec{a}}(x)$

- Construct rewrite rules exploiting the structure of integrands
- Apply rules recursively
- Sufficient set of rules avoids construction of ODEs

Rules for Mellin convolution integrals

- $$\int_x^1 \frac{f(t)}{(t-c)\sqrt{t-x}}dt = \\ = \frac{1}{\sqrt{x-c}}\int_x^1 \frac{1}{\sqrt{t-c}}\left(\frac{f(1)}{\sqrt{1-t}} - \int_t^1 \frac{f'(u)}{\sqrt{u-t}}du\right)dt$$
- $$\int_x^1 \frac{f(t)}{\sqrt{t-a}\sqrt{t-x}}dt = \\ = \int_x^1 \frac{1}{t-a}\left(\frac{\sqrt{1-a}f(1)}{\sqrt{1-t}} - \int_t^1 \frac{\sqrt{u-a}f'(u)}{\sqrt{u-t}}du\right)dt$$
- $$\int_x^1 \frac{\sqrt{t}f(t)}{(t-c)\sqrt{t-x}}dt = \\ = \int_x^1 \frac{1}{t}\left(\frac{f(1)}{\sqrt{1-t}} - \int_t^1 \frac{\sqrt{u}f'(u)}{\sqrt{u-t}}du\right)dt + \\ + \frac{c}{\sqrt{x-c}}\int_x^1 \frac{1}{t\sqrt{t-c}}\left(\frac{f(1)}{\sqrt{1-t}} - \int_t^1 \frac{\sqrt{u}f'(u)}{\sqrt{u-t}}du\right)dt$$

Patterns

Theorem

$$\binom{2N}{N}\mathbf{M}\left(\frac{\mathrm{H}^*_{a_1,\ldots,a_k}(x)}{x-a_0}\right)(N)=\frac{4^N}{\pi}\mathbf{M}\left(\frac{\mathrm{H}^*_{\mathbf{b_0},\ldots,\mathbf{b_k}}(x)}{\sqrt{x(x-a_0)}}\right)(N)$$

where $\mathbf{b_i}=\{a_i,a_{i+1}\}$ for $i\in\{0,\ldots,k-1\}$ and $\mathbf{b_k}=\{1,a_k\}$

Theorem

$$\frac{1}{N\binom{2N}{N}}\mathbf{M}\left(\frac{x\,\mathrm{H}^*_{\mathbf{b_0},\ldots,\mathbf{b_k}}(x)}{(x-c)\sqrt{x(x-a_0)}}\right)(N)=$$

$$\frac{\pi}{4^N}\mathbf{M}\left(\frac{\mathrm{H}^*_{a_0,\ldots,a_k}(x)}{x}+c\frac{\mathrm{H}^*_{\mathbf{w},a_1,\ldots,a_k}(x)}{x\sqrt{x-c}}\right)(N)$$

where $\mathbf{w}=(a_0,\{c\})$ and $\mathbf{b_0},\ldots,\mathbf{b_k}$ are as above

Example revisited

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j)$$

Example revisited

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j)$$

$$= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x-1}\right)(i)\right)$$

Example revisited

$$\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2} = \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j)$$

$$= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\left(\zeta_2+\mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x-1}\right)(i)\right)$$

$$= \sum_{i=1}^{N}\binom{2i}{i}\mathbf{M}\left(\frac{\zeta_2-\mathrm{H}_{1,0}^*(x)}{x}\right)(i)$$

Example revisited

$$\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2} = \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j)$$

$$= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\left(\zeta_2+\mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x-1}\right)(i)\right)$$

$$= \sum_{i=1}^{N}\binom{2i}{i}\mathbf{M}\left(\frac{\zeta_2-\mathrm{H}_{1,0}^*(x)}{x}\right)(i)$$

$$= \sum_{i=1}^{N}\frac{4^i}{\pi}\mathbf{M}\left(\frac{\zeta_2\,\mathrm{H}_{\{0,1\}}^*(x)-\mathrm{H}_{\{0,1\},\{0,1\},\{0,1\}}^*(x)}{x}\right)(i)$$

Example revisited

$$
\begin{aligned}
\sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\frac{1}{j^2} &= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\sum_{j=1}^{i}\mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j)\\
&= \sum_{i=1}^{N}\frac{\binom{2i}{i}}{i}\left(\zeta_2+\mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x-1}\right)(i)\right)\\
&= \sum_{i=1}^{N}\binom{2i}{i}\mathbf{M}\left(\frac{\zeta_2-\mathrm{H}_{1,0}^*(x)}{x}\right)(i)\\
&= \sum_{i=1}^{N}\frac{4^i}{\pi}\mathbf{M}\left(\frac{\zeta_2\,\mathrm{H}_{\{0,1\}}^*(x)-\mathrm{H}_{\{0,1\},\{0,1\},\{0,1\}}^*(x)}{x}\right)(i)\\
&= \frac{4^N}{\pi}\mathbf{M}\left(\frac{\zeta_2\,\mathrm{H}_{\{0,1\}}^*(x)-\mathrm{H}_{\{0,1\},\{0,1\},\{0,1\}}^*(x)}{x-1/4}\right)(N)\\
&\quad -\frac{2}{3}\zeta_3
\end{aligned}
$$

Recall

Mellin transform

$$\mathbf{M}\left(f(x)\right)(n) := \int_0^1 x^n f(x)\, dx$$

Properties

$$\mathbf{M}\left(x^k f(x)\right)(n) = \mathbf{M}\left(f(x)\right)(n+k)$$

$$\mathbf{M}\left(\frac{f(x)}{x-c}\right)(n) = c^n \left(\int_0^1 \frac{f(x)}{x-c}\, dx + \sum_{i=1}^{n} \frac{1}{c^i} \mathbf{M}\left(f(x)\right)(i-1)\right)$$

$$\mathbf{M}\left(f(x)\right)(n) = \frac{1}{n+1}\left(f(1) - \mathbf{M}\left(f'(x)\right)(n+1)\right)$$

Rules for Mellin transforms involving square roots

- $$\mathbf{M}\left(\frac{f(x)}{\sqrt{x-a}}\right)(n) = \frac{(4a)^n}{(2n+1)\binom{2n}{n}}\left(\int_0^1 \frac{f(x)}{\sqrt{x-a}}\,dx + 2\sum_{i=1}^{n}\frac{\binom{2i}{i}}{(4a)^i}\left(\sqrt{1-a}f(1) - \mathbf{M}\left(\sqrt{x-a}f'(x)\right)(i)\right)\right)$$

- $$\mathbf{M}\left(\frac{f(x)}{\sqrt{x}\sqrt{x-a}}\right)(n) = \frac{a^n}{4^n}\binom{2n}{n}\left(\int_0^1 \frac{f(x)}{\sqrt{x}\sqrt{x-a}}\,dx + \sum_{i=1}^{n}\frac{(4/a)^i}{i\binom{2i}{i}}\left(\sqrt{1-a}f(1) - \mathbf{M}\left(\frac{\sqrt{x-a}}{\sqrt{x}}f'(x)\right)(i)\right)\right)$$

Generating functions

Ordinary generating function

$$\sum_{n=0}^{\infty} f(n)x^n$$

Properties

$$\sum_{n=0}^{\infty} \frac{f(n)}{n}x^n = \int_0^x \frac{1}{t}\sum_{n=0}^{\infty} f(n)t^n\,dt$$

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n} f(i)x^n = \frac{1}{1-x}\sum_{n=0}^{\infty} f(n)x^n$$

Rules for generating functions containing $\binom{2n}{n}$

- $$\sum_{n=0}^{\infty} x^n \binom{2n}{n} \sum_{i=0}^{n} f(i) = \\ = \frac{1}{\sqrt{\frac{1}{4}-x}} \left(\frac{f(0)}{2} + \frac{1}{4} \int_0^x \frac{1}{t\sqrt{\frac{1}{4}-t}} \sum_{n=0}^{\infty} t^n n \binom{2n}{n} f(n)\, dt \right)$$

- $$\sum_{n=0}^{\infty} x^n \frac{1}{(2n+1)\binom{2n}{n}} \sum_{i=0}^{n} f(i) = \\ = \frac{2}{\sqrt{x}\sqrt{4-x}} \int_0^x \frac{1}{\sqrt{t}\sqrt{4-t}} \sum_{n=0}^{\infty} t^n \frac{f(n)}{\binom{2n}{n}}\, dt$$

- $$\sum_{n=1}^{\infty} x^n \frac{1}{n\binom{2n}{n}} \sum_{i=0}^{n} f(i) = \\ = \sum_{n=1}^{\infty} x^n \frac{f(n)}{n\binom{2n}{n}} + \frac{\sqrt{x}}{\sqrt{4-x}} \int_0^x \frac{1}{\sqrt{t}\sqrt{4-t}} \sum_{n=0}^{\infty} t^n \frac{f(n)}{\binom{2n}{n}}\, dt$$

Rules for generating functions of moments

$$\sum_{n=k}^{\infty} c_n x^n = \int_0^1 \frac{(tx)^k}{1-tx} f(t)\,dt \quad \text{where} \quad c_n := \int_0^1 t^n f(t)\,dt$$

- $$\int_0^1 \frac{(tx)^k}{1-tx}\frac{t}{t-c} f(t)\,dt =$$
$$= \frac{1}{1-cx}\left(c_k x^k + \frac{1}{x}\int_0^x \left(f(1)\frac{t^{k+1}}{1-t} - \int_0^1 \frac{(tu)^{k+1}}{1-tu} u f'(u)\,du\right)dt\right)$$

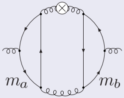
- $$\int_0^1 \frac{(tx)^k}{1-tx}\frac{1}{\sqrt{t}\sqrt{t-a}} f(t)\,dt =$$
$$= \frac{1}{\sqrt{1-ax}}\left(k c_k \int_0^x \frac{t^{k-1}}{\sqrt{1-at}}\,dt + \int_0^x \frac{1}{\sqrt{1-at}}\left(\sqrt{1-a} f(1)\frac{t^k}{1-t} - \int_0^t \frac{(tu)^k}{1-tu}\sqrt{u}\sqrt{u-a} f'(u)\,du\right)dt\right)$$

Two masses

Master integrals for the case of two masses

Two massive quark lines

Integrals depend on an additional parameter $y = \frac{m_b^2}{m_a^2}$



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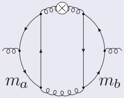

Compute the master integrals

joint work in progress with Andreas von Manteuffel, Moulay Barkatou, Suzy Maddah, Jacques-Arthur Weil

1. Differential systems w.r.t. y for master integrals
2. Simplify and decouple the systems
3. Initial conditions by equal mass case
4. Solve in terms of nested integrals

Conclusion

Nested (inverse) binomial sums and their associated integrals

- Conversion between sum and integral representations
- Asymptotic expansion of the sums
- Basis of new iterated integrals involving square roots
- Conditions for Mellin transform as nested binomial sums

Parametric integration

Compute ODEs for parameter integrals (e.g. Mellin convolutions)

Rewrite rules

- Patterns in Mellin convolutions
- Reduce computational effort by available rules and patterns
- Also for Mellin transforms and generating functions