Resummation and theoretical uncertainties

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Soft and collinear branching

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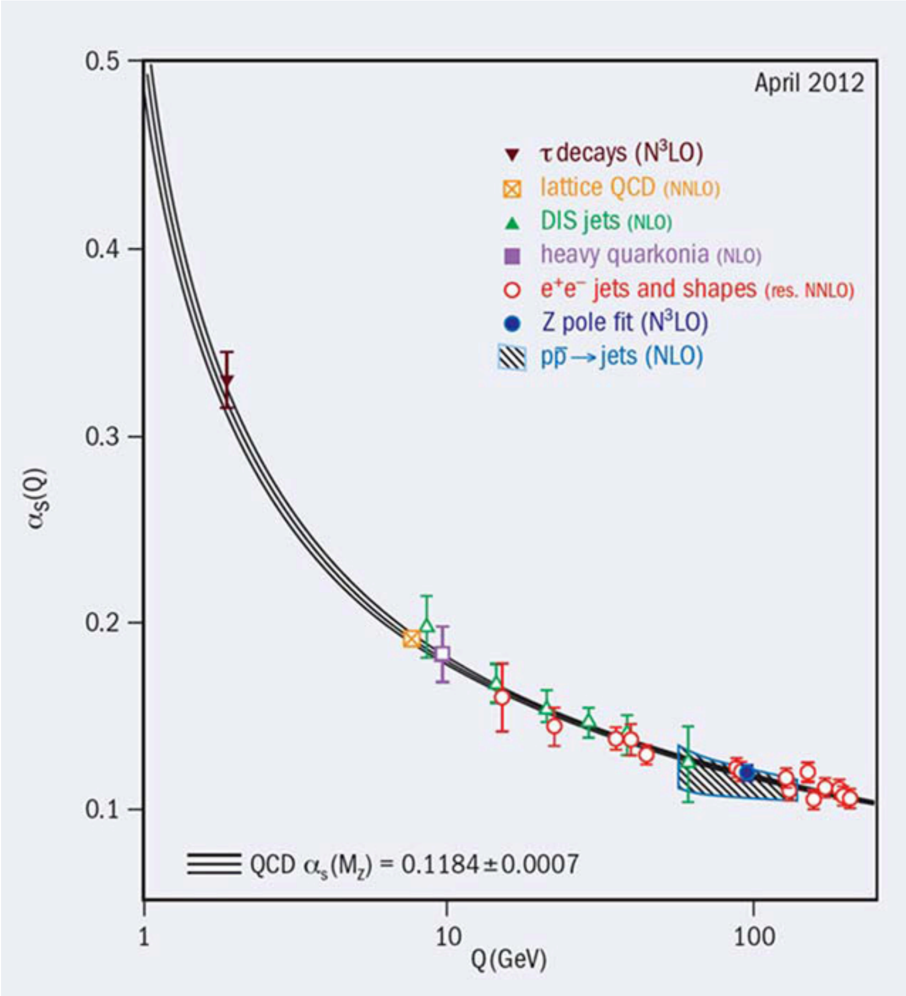
Short-distance observables

We know that QCD is an asymptotically free theory

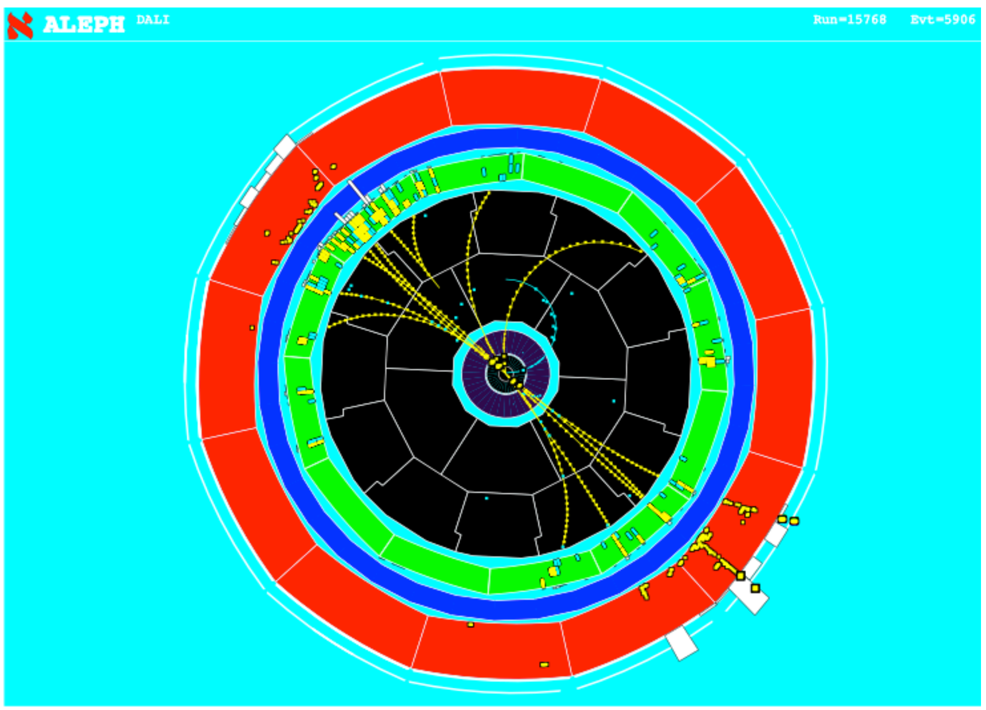


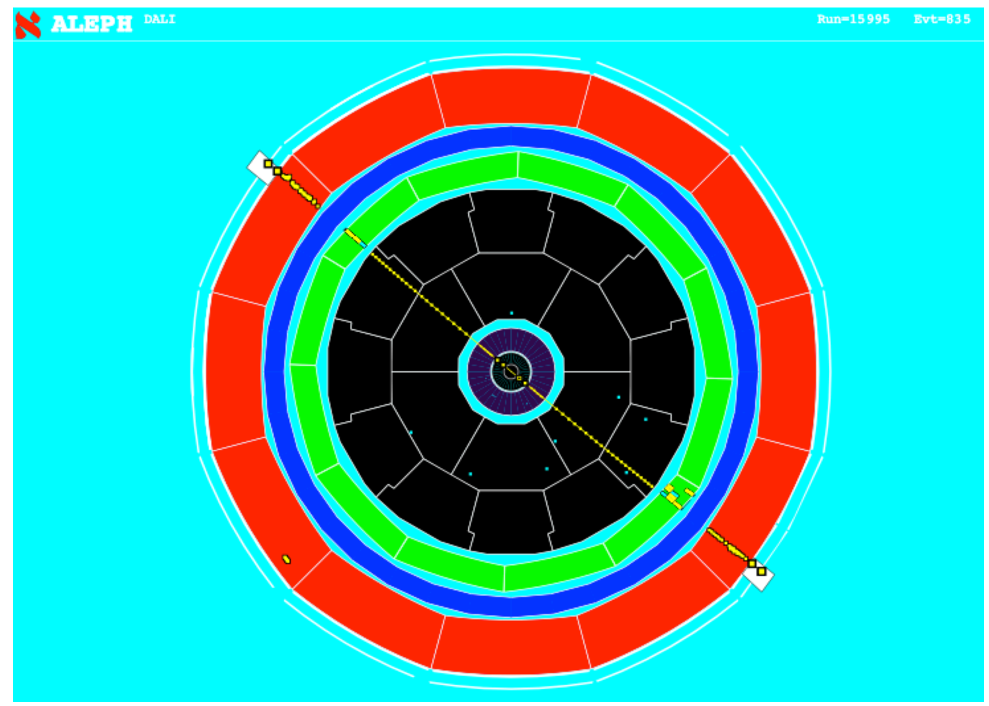
We can safely apply perturbation theory to observables dominated by large momentum scales: how do we find such observables?

Short-distance observables

Consider a simple counting observable in $e^+ e^-$ annihilation, the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$





Short-distance observables

R in general is a function of the centre of mass energy squared s, of the quark masses m_q^2, and of the QCD coupling $\alpha_s(\mu^2)$, where μ^2 is the QCD renormalisation scale

$$R = R\left(\alpha_s(\mu^2), \frac{s}{\mu^2}, \frac{m_q^2}{\mu^2}\right)$$

In massless QCD, we can compute safely the quantity

$$\hat{R}\left(\alpha_s(\mu^2), \frac{s}{\mu^2}\right) = \frac{\sigma(e^+e^- \to \text{quarks, gluons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

The partonic quantity $\hat{R}$ admits an expansion in powers of α_s

$$\hat{R}\left(\alpha_s(\mu^2), \frac{s}{\mu^2}\right) = R_0\left(1 + \sum_{n=1}^{\infty} c_n\left(\frac{s}{\mu^2}\right)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n\right)$$

Short-distance observables

Under the assumption that the sum over all *partonic* final states is the same as that over *hadronic* final states (local parton-hadron duality), the existence of a massless limit implies

$$R\left(\alpha_s(\mu^2), \frac{s}{\mu^2}, \frac{m_q^2}{\mu^2}\right) = \hat{R}\left(\alpha_s(\mu^2), \frac{s}{\mu^2}\right) + \mathcal{O}\left(\frac{m_q^2}{s}\right)^p$$

Renormalisation group analysis (the fact that R does not depend on μ) gives us

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right)\hat{R}\left(\alpha_s(\mu^2), \frac{s}{\mu^2}\right) = 0 \qquad \Rightarrow \qquad s\frac{\partial \hat{R}}{\partial s} = \beta(\alpha_s)\frac{\partial \hat{R}}{\partial \alpha_s}$$

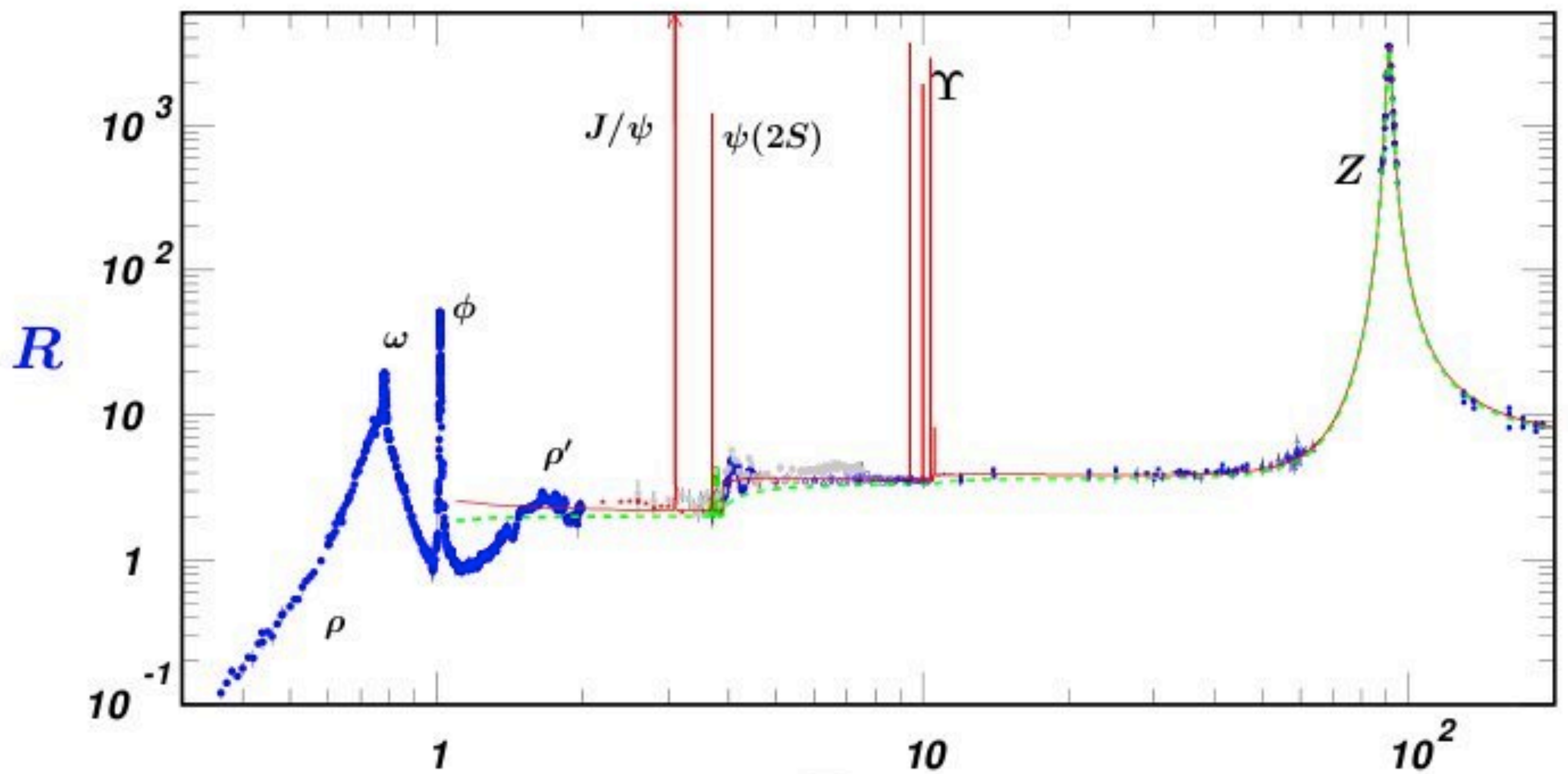
The formal solution of this equation is

$$\hat{R}\left(\alpha_s(\mu^2), \frac{s}{\mu^2}\right) = \hat{R}\left(\alpha_s(s), 1\right)$$

If s is sufficiently large the use of perturbation theory is legitimate

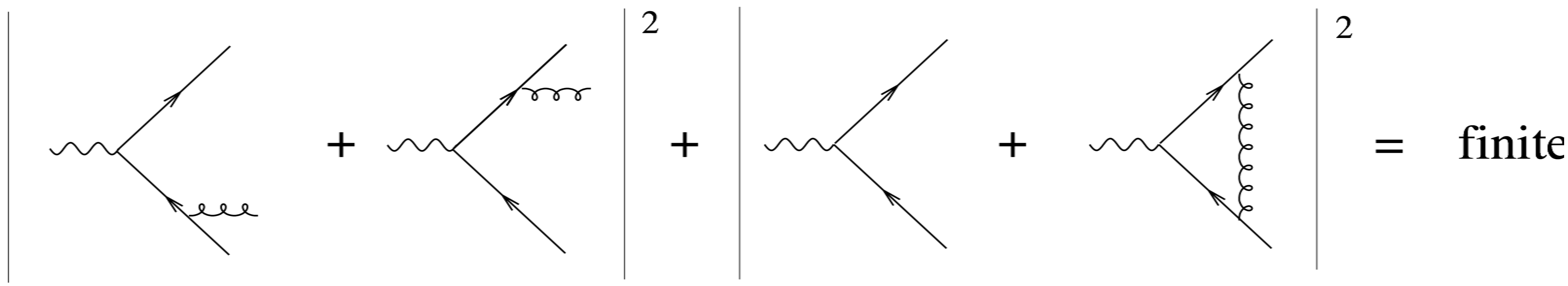
Short-distance observables

Perturbation theory really works, except close to hadronic resonances, where local parton-hadron duality is not a reasonable assumption



Soft and collinear divergences

In the partonic ratio $\hat{R}$, IRC singularities cancel in the inclusive sum of real and virtual contributions



$\hat{R}$ is and example of an infrared and collinear safe observable.

Furthermore, it exhibits a complete cancellation of soft and collinear emission contributions, i.e. is insensitive to all emissions up to the scale $Q^2 = s$

The origin of this complete cancellation is due to

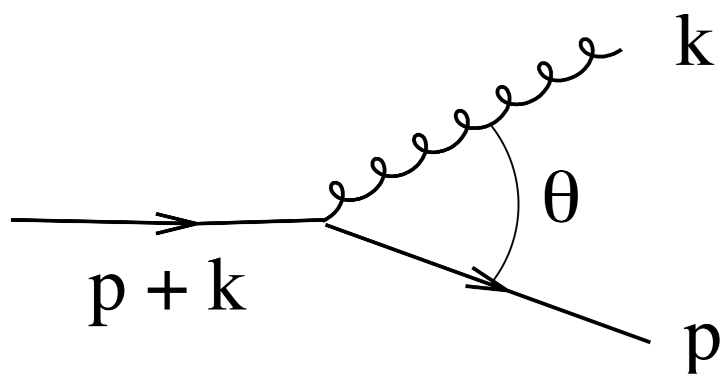
1. real and virtual matrix elements are equal but opposite in sign

2. the observable assigns the same weight to real and virtual corrections

Soft and collinear divergences

The applicability of perturbation theory is deeply related to the existence of the massless limit of partonic observables

Divergences can occur whenever a propagator goes on shell



Emission of a gluon (energy ω) off a hard quark (energy E)

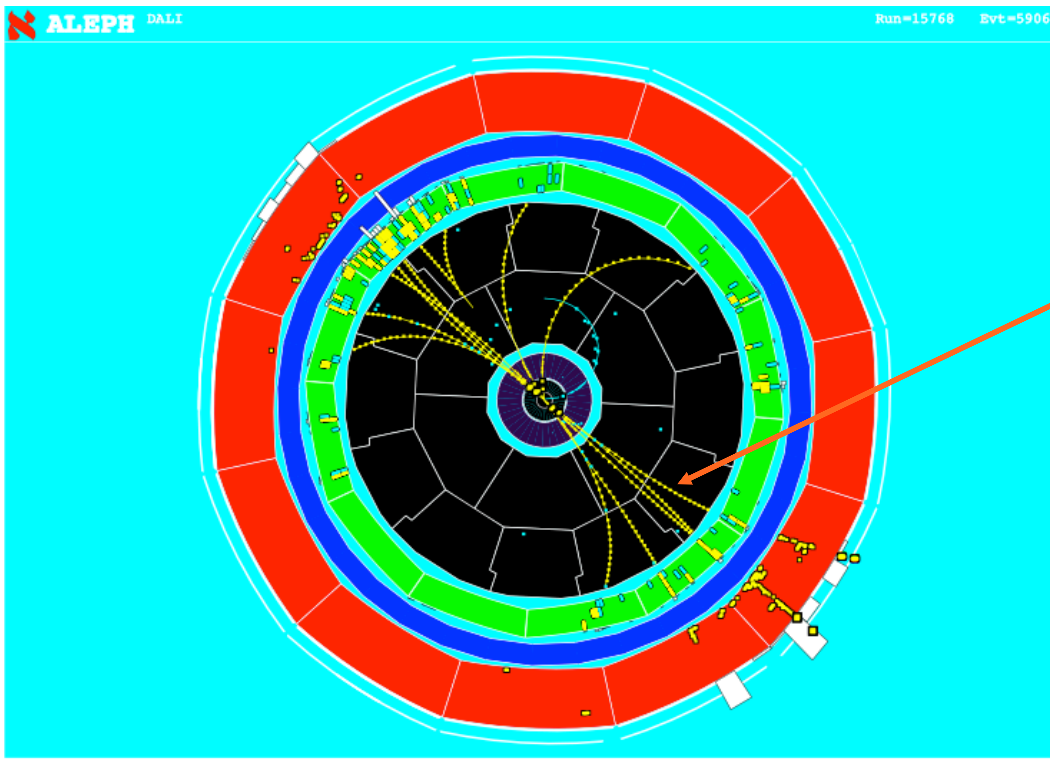
$$(p+k)^2 = 2E\omega(1-\cos\theta) \simeq E\omega\theta^2$$

singular for soft ($\omega \to 0$) and collinear ($\theta \to 0$) emissions

Infrared (IR) and collinear (together IRC) singularities are present in both real and virtual corrections

Factorisation of collinear divergences

The massless limit of the one hadron inclusive cross section does not exist $\Rightarrow$ dependence on the quark masses is not power suppressed



pick up any hadron and bin the energy fraction

$$x = \frac{2E_h}{\sqrt{s}}$$

$$\frac{1}{\sigma}\frac{d\sigma}{dx} = F\left(\alpha_s(\mu^2), \frac{s}{\mu^2}, \frac{m_q^2}{\mu^2}\right) = F\left(\alpha_s(s), \frac{s}{m_q^2}\right)$$

The dependence on s/m_q^2 reflects the structure of the collinear divergences, i.e. it is logarithmic

Factorisation of collinear divergences

Luckily, for the one hadron inclusive cross section, collinear singularities factorise

$$F\left(\alpha_s(s), \frac{s}{m_q^2}\right) = C\left(\alpha_s(s), \frac{s}{\mu_F^2}\right) \otimes D\left(\alpha_s(\mu_F^2), \frac{\mu_F^2}{m_q^2}\right)$$

- The coefficient function $C(\alpha_s(s), s/\mu_F^2)$ has a massless limit, hence can be computed at all orders in perturbation theory
- The fragmentation function $D(\alpha_s(\mu_F^2), \mu_F^2/m_q^2)$ is universal (does not depend on the hard process) and contains all dependence on the quark masses

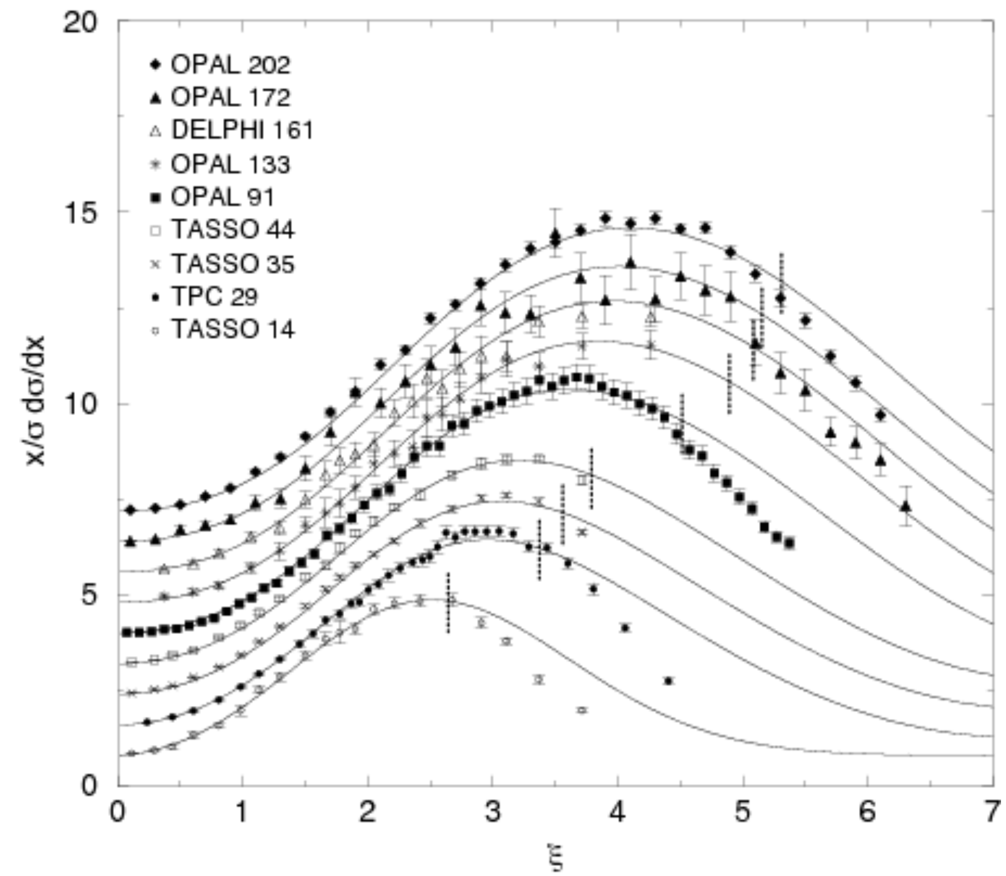
Factorisation of collinear singularities require the introduction of a factorisation scale, that separates short distances ($s \gg \mu_F^2$) from long distances ($\mu_F^2 \gg m_q^2$)

Factorisation of collinear divergences

If $m_q \sim \Lambda_{\rm QCD}$ the fragmentation function cannot be computed in perturbation theory, because the QCD coupling runs in the infrared. Nevertheless, the dependence on μ_F^2 can be computed via the DGLAP equation

$$\mu_F^2 \frac{\partial D\left(\alpha_s(\mu_F^2), \frac{\mu_F^2}{m_q^2}\right)}{\partial \mu_F^2}$$

$$= P(\alpha_s(\mu_F)) \otimes D\left(\alpha_s(\mu_F^2), \frac{\mu_F^2}{m_q^2}\right)$$



Factorisation of collinear divergences

If $m_q \gg \Lambda_{\rm QCD}$ the fragmentation function can be computed in perturbation theory

$$D\left(\alpha_s(\mu_F^2), \frac{\mu_F^2}{m_q^2}\right) = \sum_{n=0}^{\infty} \alpha_s^n(\mu_F^2)\, D_n\!\left(\frac{\mu_F^2}{m_q^2}\right) + \text{power corrections}$$

Each coefficient D_n contains logarithmically enhanced contributions, at most $\ln^n(\mu_F^2/m_q^2)$

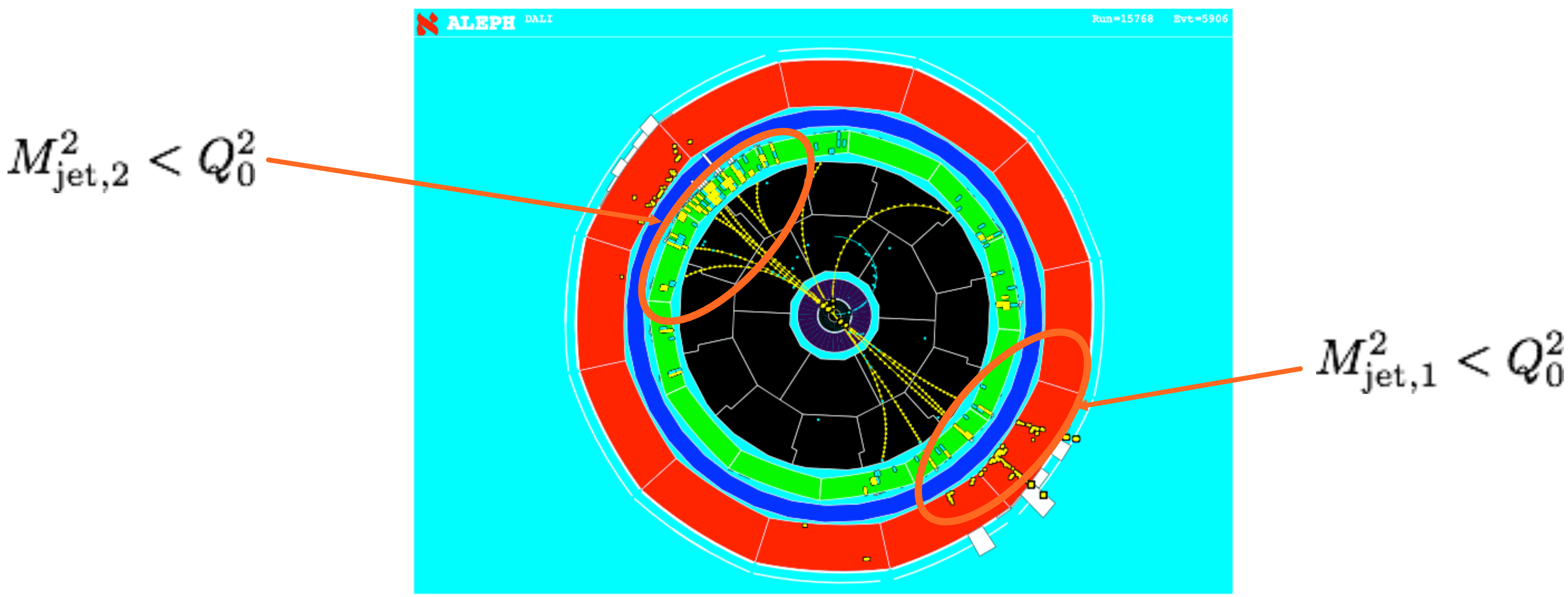
Even if $\alpha_s(\mu_F^2)$ is small, if $\mu_F^2 \gg m_q^2$ we can have $\alpha_s^n(\mu_F^2)\ln^n(\mu_F^2/m_q^2) \sim 1$, thus spoiling the validity of perturbation theory $\Rightarrow$ all order resummation needed

Resummation of collinear logarithms is provided by the DGLAP equation

Note. One can eliminate these logs by choosing $\mu_F \sim m_q$, but this introduces large logarithms $\ln(s/\mu_F^2)$ in the coefficient function

Two-scale problems

A case similar to the fragmentation function is when we want the invariant mass of any jet to be small, i.e. only pencil-like events are allowed



The fraction of events for which any jet mass is below Q_0^2 depends on two scales

$$\Sigma(Q_0^2) = \Sigma\left(\alpha_s(s), \frac{s}{Q_0^2}\right) = 1 + \sum_{n=1}^{\infty} \Sigma_n(s/Q_0^2)\,\alpha_s^n(s) + \text{power corrections}$$

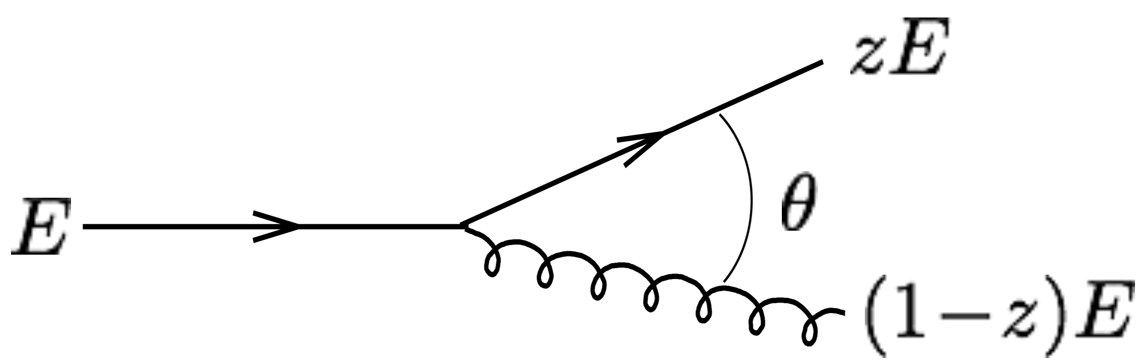
This observable has a massless limit, but one can expect large logarithms $\ln(s/Q_0^2)$

Collinear splitting

The occurrence of large logarithms is deeply related to IR and collinear divergences

The matrix elements for collinear splitting factorise from hard matrix elements, and depend only on the nature of the splitting

Example: collinear radiation of a gluon from a hard quark of energy E



$$dP_{q\to qg} = \frac{d\theta^2}{\theta^2} dz \frac{\alpha_s[(1-z)\theta E]}{2\pi} C_F \frac{1+z^2}{1-z}$$

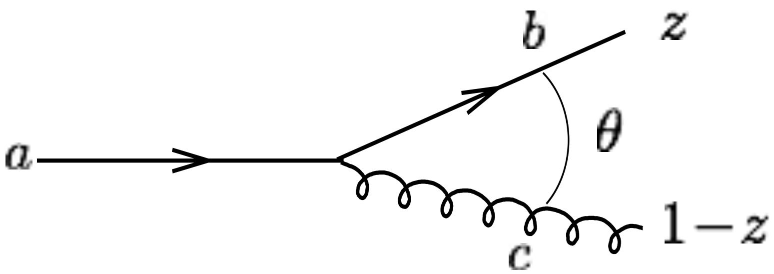
collinear divergence

soft divergence

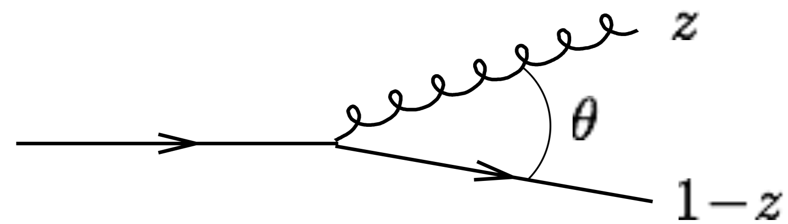
Collinear splitting

Similarly, we can associate to any splitting process a splitting probability

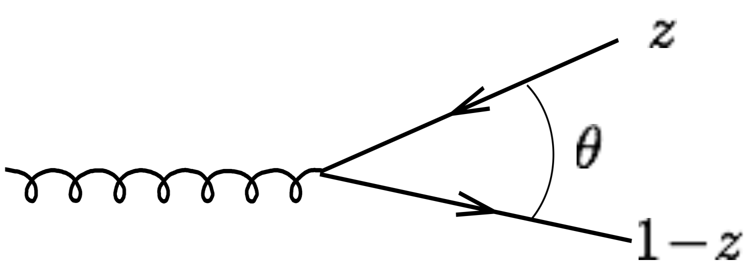
$$dP_{a\to bc} = \frac{d\theta^2}{\theta^2} dz \frac{\alpha_s[(1-z)\theta E]}{2\pi} p_{ab}(z)$$



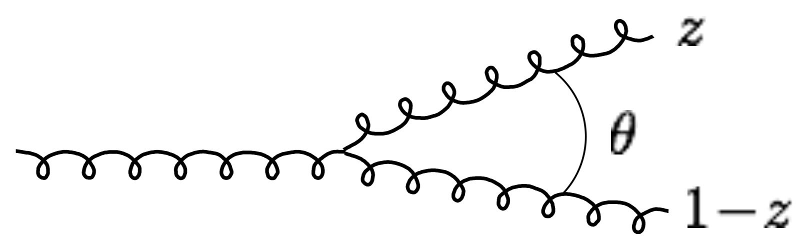
$$p_{qq}(z) = C_F \frac{1+z^2}{1-z}$$



$$p_{qg}(z) = C_F \frac{1+(1-z)^2}{z}$$



$$p_{gq}(z) = T_R n_f [z^2 + (1-z)^2]$$



$$p_{gg}(z) = C_A \left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right)$$

Collinear branching

Resummation of collinear logarithms for the fragmentation function can be performed with the DGLAP equation. For a single splitting we have

$$Q^2 \frac{\partial}{\partial Q^2} D(x, Q^2) = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} [p(z)]_+ D\left(\frac{x}{z}, Q^2\right)$$

where we have introduced the plus distribution

$$\int_0^1 dz\, [p(z)]_+ f(z) = \int_0^1 dz\, p(z)[f(z) - f(1)]$$

Introducing a cutoff ϵ to avoid the soft singularity we obtain

$$Q^2 \frac{\partial}{\partial Q^2} D(x, Q^2) = \int_x^{1-\epsilon} \frac{dz}{z} \frac{\alpha_s}{2\pi} p(z) D\left(\frac{x}{z}, Q^2\right) - \int_0^{1-\epsilon} \frac{dz}{z} \frac{\alpha_s}{2\pi} p(z) D\left(x, Q^2\right)$$

Collinear branching

Introducing the Sudakov form factor

$$\Delta(Q^2, Q_0^2) = \exp\left[-\int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \int_0^{1-\epsilon} dz \frac{\alpha_s}{2\pi} p(z)\right]$$

the DGLAP equation with a cutoff can be rewritten as

$$Q^2 \frac{\partial}{\partial Q^2} \frac{D(x, Q^2)}{\Delta(Q^2, Q_0^2)} = \frac{1}{\Delta(Q^2, Q_0^2)} \int_x^{1-\epsilon} \frac{dz}{z} \frac{\alpha_s}{2\pi} p(z) D\left(\frac{x}{z}, Q^2\right)$$

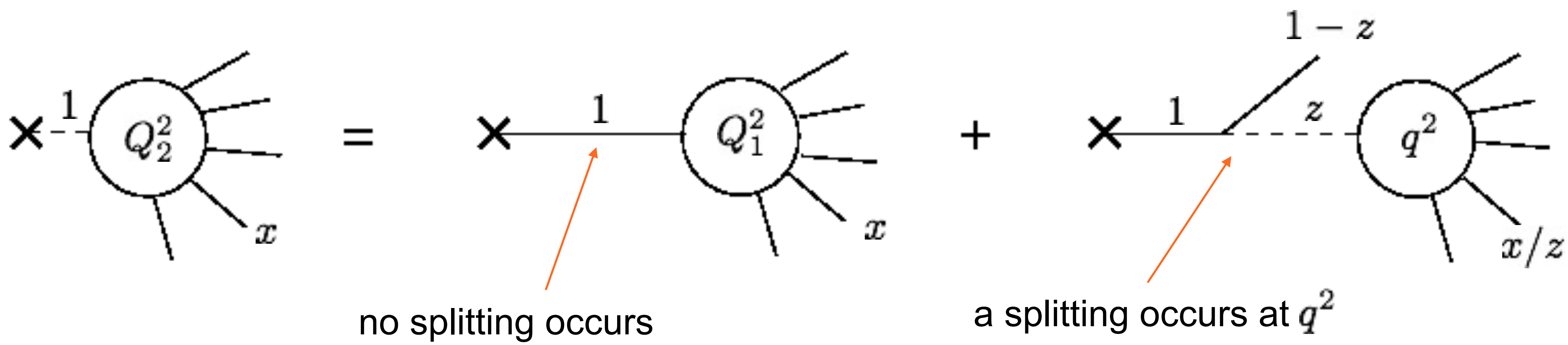
This equation can be recast in an integral form

$$D(x, Q_2^2) = D(x, Q_1^2) \frac{\Delta(Q_2^2, Q_0^2)}{\Delta(Q_1^2, Q_0^2)} + \int_{Q_1^2}^{Q_2^2} \frac{dq^2}{q^2} \frac{\Delta(Q_2^2, Q_0^2)}{\Delta(q^2, Q_0^2)} \int_0^{1-\epsilon} \frac{\alpha_s}{2\pi} p(z) D\left(\frac{x}{z}, q^2\right)$$

Collinear branching

The evolution of the fragmentation function has a probabilistic interpretation

$$D(x,Q_2^2) = D(x,Q_1^2)\frac{\Delta(Q_2^2,Q_0^2)}{\Delta(Q_1^2,Q_0^2)} + \int_{Q_1^2}^{Q_2^2}\frac{dq^2}{q^2}\frac{\Delta(Q_2^2,Q_0^2)}{\Delta(q^2,Q_0^2)}\int_0^{1-\epsilon}\frac{\alpha_s}{2\pi}p(z)D\left(\frac{x}{z},q^2\right)$$



The ratio of Sudakov form factors $\Delta(Q_2^2,Q_0^2)/\Delta(Q_1^2,Q_0^2)$ is the probability of not emitting anything between Q_1^2 and Q_2^2

Sudakov form factor

The Sudakov form factor contains virtual corrections and unresolved real emissions up to the scale Q_0^2

$$\Delta(Q^2, Q_0^2) = \exp\left[-\int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \int_0^{1-\epsilon} dz \frac{\alpha_s}{2\pi} p(z)\right]$$

The properties of the Sudakov form factor are

- $\Delta(Q^2, Q^2) = 1 \Rightarrow$ inclusive real-virtual cancellation, the sum of real and virtual corrections give the total cross section
- $\Delta(Q^2, 0) = 0 \Rightarrow$ it is impossible to find any quark or gluon without perturbative radiation

Emission probability

Using the Sudakov form factor we can construct an actual splitting probability that is the basis of exclusive QCD studies and Monte Carlo parton shower event generators

$$dP[q^2,z] = \frac{dq^2}{q^2}\frac{\Delta(Q^2,Q_0^2)}{\Delta(q^2,Q_0^2)}\,dz\,\frac{\alpha_s}{2\pi}\,p(z)/\int_0^{1-\epsilon}dz\frac{\alpha_s}{2\pi}p(z)$$

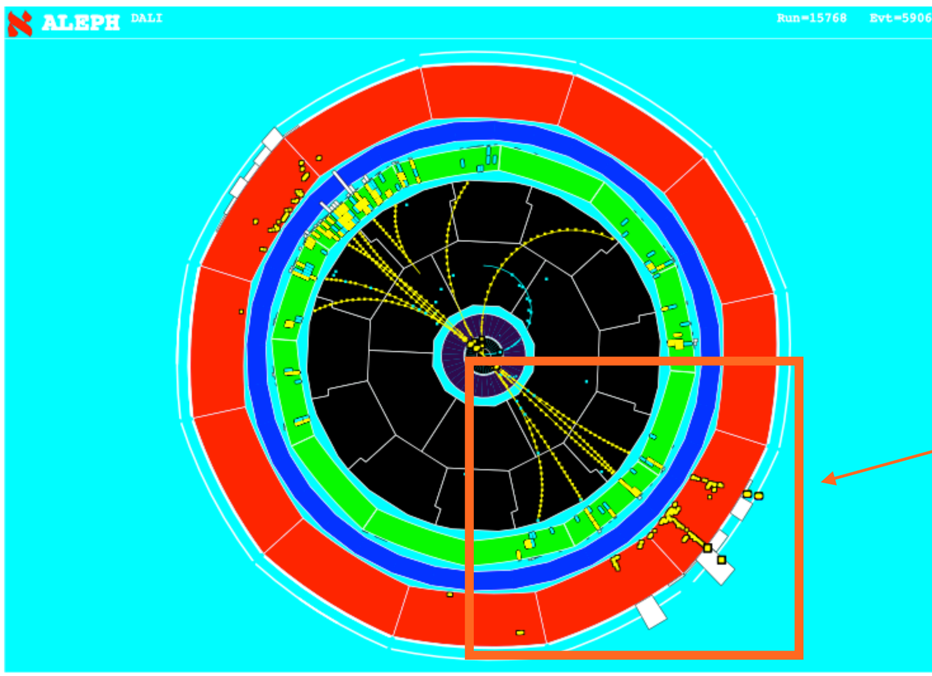
The total emission probability is not normalised to 1 because we do not allow any radiation below the scale Q_0^2

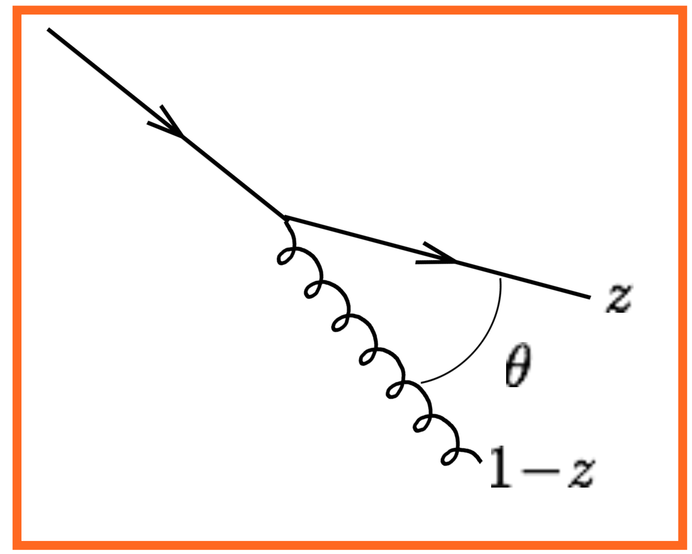
$$\int_{Q_0^2}^{Q^2} dP[q^2,z] = 1 - \Delta(Q^2,Q_0^2)$$

From this probability one can construct iteratively a sequence of soft/collinear emissions ordered in the variable $q^2 \sim \theta^2$

Sudakov logarithms

The Sudakov form factor is deeply related to resummation.





At first order a jet is made up of a single gluon. Then the fraction of events for which the invariant mass of any jet is less than Q_0^2 is given by

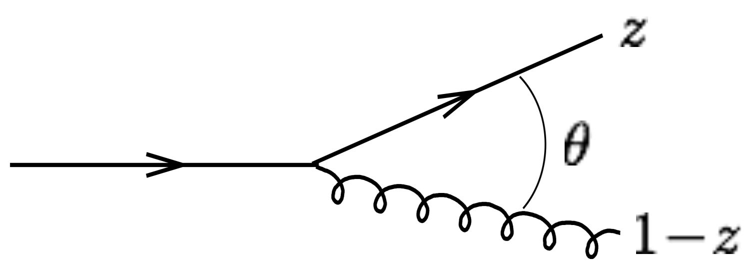
$$\Sigma(Q_0^2) \simeq 1 - 2 \int dP_{q\to qg}\, \Theta(4z(1-z)Q^2\theta^2 - Q_0^2)$$

two jets

invariant mass of each jet $\simeq E_q E_g \theta^2$

Sudakov logarithms

In the soft limit we can approximate $z \simeq 1$ whenever possible



$$\int dP_{q\to qg}\,\Theta(4z(1-z)Q^2\theta^2 - Q_0^2) \simeq 2C_F\frac{\alpha_s}{2\pi}\int_0^1 \frac{d\theta^2}{\theta^2}\int_0^1 \frac{dz}{1-z}\Theta((1-z)\theta^2 Q^2 - Q_0^2)$$

$$\simeq 2C_F\frac{\alpha_s}{2\pi}\int_{Q_0^2/Q^2}^1 \frac{d\theta^2}{\theta^2}\int_0^{1-\epsilon}\frac{dz}{1-z} \qquad \epsilon = \frac{Q_0^2}{Q^2\theta^2}$$

Performing the integral gives the leading order mass distribution

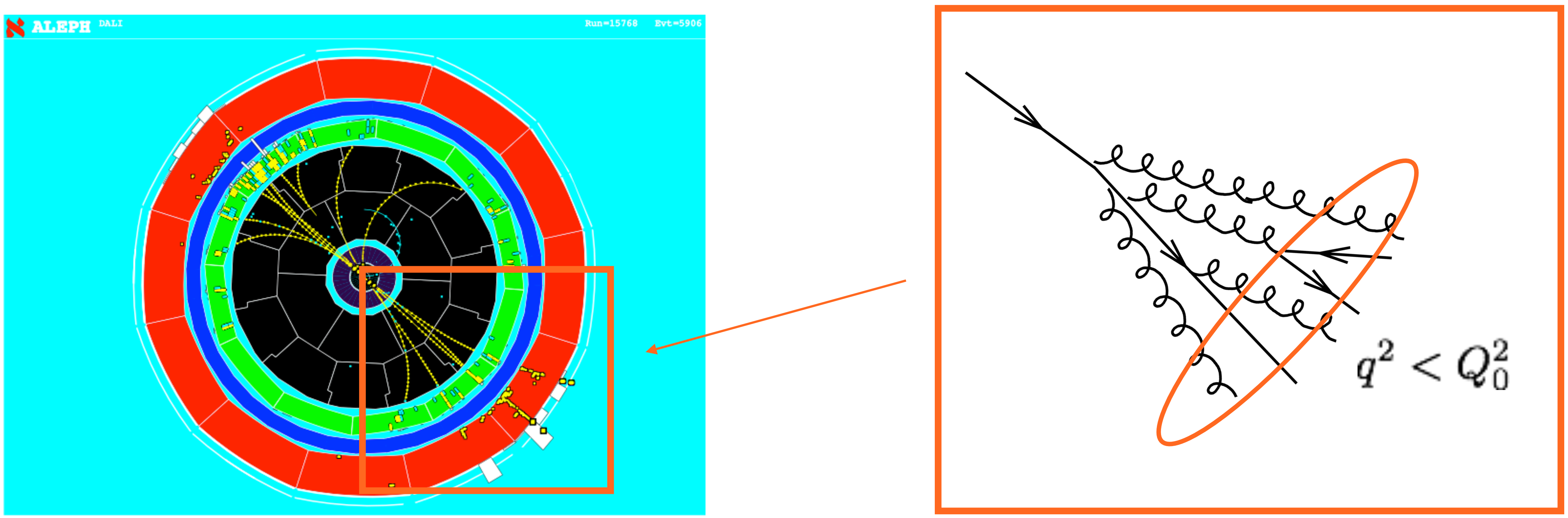
$$\Sigma(Q_0^2) \simeq 1 - C_F\frac{\alpha_s}{\pi}\ln^2\left(\frac{Q^2}{Q_0^2}\right)$$

The double log arises from an incomplete cancellation of real and virtual corrections

Large logarithms appear at all orders and need to be resummed

Leading logarithmic resummation

At all orders, the invariant mass distribution is just the probability that in any jet there are no resolved collinear splitting above the scale Q_0^2



But this probability is just the product of two Sudakov form factors, one for each jet

$$\Sigma(Q_0^2) \simeq [\Delta_q(Q^2, Q_0^2)]^2 \qquad \Delta_q(Q^2, Q_0^2) = \exp\left[-\int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \int_0^{1-\epsilon} dz \, \frac{\alpha_s}{2\pi} p_{qq}(z)\right]$$

Leading logarithmic resummation

In the soft limit, using $q^2 \simeq (1-z)\theta^2 Q^2$ and $\theta^2 \lesssim 1$ we obtain

$$\Delta_q(Q^2, Q_0^2) \simeq \exp\left[-\int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \int_0^1 dz \, \frac{\alpha_s}{2\pi} \frac{2}{1-z} \Theta\left(1 - z - \frac{q^2}{Q^2}\right)\right] \simeq e^{-C_F \frac{\alpha_s}{2\pi} \ln^2\left(\frac{Q^2}{Q_0^2}\right)}$$

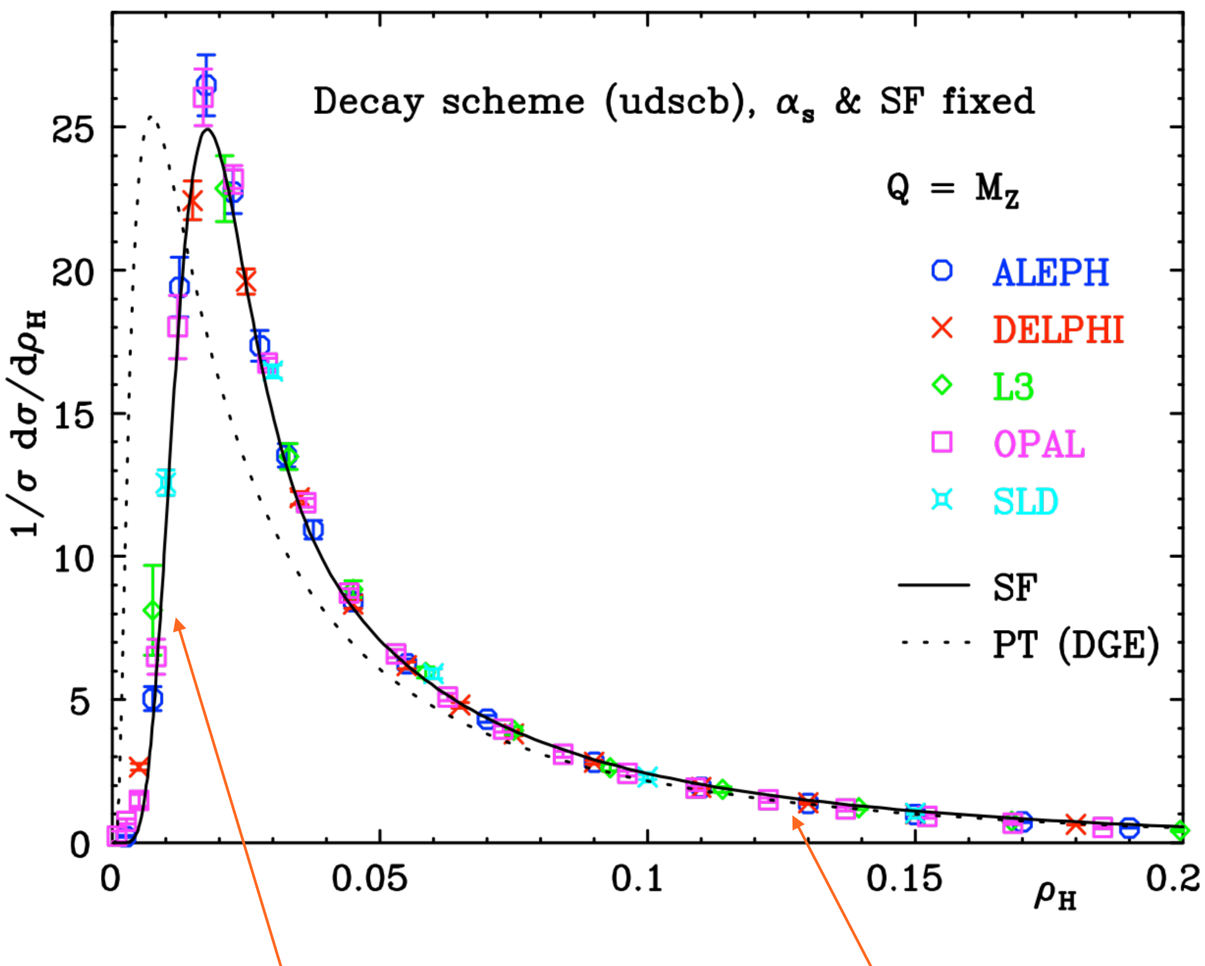
This gives us that the invariant mass distribution is just the exponentiated version of the one-gluon result

$$\Sigma(Q_0^2) \simeq [\Delta_q(Q^2, Q_0^2)]^2 \simeq e^{-C_F \frac{\alpha_s}{\pi} \ln^2\left(\frac{Q^2}{Q_0^2}\right)}$$

This expression resums the leading logarithms: it implicitly assumes that all emissions are strongly ordered in the invariant mass $q_1^2 \gg q_2^2 \gg \cdots \gg q_n^2$

Leading logarithmic resummation

This leading logarithmic behaviour is reflected qualitatively in actual data



Sudakov suppression

fixed order region

$$\Sigma(Q_0^2) \simeq e^{-C_F \frac{\alpha_s}{\pi} \ln^2\left(\frac{Q^2}{Q_0^2}\right)}$$

$$\frac{1}{\sigma}\frac{d\sigma}{d\rho_H} = \frac{d}{d\rho_H}\Sigma(\rho_H Q^2)$$

Leading logarithmic resummation

Exercise 1. Compute the Sudakov form factor with the full splitting function

$$\Delta_q(Q^2, Q_0^2) \simeq \exp\left[-\int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \int_0^1 dz \, \frac{\alpha_s}{2\pi} C_F \frac{1+z^2}{1-z} \Theta\left(1 - z - \frac{q^2}{Q^2}\right)\right]$$

Exercise 2. Compute the Sudakov form factor with the running coupling

$$\Delta_q(Q^2, Q_0^2) \simeq \exp\left[-\int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \int_0^1 dz \, \frac{\alpha_s((1-z)q^2)}{2\pi} C_F \frac{1+z^2}{1-z} \Theta\left(1 - z - \frac{q^2}{Q^2}\right)\right]$$

Hint: use the one loop expression for the running coupling and express the result as a function of $\lambda = \alpha_s(Q^2)\beta_0 \ln(Q^2/Q^2)$

Resummation accuracy

The peak of the invariant mass distribution corresponds to $\alpha_s L \sim 1$, where

$$\alpha_s \equiv \alpha_s(Q^2) \qquad\qquad L \equiv \ln(Q^2/Q_0^2)$$

Resummation is the reorganisation of the perturbative series in the region $\alpha_s L \sim 1$

$$\Sigma(Q_0^2) \simeq e^{\underbrace{Lg_1(\alpha_s L)}_{\mathrm{LL}} + \underbrace{g_2(\alpha_s L)}_{\mathrm{NLL}} + \underbrace{\alpha_s g_3(\alpha_s L)}_{\mathrm{NNLL}} + \ldots}$$

$$\simeq e^{\underbrace{Lg_1(\alpha_s L)}_{\mathrm{LL}}} \times \left(\begin{matrix} 1 & + & \alpha_s & + \ldots \\ \underbrace{G_2(\alpha_s L)}_{\mathrm{NLL}} & + & \underbrace{\alpha_s G_3(\alpha_s L)}_{\mathrm{NNLL}} & + \ldots \end{matrix} \right)$$

Only the leading logarithms are required to exponentiate, the subleading logarithms have to factorise, and give rise to a well-defined perturbative series

Summary

In this lecture we have learnt

- IRC collinear safety is the property that makes it possible to use safely QCD perturbation theory
- QCD has soft and collinear divergences, whose incomplete cancellation leads to large logarithms that need to be resummed
- Resummation of collinear logarithms (DGLAP) can be seen as a probabilistic process. From that the Sudakov form factor emerges as the probability of having no emissions between two scales
- The Sudakov form factor gives a straightforward handle to perform the all-order resummation of double logarithms