Colorful NNLO – Completely local subtractions for fully differential predicitions at NNLO

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Message

Message: can compute NNLO cross sections like you always though you would

1. Compute relevant IR factorization formulae
2. Use them to construct general, explicit, local subtractions (see Zoltán's talk)
3. Integrate subtractions once and for all, verify pole cancellation (this talk)
4. Apply the generic scheme to specific process (this talk)

Subtraction at NNLO – a quick overview

IR factorization formulae

Collinear and soft currents at NNLO are known

- Tree level 3-parton splitting functions and double soft gg and $q\bar{q}$ currents

(Campbell, Glover 1997; Catani, Grazzini 1998; Del Duca, Frizzo, Maltoni 1999; Kosower 2002)

- One-loop 2-parton splitting functions and soft gluon current

(Bern, Dixon, Dunbar, Kosower 1994; Bern, Del Duca, Kilgore, Schmidt 1998-9; Kosower, Uwer 1999; Catani, Grazzini 2000; Kosower 2003)

Subtraction at NNLO – structure

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ regularizes the double unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ regularizes the single unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
3. $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$ accounts for the overlap of $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$
4. $\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}$ regularizes the single unresolved limits of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$
5. $\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}}$ regularizes the singly-unresolved limit of $\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$

Colorful NNLO – general features

Colorful NNLO: Completely local subtractions for fully differential predictions at NNLO

Construct subtractions starting from universal IR limit formulae

- tree and one-loop splitting functions, soft currents
- simple and general procedure for matching of limits to avoid multiple subtractions
- extension over full phase space based on momentum mappings that can be generalized to any number of unresolved partons

Fully local in color, spin and momentum space

- no need to consider the color decomposition of real emission ME's
- azimuthal correlations correctly taken into account in gluon splitting
- can check explicitly that the ratio of subtractions to the real emission cross section tends to unity in any IR limit

Straightforward to constrain subtractions to near singular regions

- gain in efficiency
- independence of physical results on phase space cut is strong check

Integrating the subtractions

Integrating the subtractions

Momentum mappings used to define the counterterms

$$\{p\}_{n+p} \xrightarrow{R} \{\tilde{p}\}_n \quad \Rightarrow \quad \mathrm{d}\phi_{n+p}(\{p\}; Q) = \mathrm{d}\phi_n(\{\tilde{p}\}_n^{(R)}; Q)[\mathrm{d}p_{p,n}^{(R)}]$$

- implement exact momentum conservation, recoil distributed democratically (can be generalized to any p)
- different collinear and soft mappings (R labels precise limit)
- exact factorization of phase space

Counterterms are products (in color and spin space) of

- factorized ME's independent of variables in $[\mathrm{d}p_{p,n}^{(R)}]$
- singular factors (AP functions, soft currents), to be integrated over $[\mathrm{d}p_{p,n}^{(R)}]$

$$\mathcal{X}_R(\{p\}_{n+p}) = (8\pi\alpha_\mathrm{s}\mu^{2\epsilon})^p\, Sing_R(p_p^{(R)}) \otimes |\mathcal{M}_n^{(0)}(\{\tilde{p}\}_n^{(R)})|^2$$

Can compute once and for all the integral over unresolved partons

$$\int_p \mathcal{X}_R(\{p\}_{n+p}) = (8\pi\alpha_\mathrm{s}\mu^{2\epsilon})^p \left[\int_p Sing_R(p_p^{(R)})\right] \otimes |\mathcal{M}_n^{(0)}(\{\tilde{p}\}_n^{(R)})|^2$$

List of master integrals

Int	status
$\mathcal{I}^{(k)}_{1\mathcal{C},0}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},1}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},2}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},3}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},4}$	✔
$\mathcal{I}^{(k,l)}_{1\mathcal{C},5}$	✔
$\mathcal{I}^{(k,l)}_{1\mathcal{C},6}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},7}$	✔
$\mathcal{I}_{1\mathcal{C},8}$	✔

Int	status
$\mathcal{I}_{1\mathcal{S},0}$	✔
$\mathcal{I}_{1\mathcal{S},1}$	✔
$\mathcal{I}_{1\mathcal{S},2}$	$(m>3)$ ✘
$\mathcal{I}^{(k)}_{1\mathcal{S},3}$	✔
$\mathcal{I}_{1\mathcal{S},4}$	✔
$\mathcal{I}_{1\mathcal{S},5}$	✔
$\mathcal{I}_{1\mathcal{S},6}$	✔
$\mathcal{I}_{1\mathcal{S},7}$	✔

Int	status
$\mathcal{I}_{1\mathcal{CS},0}$	✔
$\mathcal{I}_{1\mathcal{CS},1}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{CS},2}$	✔
$\mathcal{I}_{1\mathcal{CS},3}$	✔
$\mathcal{I}_{1\mathcal{CS},4}$	✔

Int	status
$\mathcal{I}^{(k,l)}_{12\mathcal{C},1}$	✔
$\mathcal{I}^{(k,l)}_{12\mathcal{C},2}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},3}$	✔
$\mathcal{I}^{(k,l)}_{12\mathcal{C},4}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},5}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},6}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},7}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},8}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},9}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},10}$	✔

Int	status
$\mathcal{I}_{2\mathcal{S},1}$	✔
$\mathcal{I}_{2\mathcal{S},2}$	✔
$\mathcal{I}_{2\mathcal{S},3}$	✔
$\mathcal{I}_{2\mathcal{S},4}$	✔
$\mathcal{I}_{2\mathcal{S},5}$	✔
$\mathcal{I}_{2\mathcal{S},6}$	✔
$\mathcal{I}_{2\mathcal{S},7}$	✔
$\mathcal{I}_{2\mathcal{S},8}$	✔
$\mathcal{I}_{2\mathcal{S},9}$	✔
$\mathcal{I}_{2\mathcal{S},10}$	✔
$\mathcal{I}_{2\mathcal{S},11}$	✔
$\mathcal{I}_{2\mathcal{S},12}$	✔
$\mathcal{I}_{2\mathcal{S},13}$	✔
$\mathcal{I}_{2\mathcal{S},14}$	✔
$\mathcal{I}_{2\mathcal{S},15}$	✔
$\mathcal{I}_{2\mathcal{S},16}$	✔
$\mathcal{I}_{2\mathcal{S},17}$	✔
$\mathcal{I}_{2\mathcal{S},18}$	✔
$\mathcal{I}_{2\mathcal{S},19}$	✔
$\mathcal{I}_{2\mathcal{S},20}$	✔
$\mathcal{I}_{2\mathcal{S},21}$	✔
$\mathcal{I}_{2\mathcal{S},22}$	✔
$\mathcal{I}_{2\mathcal{S},23}$	✔

Int	status
$\mathcal{I}^{(k)}_{12\mathcal{S},1}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{S},2}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{S},3}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{S},4}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{S},5}$	✔
$\mathcal{I}_{12\mathcal{S},6}$	✔
$\mathcal{I}_{12\mathcal{S},7}$	✔
$\mathcal{I}_{12\mathcal{S},8}$	✔
$\mathcal{I}_{12\mathcal{S},9}$	✔
$\mathcal{I}_{12\mathcal{S},10}$	✔
$\mathcal{I}_{12\mathcal{S},11}$	✔
$\mathcal{I}_{12\mathcal{S},12}$	✔
$\mathcal{I}_{12\mathcal{S},13}$	✔

Int	status
$\mathcal{I}^{(k)}_{12\mathcal{CS},1}$	✔
$\mathcal{I}_{12\mathcal{CS},2}$	✔
$\mathcal{I}_{12\mathcal{CS},3}$	✔

Int	status
$\mathcal{I}^{(j,k,l,m)}_{2\mathcal{C},1}$	✔
$\mathcal{I}^{(j,k,l,m)}_{2\mathcal{C},2}$	✔
$\mathcal{I}^{(j,k,l,m)}_{2\mathcal{C},3}$	✔
$\mathcal{I}^{(j,k,l,m)}_{2\mathcal{C},4}$	✔
$\mathcal{I}^{(-1,-1,-1,-1)}_{2\mathcal{C},5}$	✔/✘
$\mathcal{I}^{(k,l)}_{2\mathcal{C},6}$	✔

Int	status
$\mathcal{I}^{(k)}_{2\mathcal{CS},1}$	✔
$\mathcal{I}^{(k)}_{2\mathcal{CS},2}$	✔
$\mathcal{I}^{(2)}_{2\mathcal{CS},2}$	✔/✘
$\mathcal{I}^{(k)}_{2\mathcal{CS},3}$	✔
$\mathcal{I}^{(k)}_{2\mathcal{CS},4}$	✔
$\mathcal{I}^{(k)}_{2\mathcal{CS},5}$	✔

✔: pole coefficients known analytically, finite numerically

✘: pole coefficients known analytically up to $\frac{1}{\epsilon^2}$, finite and $\frac{1}{\epsilon}$ numerically

List of master integrals

Note

- not the usual notion of master integrals: no IBPs used
- algebraic and symmetry relations exploited to reduce to this basic set
- but set is not linearly independent, known relations used for checks

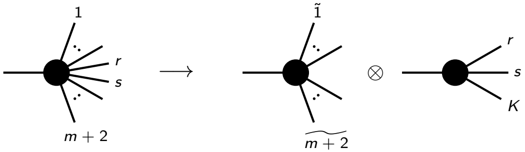
Phase space integrals - an example

Abelian double soft counterterm: among many others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2}$ we find

$$\left(\mathcal{S}_{rs}^{(0,0)}\right)^{\mathrm{ab}} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \sum_{i,k,j,l} \frac{1}{4} \frac{s_{ik}}{s_{ir}s_{kr}} \frac{s_{jl}}{s_{js}s_{ls}} |\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$
$$\times (1 - y_{rQ} - y_{sQ} + y_{rs})^{d_0' - m(1-\epsilon)} \Theta(y_0 - y_{rQ} - y_{sQ} + y_{rs})$$

The set of m momenta, $\{\tilde{p}\}$, is obtained by a momentum mapping which leads to an exact factorization of phase space

$$\{p\}_{m+2} \xrightarrow{\mathsf{S}_{rs}} \{\tilde{p}\} : \quad \mathrm{d}\phi_{m+2}(\{p\}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}; Q)[\mathrm{d}p_{2,m}^{(rs)}]$$



Phase space integrals - an example

Abelian double soft counterterm: among many others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$ we find

$$\left(\mathcal{S}_{rs}^{(0,0)}\right)^{\mathrm{ab}} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \sum_{i,k,j,l} \frac{1}{4} \frac{s_{ik}}{s_{ir}s_{kr}} \frac{s_{jl}}{s_{js}s_{ls}} |\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$
$$\times (1 - y_{rQ} - y_{sQ} + y_{rs})^{d_0' - m(1-\epsilon)} \Theta(y_0 - y_{rQ} - y_{sQ} + y_{rs})$$

The set of m momenta, $\{\tilde{p}\}$, is obtained by a momentum mapping which leads to an exact factorization of phase space

$$\{p\}_{m+2} \xrightarrow{\mathrm{S}_{rs}} \{\tilde{p}\} : \quad \mathrm{d}\phi_{m+2}(\{p\}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}; Q)[\mathrm{d}p_{2,m}^{(rs)}]$$

Then we must compute

$$\int [\mathrm{d}p_{2,m}^{(rs)}] \left(\mathcal{S}_{rs}^{(0,0)}\right)^{\mathrm{ab}} \equiv \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i,k,j,l} [\mathrm{S}_{rs}^{(0)}]^{(i,k),(j,l)} |\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$

where $[\mathrm{S}_{rs}^{(0)}]^{(i,k),(j,l)} \equiv [\mathrm{S}_{rs}^{(0)}]^{(i,k),(j,l)}(p_i, p_k, p_j, p_l, \epsilon, y_0, d_0')$ is a kinematics dependent function.

Abelian double soft integral

For simplicity, consider the terms in the sum where $j = i$ and $l = k$: $[\mathrm{S}_{rs}^{(0)}]^{(i,k),(i,k)}$.
Kinematical dependence is through $\cos\chi_{ik} = \measuredangle(p_i, p_k)$, we set $\cos\chi_{ik} = 1 - 2Y_{ik,Q}$, i.e., $Y_{ik,Q}$ is between zero and one.

Using angles and energies in the Q rest frame with some specific orientation to parametrize the factorized phase space measure, $[\mathrm{d}p_{2,m}^{(rs)}]$, we find that $[\mathrm{S}_{rs}^{(0)}]^{(i,k),(i,k)}$ is proportional to

$$\mathcal{I}_{2\mathcal{S},2}(Y_{ik,Q};\epsilon, y_0, d_0') = -\frac{4\Gamma^4(1-\epsilon)}{\pi\Gamma^2(1-\epsilon)}\frac{B_{y_0}(-2\epsilon, d_0')}{\epsilon} Y_{ik,Q}\int_0^{y_0} \mathrm{d}y\, y^{-1-2\epsilon}(1-y)^{d_0'-1+\epsilon}$$
$$\times \int_{-1}^{1} \mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1}\mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon}\left[f(\vartheta,\varphi;0)\right]^{-1}\left[f(\vartheta,\varphi;Y_{ik,Q})\right]^{-1}$$
$$\times \left[Y(y,\vartheta,\varphi;Y_{ik,Q})\right]^{-\epsilon}{}_2F_1\big(-\epsilon,-\epsilon,1-\epsilon,1-Y(y,\vartheta,\varphi;Y_{ik,Q})\big)$$

where

$$f(\vartheta,\varphi;Y_{ik,Q}) = 1 - 2\sqrt{Y_{ik,Q}(1-Y_{ik,Q})}\sin\vartheta\cos\varphi - (1-2Y_{ik,Q})\chi\cos\vartheta$$

$$Y(y,\vartheta,\varphi;\chi) = \frac{4(1-y)Y_{ik,Q}}{[2(1-y)+y\,f(\vartheta,\varphi;0)][2(1-y)+y\,f(\vartheta,\varphi;Y_{ik,Q})]}$$

Solving the integrlas

Strategy for computing the master integrals

1. write phase space in terms of angles and energies
2. angular integrals in terms of Mellin-Barnes representations
3. resolve the ϵ poles by analytic continuation
4. MB integrals to Euler-type integrals, pole coefficients are finite parametric integrals

1. choose explicit parametrization of phase space
2. write the parametric integral representation in chosen variables
3. resolve the ϵ poles by sector decomposition
4. pole coefficients are finite parametric integrals

5. evaluate the parametric integrals in terms of multiple polylogs
6. simplify result (optional)

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

This admits the following Mellin-Barnes representation $(j = j_1 + \ldots + j_n)$

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon} \frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)}$$
$$\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left[\prod_{k=1}^{n}\prod_{l=k}^{n} \frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}} \Gamma(-z_{kl})\ (v_{kl})^{z_{kl}}\right] \left[\prod_{k=1}^{n}\Gamma(j_k+z_k)\right] \Gamma(1-j-\epsilon-z)\,.$$

where $v_{kl} = \dfrac{p_k \cdot p_l}{2}$ for $k \neq l$ and $v_{kk} = \dfrac{p_k^2}{4}$ while

$$z = \sum_{k=1}^{n}\sum_{l=k}^{n} z_{kl} \qquad \text{and} \qquad z_k = \sum_{l=1}^{k} z_{lk} + \sum_{l=k}^{n} z_{kl}\,.$$

Methods of integration - MB to parametric integrals

Basic idea is to express products of gamma functions as real integrals

$$
\begin{aligned}
I &= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \cdots \Gamma[a+z_1+z_2]\Gamma[b-z_1-z_2] \cdots v_1^{z_1} v_2^{z_2} \\
&= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \cdots \Gamma[a+b] \int_0^1 \mathrm{d}t\, t^{a-1+z_1+z_2}(1-t)^{b-1-z_1-z_2} \cdots v_1^{z_1} v_2^{z_2}
\end{aligned}
$$

if $\Re(a+z_1+z_2) > 0$ and $\Re(b-z_1-z_2) > 0$ so the t integral converges

Eliminate enough gamma functions to be able to perform the MB integrals

- can eliminate all gamma functions for real integrals, then use

$$
\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z}{2\pi\mathrm{i}} v^z = \delta(1-v)\,, \qquad v > 0
$$

- For multidimensional MB integrals, sometimes it is more useful to eliminate just the gamma functions that couple the MB integrations. This turns the multidimensional MB integral into products of 1d MB integrals.

After solving the remaining MB integrals, we get the desired parametric representation.

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\ldots\,\frac{P(x,t_1,t_2,t_3,\ldots)}{Q(x,t_1,t_2,t_3,\ldots)}$$

The t_1 integration

- assuming the denominator is a product of factors all linear in t_1, after partial fractioning, we will need to compute

$$\int_0^1 \frac{\mathrm{d}t_1}{t_1^n}\,, \qquad \int_0^1 \frac{\mathrm{d}t_1}{[t_1 - a(x,t_2,\ldots)]^n}\,,$$

- $n = 1$ is non-trivial

$$\int \frac{\mathrm{d}t_1}{t_1} = \ln t_1\,, \qquad \int \frac{\mathrm{d}t_1}{t_1 - a(x,t_2,\ldots)} = \ln[t_1 - a(x,t_2,\ldots)]$$

- e.g., we have

$$\int_0^1 \frac{\mathrm{d}t_1}{t_1 - a(x,t_2,\ldots)} = \ln\left[1 - \frac{1}{a(x,t_2,\ldots)}\right]$$

- this is elementary, although there is some fine print for definite integrals

Methods of integration - symbolic integration

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\ldots\,\frac{P(x,t_1,t_2,t_3,\ldots)}{Q(x,t_1,t_2,t_3,\ldots)}$$

The t_2 integration

- assuming the new denominator is a product of factors all linear in t_2, after partial fractioning — aside from the integrals we already encountered — we will have to compute

$$\int_0^1 \frac{\mathrm{d}t_2}{t_2^n} \ln\left[1-\frac{1}{a(x,t_2,\ldots)}\right], \qquad \int_0^1 \frac{\mathrm{d}t_2}{[t_2-b(x,t_3,\ldots)]^n} \ln\left[1-\frac{1}{a(x,t_2,\ldots)}\right],$$

- if $a(x,t_2,\ldots)$ is also linear in t_2, we can use the functional identities for the logarithm $[\ln(ab)=\ln a+\ln b,\ \ln(1/a)=-\ln a]$ to write

$$\ln\left[1-\frac{1}{a(x,t_2,\ldots)}\right] = \ln[a_1(x,t_3,\ldots)-t_2] - \ln[a_2(x,t_3,\ldots)-t_2]$$

- again, $n=1$ is non-trivial

$$\int \frac{\mathrm{d}t_2}{t_2}\ln t_2 = \frac{1}{2}\ln^2(t_2)\,, \qquad \int \frac{\mathrm{d}t_2}{t_2}\ln(1-t_2) = -\mathrm{Li}_2(t_2)$$

Methods of integration - symbolic integration

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}t_3\, \ldots \frac{P(x, t_1, t_2, t_3, \ldots)}{Q(x, t_1, t_2, t_3, \ldots)}$$

The t_2 integration (cont.)

- e.g., we have

$$\int_0^1 \frac{\mathrm{d}t_2}{t_2 - b(x, t_3, \ldots)} \ln(t_2) = \mathrm{Li}_2 \left[\frac{1}{b(x, t_3, \ldots)}\right]$$

Methods of integration - symbolic integration

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}t_3\, \dots \frac{P(x, t_1, t_2, t_3, \dots)}{Q(x, t_1, t_2, t_3, \dots)}$$

Before going to the t_3 integration, notice

1. at each step, we needed to introduce a new transcendental function, ln, Li_2
2. we needed to know the functional identities for ln to proceed

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\ldots\,\frac{P(x,t_1,t_2,t_3,\ldots)}{Q(x,t_1,t_2,t_3,\ldots)}$$

The t_3 integration

- assuming the new denominator is a product of factors all linear in t_3, after partial fractioning — aside from the integrals we already encountered — we will have to compute

$$\int_0^1 \frac{\mathrm{d}t_3}{t_3^n}\,\mathrm{Li}_2\left[\frac{p(x,t_3,\ldots)}{q(x,t_3,\ldots)}\right], \qquad \int_0^1 \frac{\mathrm{d}t_3}{[t_3 - c(x,t_4,\ldots)]^n}\,\mathrm{Li}_2\left[\frac{p(x,t_3,\ldots)}{q(x,t_3,\ldots)}\right],$$

- will need to introduce new transcendental functions $\Rightarrow$ multiple polylogs
- will need to use the functional identities for Li_2 to reduce to some standard form $\Rightarrow$ symbols, coporoducts, Hopf algebra of multiple polylogs

Multiple polylogarithms

The appropriate generalization of log and classical polylogs (Goncharov 1998, 2001)

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{\mathrm{d}t}{t-a_1} G(a_2,\ldots,a_n;t) \qquad \text{with} \qquad G(z)=1$$

$$G(\underbrace{0,\ldots,0}_{n};z) = \frac{1}{n!}\ln^n(z)$$

Logarithms and classical polylogs are special cases, e.g.,

$$G(\underbrace{a,\ldots,a}_{n};z) = \frac{1}{n!}\ln^n\left(1-\frac{z}{a}\right), \qquad G(\underbrace{0,\ldots,0}_{n-1},a;z) = -\,\mathrm{Li}_n\left(\frac{z}{a}\right)$$

Functional relations among Gs

- Problem: after the $(n-1)$-st integration, the n-th variable can appear in the a_i

$$\int \frac{\mathrm{d}t_n}{t_n-b} G(a_1(t_n,\ldots),\ldots,a_{n-1}(t_n,\ldots);z(t_n,\ldots))$$

  Must reduce to 'canonical' form, where t_n is only in the last entry.
- Unfortunately the functional equations among Gs that would be needed to do this are often unknown and need to be derived.

Symbols, coproducts

Symbols are a tool for obtaining functional equations among Gs

(Goncharov 2009; Goncharov, Spradlin, Vergu, Volovich 2010; Duhr, Gangl, Rhodes 2011)

- The symbol is a way of associating to a multiple polylog a tensor in a certain tensor space.

$$\mathcal{S}(G(a_{n-1},\ldots,a_1;a_n)) = \sum_{i=1}^{n-1} \mathcal{S}(G(a_{n-1},\ldots,a_{i-1},a_{i+1},\ldots,a_1;a_n)) \otimes \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}}\right)$$

e.g.,

$$\mathcal{S}\left(\frac{1}{n!}\ln^n(z)\right) = \underbrace{z\otimes\ldots\otimes z}_{n \text{ times}}, \qquad \mathcal{S}(\mathrm{Li}_n(z)) = -(1-z)\otimes\underbrace{z\otimes\ldots\otimes z}_{(n-1) \text{ times}}$$

- Functional equations between multiple polylogs become algebraic equations between tensors.

The idea of symbols can be refined based on the Hopf algebra structure of multiple polylogs $\Rightarrow$ coproduct (Duhr 2012)

- With these refinements one can build algorithms to reduce multiple polylogs to ‘canonical’ form.

Abelian double soft integral

For simplicity, consider the terms in the sum where $j = i$ and $l = k$: $[\mathrm{S}^{(0)}_{rs}]^{(i,k),(i,k)}$. Kinematical dependence is through $\cos\chi_{ik} = \measuredangle(p_i, p_k)$, we set $\cos\chi_{ik} = 1 - 2Y_{ik,Q}$, i.e., $Y_{ik,Q}$ is between zero and one.

Using angles and energies in the Q rest frame with some specific orientation to parametrize the factorized phase space measure, $[\mathrm{d}p^{(rs)}_{2,m}]$, we find that $[\mathrm{S}^{(0)}_{rs}]^{(i,k),(i,k)}$ is proportional to

$$\mathcal{I}_{2\mathcal{S},2}(Y_{ik,Q};\epsilon,y_0,d_0') = -\frac{4\Gamma^4(1-\epsilon)}{\pi\Gamma^2(1-\epsilon)}\frac{B_{y_0}(-2\epsilon,d_0')}{\epsilon}Y_{ik,Q}\int_0^{y_0}\mathrm{d}y\,y^{-1-2\epsilon}(1-y)^{d_0'-1+\epsilon}$$
$$\times\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1}\mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon}\big[f(\vartheta,\varphi;0)\big]^{-1}\big[f(\vartheta,\varphi;Y_{ik,Q})\big]^{-1}$$
$$\times\big[Y(y,\vartheta,\varphi;Y_{ik,Q})\big]^{-\epsilon}{}_2F_1\big(-\epsilon,-\epsilon,1-\epsilon,1-Y(y,\vartheta,\varphi;Y_{ik,Q})\big)$$

where

$$f(\vartheta,\varphi;Y_{ik,Q}) = 1 - 2\sqrt{Y_{ik,Q}(1-Y_{ik,Q})}\sin\vartheta\cos\varphi - (1-2Y_{ik,Q})\chi\cos\vartheta$$

$$Y(y,\vartheta,\varphi;\chi) = \frac{4(1-y)Y_{ik,Q}}{[2(1-y)+y\,f(\vartheta,\varphi;0)][2(1-y)+y\,f(\vartheta,\varphi;Y_{ik,Q})]}$$

Abelian double soft integral

This integral is equal to $\left(y_0 = 1,\ d_0' = 3 - 3\epsilon\right)$

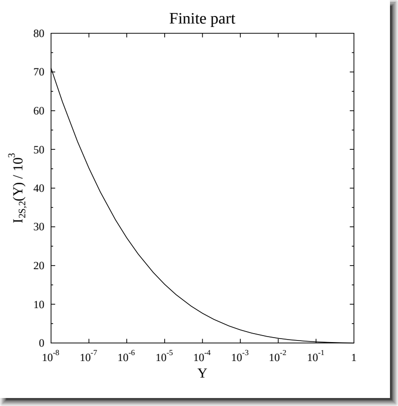
$$
\begin{aligned}
&\mathcal{I}_{2\mathcal{S},2}(Y;\epsilon,1,3-3\epsilon) = \\
&= \frac{1}{2\epsilon^4} - \frac{1}{\epsilon^3}\Big[\ln(Y) - 3\Big] + \frac{1}{\epsilon^2}\Big[2\,\mathrm{Li}_2(1-Y) + \ln^2(Y) - \pi^2 - \bigg(\frac{2}{1-Y} \\
&- \frac{1}{2(1-Y)^2} + \frac{9}{2}\bigg)\ln(Y) + \frac{1}{2(1-Y)} + 16\Big] + \frac{1}{\epsilon}\bigg[\frac{5}{3}\bigg(\frac{18\,\mathrm{Li}_3(1-Y)}{5} + \frac{6\,\mathrm{Li}_3(Y)}{5} \\
&- \frac{6\,\mathrm{Li}_2(1-Y)\ln(Y)}{5} - \frac{2}{5}\ln^3(Y) + \frac{3}{5}\ln(1-Y)\ln^2(Y) + \pi^2\ln(Y) - \frac{78\zeta_3}{5}\bigg) \\
&+ \bigg(\frac{3}{1-Y} - \frac{3}{4(1-Y)^2} + \frac{15}{4}\bigg)\Big(2\,\mathrm{Li}_2(1-Y) + \ln^2(Y)\Big) - 6\pi^2 - \bigg(\frac{27}{2(1-Y)} \\
&- \frac{13}{4(1-Y)^2} + \frac{91}{4}\bigg)\ln(Y) + \frac{19}{4(1-Y)} + \frac{163}{2}\bigg] + \mathrm{O}(\epsilon^0)
\end{aligned}
$$

Note the $Y \to 1$ limit is finite

$$
\lim_{Y\to 1}\mathcal{I}_{2\mathcal{S},2}(Y;\epsilon,1,3-3\epsilon) = \frac{1}{2\epsilon^4} + \frac{3}{\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{71}{4} - \pi^2\right) + \frac{1}{\epsilon}\left(\frac{393}{4} - 6\pi^2 - 24\zeta_3\right) + \mathrm{O}(\epsilon^0)
$$

Abelian double soft integral

Finite term is computed numerically $(y_0 = 1,\ d_0' = 3 - 3\epsilon)$



Analytic vs. numeric

As a matter of principle

- A rigorous proof of cancellation of IR poles requires the poles of integrated counterterms in analytic form.

However

- An actual implementation needs numbers for the finite parts of the integrated counterterms.
- These finite parts are smooth functions of kinematic variables.

Hence

- Numerical forms of the finite parts are sufficient for practical purposes. The final results can be conveniently given by interpolating tables or approximating functions computed once and for all.
- In particular, suitable approximating functions may be obtained by fitting.

Example of approximation by fitting

Doubly-unresolved double-collinear master integral $\mathcal{I}_{2\mathcal{C},6}(x_{ir}, x_{js}; \epsilon, 1, 3-3\epsilon, k, l)$

$$\mathcal{I}_{2\mathcal{C},6}(x_{ir}, x_{js}; \epsilon, \alpha_0, d_0; k, l) = x_{ir}x_{js} \int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta \int_0^1 \mathrm{d}v\,\mathrm{d}u\,\alpha^{-1-\epsilon}\beta^{-1-\epsilon}(1-\alpha-\beta)^{2d_0-2(1-\epsilon)}$$
$$\times [\alpha + (1-\alpha-\beta)x_{ir}]^{-1-\epsilon}[\beta + (1-\alpha-\beta)x_{js}]^{-1-\epsilon} v^{-\epsilon}(1-v)^{-\epsilon}u^{-\epsilon}(1-u)^{-\epsilon}$$
$$\times \left(\frac{\alpha + (1-\alpha-\beta)x_{ir}v}{2\alpha + (1-\alpha-\beta)x_{ir}}\right)^k \left(\frac{\beta + (1-\alpha-\beta)x_{js}u}{2\beta + (1-\alpha-\beta)x_{js}}\right)^l \Theta(\alpha_0 - \alpha - \beta)$$

Example of approximation by fitting

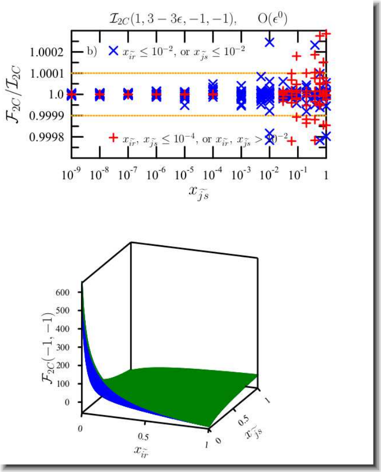
Doubly-unresolved double-collinear master integral $\mathcal{I}_{2\mathcal{C},6}(x_{ir}, x_{js}; \epsilon, 1, 3-3\epsilon, k, l)$

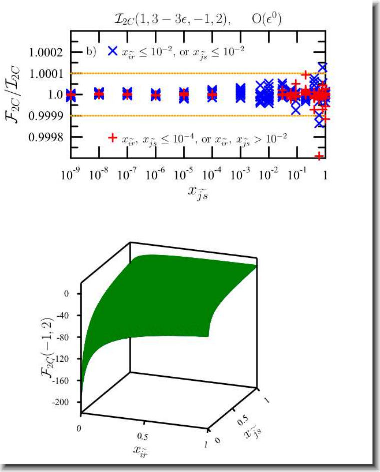
- poles (up to $\frac{1}{\epsilon^4}$) extracted via sector decomposition
- numerical values of pole coefficients computed for a 17×17 grid with precision of $\sim 10^{-7}$
- define three regions (note: result is symmetric in x_{ir}, x_{js})
  - asymptotic: $x_{ir}, x_{js} < 10^{-4}$
  - non-asymptotic: $x_{ir}, x_{js} > 10^{-2}$
  - border: $x_{ir} < 10^{-2}$ or $x_{js} < 10^{-2}$
- in each region, fit with ansatz

$$\mathcal{F}(x_{ir}, x_{js}) = \sum_{p_i, l_i} C_{m;p_1,p_2;l_1,l_2} (x_{ir}^{p_1} x_{js}^{p_2})(\log^{l_1}(x_{ir}) \log^{l_2}(x_{js}))$$

where $p_1 + p_2 \leq m$ with m a free parameter, while $l_1 + l_2 \leq n$ and n is predicted

Doubly-unresolved double-collinear master integral $\mathcal{I}_{2\mathcal{C},6}(x_{ir}, x_{js}; \epsilon, 1, 3-3\epsilon, k, l)$





Integrated approximate cross sections

Recall the NNLO correction is a sum of three terms

$$\sigma^{\rm NNLO} = \sigma^{\rm RR}_{m+2} + \sigma^{\rm RV}_{m+1} + \sigma^{\rm VV}_{m} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m}$$

each integrable in four dimensions

$$\sigma^{\rm NNLO}_{m+2} = \int_{m+2} \Big\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \Big[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \Big] \Big\}$$

$$\sigma^{\rm NNLO}_{m+1} = \int_{m+1} \Big\{ \Big[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big] J_{m+1} - \Big[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] J_m \Big\}$$

$$\sigma^{\rm NNLO}_{m} = \int_{m} \Big\{ {\rm d}\sigma^{\rm VV}_{m} + \int_2 \Big[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \Big] + \int_1 \Big[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] \Big\} J_m$$

Integrated approximate cross sections

- After summing over unobserved flavors, all integrated approximate cross sections can be written as products (in color space) of various insertion operators with lower point cross sections.
- Can be computed once and for all (though admittedly lots of tedious work).

The double virtual contribution

After adding all integrated approximate cross sections the double virtual contribution must be finite in ϵ.

$$\sigma_m^{\rm NNLO} = \int_m \left\{ {\rm d}\sigma_m^{\rm VV} + \int_2 \left[{\rm d}\sigma_{m+2}^{\rm RR,A_2} - {\rm d}\sigma_{m+2}^{\rm RR,A_{12}} \right] + \int_1 \left[{\rm d}\sigma_{m+1}^{\rm RV,A_1} + \left(\int_1 {\rm d}\sigma_{m+2}^{\rm RR,A_1} \right)^{\rm A_1} \right] \right\} J_m$$

- Have checked the cancellation of the $\frac{1}{\epsilon^4}$ and $\frac{1}{\epsilon^3}$ poles analytically for any number of jets (i.e., with m symbolic).
- Have checked $m = 2$ ($e^+e^- \to q\bar{q}$, $H \to b\bar{b}$) explicitly and we find that all poles cancel.
- Have checked $m = 3$ ($e^+e^- \to q\bar{q}g$) explicitly and we find that all poles cancel.

The double virtual contribution

After adding all integrated approximate cross sections the double virtual contribution must be finite in ϵ.

$$\sigma_m^{\mathrm{NNLO}} = \int_m \left\{ \mathrm{d}\sigma_m^{\mathrm{VV}} + \int_2 \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}} \right] + \int_1 \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

- Have checked the cancellation of the $\frac{1}{\epsilon^4}$ and $\frac{1}{\epsilon^3}$ poles analytically for any number of jets (i.e., with m symbolic).
- Have checked $m = 2$ ($e^+e^- \to q\bar{q}$, $H \to b\bar{b}$) explicitly and we find that all poles cancel.
- Have checked $m = 3$ ($e^+e^- \to q\bar{q}g$) explicitly and we find that all poles cancel.

Message: the method works, try and apply

Application: $H \to b\bar{b}$

Cancellation of poles

Consider $H \to b\bar{b}$ decay at NNLO.

- admittedly the simplest case
- but this just amounts to having to sum less terms in a general formula

The double virtual contribution has the following pole structure ($\mu^2 = m_H^2$)

$$
\begin{aligned}
\mathrm{d}\sigma^{\mathrm{VV}}_{H\to b\bar{b}} = \left(\frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\right)^2 \mathrm{d}\sigma^{\mathrm{B}}_{H\to b\bar{b}} \Bigg\{ & \frac{2C_{\mathrm{F}}^2}{\epsilon^4} + \left(\frac{11 C_{\mathrm{A}} C_{\mathrm{F}}}{4} + 6C_{\mathrm{F}}^2 - \frac{C_{\mathrm{F}} n_{\mathrm{f}}}{2}\right)\frac{1}{\epsilon^3} \\
& + \left[\left(\frac{8}{9} + \frac{\pi^2}{12}\right) C_{\mathrm{A}} C_{\mathrm{F}} + \left(\frac{17}{2} - 2\pi^2\right) C_{\mathrm{F}}^2 - \frac{2 C_{\mathrm{F}} n_{\mathrm{f}}}{9}\right]\frac{1}{\epsilon^2} \\
& + \left[\left(-\frac{961}{216} + \frac{13\zeta_3}{2}\right) C_{\mathrm{A}} C_{\mathrm{F}} + \left(\frac{109}{8} - 2\pi^2 - 14\zeta_3\right) C_{\mathrm{F}}^2 + \frac{65 C_{\mathrm{F}} n_{\mathrm{f}}}{108}\right]\frac{1}{\epsilon} \Bigg\}
\end{aligned}
$$

Cancellation of poles

Consider $H \to b\bar{b}$ decay at NNLO.

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The double virtual contribution has the following pole structure ($\mu^2 = m_H^2$)

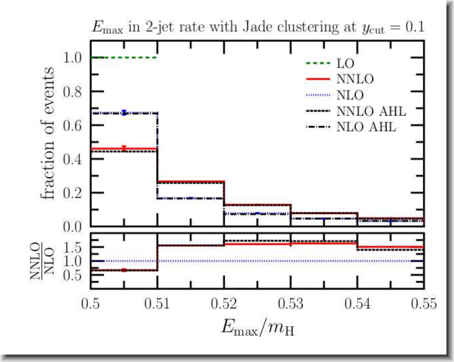
$$
\begin{aligned}
\mathrm{d}\sigma^{\mathrm{VV}}_{H\to b\bar{b}} = \left(\frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\right)^2 \mathrm{d}\sigma^{\mathrm{B}}_{H\to b\bar{b}} \Bigg\{ & \frac{2C_{\mathrm{F}}^2}{\epsilon^4} + \left(\frac{11C_{\mathrm{A}}C_{\mathrm{F}}}{4} + 6C_{\mathrm{F}}^2 - \frac{C_{\mathrm{F}}n_{\mathrm{f}}}{2}\right)\frac{1}{\epsilon^3} \\
& + \left[\left(\frac{8}{9} + \frac{\pi^2}{12}\right)C_{\mathrm{A}}C_{\mathrm{F}} + \left(\frac{17}{2} - 2\pi^2\right)C_{\mathrm{F}}^2 - \frac{2C_{\mathrm{F}}n_{\mathrm{f}}}{9}\right]\frac{1}{\epsilon^2} \\
& + \left[\left(-\frac{961}{216} + \frac{13\zeta_3}{2}\right)C_{\mathrm{A}}C_{\mathrm{F}} + \left(\frac{109}{8} - 2\pi^2 - 14\zeta_3\right)C_{\mathrm{F}}^2 + \frac{65C_{\mathrm{F}}n_{\mathrm{f}}}{108}\right]\frac{1}{\epsilon}\Bigg\}
\end{aligned}
$$

The sum of the integrated approximate cross sections gives ($\mu^2 = m_H^2$)

$$
\begin{aligned}
\sum\int \mathrm{d}\sigma^{\mathrm{A}} = \left(\frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\right)^2 \mathrm{d}\sigma^{\mathrm{B}}_{H\to b\bar{b}} \Bigg\{ & \frac{-2C_{\mathrm{F}}^2}{\epsilon^4} + \left(-\frac{11C_{\mathrm{A}}C_{\mathrm{F}}}{4} - 6C_{\mathrm{F}}^2 + \frac{C_{\mathrm{F}}n_{\mathrm{f}}}{2}\right)\frac{1}{\epsilon^3} \\
& + \left[\left(-\frac{8}{9} - \frac{\pi^2}{12}\right)C_{\mathrm{A}}C_{\mathrm{F}} + \left(-\frac{17}{2} + 2\pi^2\right)C_{\mathrm{F}}^2 + \frac{2C_{\mathrm{F}}n_{\mathrm{f}}}{9}\right]\frac{1}{\epsilon^2} \\
& + \left[\left(\frac{961}{216} - \frac{13\zeta_3}{2}\right)C_{\mathrm{A}}C_{\mathrm{F}} + \left(-\frac{109}{8} + 2\pi^2 + 14\zeta_3\right)C_{\mathrm{F}}^2 - \frac{65C_{\mathrm{F}}n_{\mathrm{f}}}{108}\right]\frac{1}{\epsilon}\Bigg\}
\end{aligned}
$$

Energy spectrum of jets in $H \to b\bar{b}$ at NNLO

Energy spectrum of the leading jet in the rest frame of the Higgs boson. Jets are clustered using the Jade algorithm with $y_{\mathrm{cut}} = 0.1$

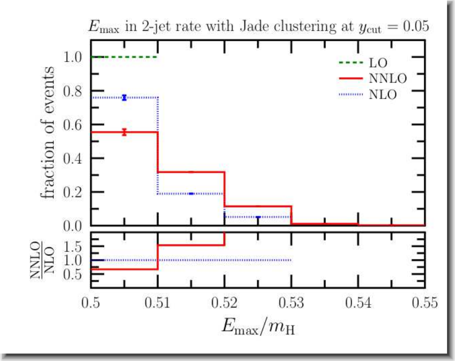


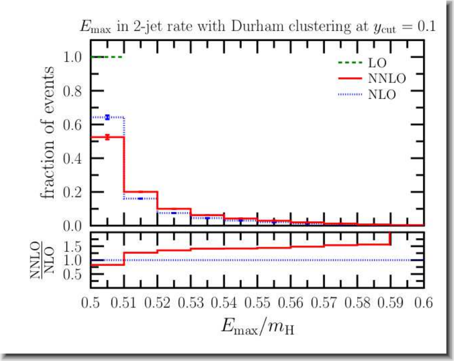
(AHL: Anastasiou, Herzog, Lazopoulos, JHEP **1203** (2012) 035)

Energy spectrum of jets in $H \to b\bar{b}$ at NNLO

Energy spectrum of the leading jet in the rest frame of the Higgs boson.

- right: jets are clustered using the Jade algorithm with $y_{\mathrm{cut}} = 0.05$
- left: jets are clustered using the Durham algorithm with $y_{\mathrm{cut}} = 0.1$





Conclusions

Have set up

- completely local subtractions for fully differential predictions at NNLO
- construction of subtraction terms based on IR limit formulae
- analytic integration of subtraction terms is feasible with modern integration techniques
- demonstrated cancellation of ϵ poles for $m = 2$ and $m = 3$
- worked out in full detail for processes with no colored particles in the initial state

First application: Higgs boson decay into a b and anti-b quark

Acknowledgments

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