Resummation and theoretical uncertainties

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Single-logarithmic resummation

Terascale Monte Carlo School 2014 - DESY Hamburg 12/03/2014

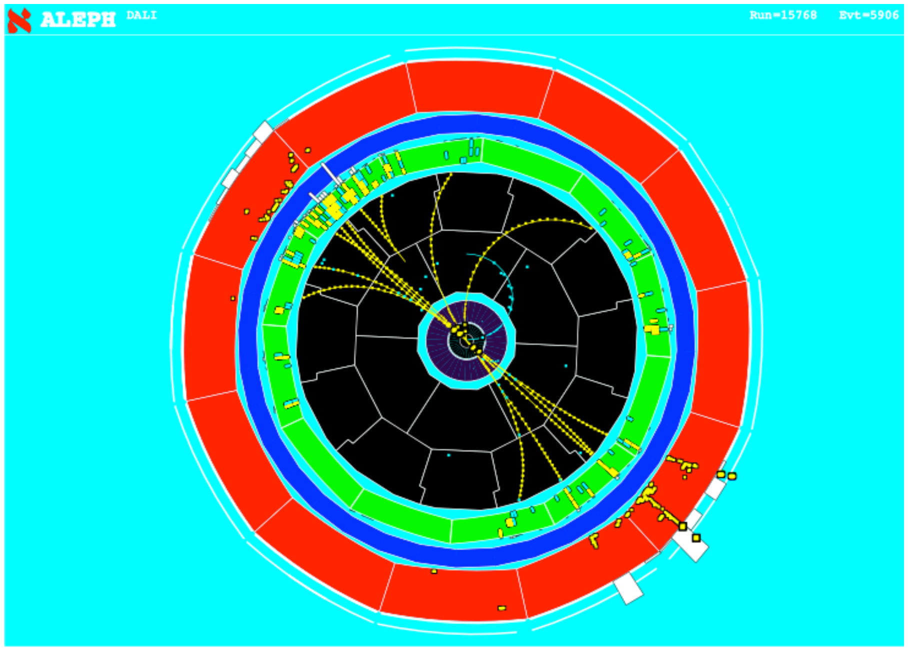
Event-shape variables

Event-shape variables $V(p_1, \ldots, p_n)$ are combinations of hadron final-state momenta in a number which gives insight on event geometry

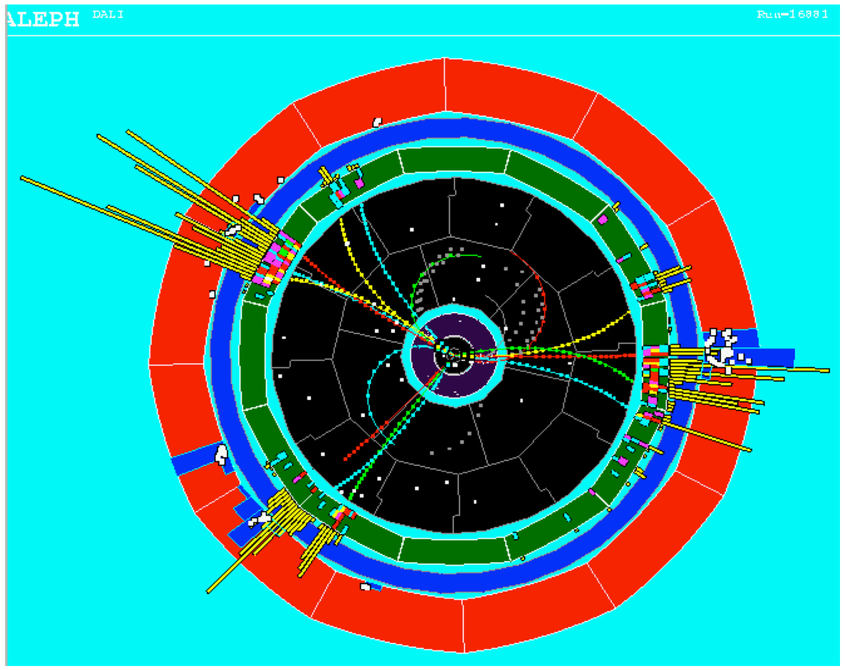
Example: Thrust, longitudinal particle alignment

$$T \equiv \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|} \qquad \tau \equiv 1 - T$$

Pencil-like event: $T \lesssim 1$



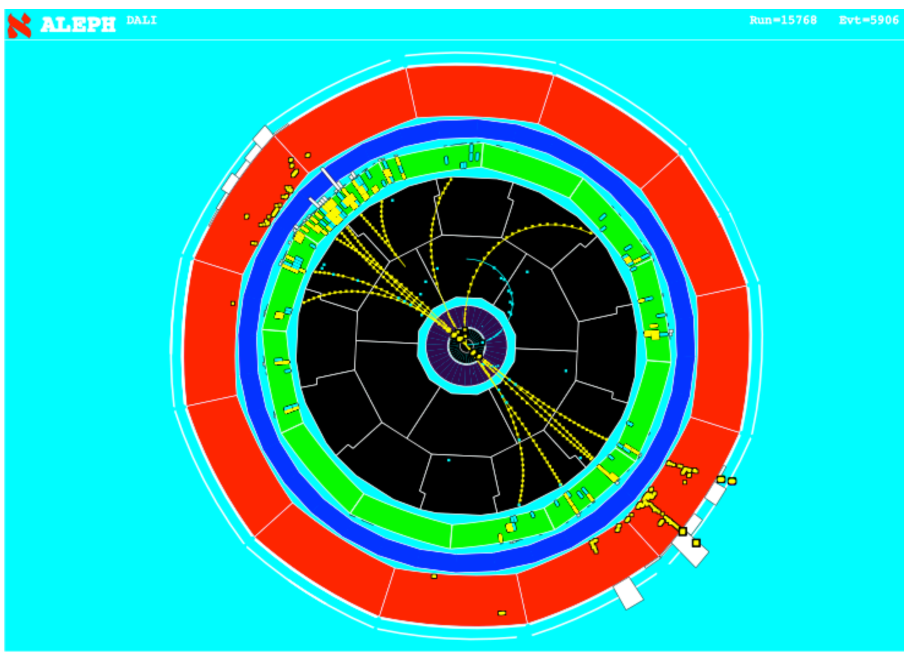
Planar event: $T \gtrsim 2/3$

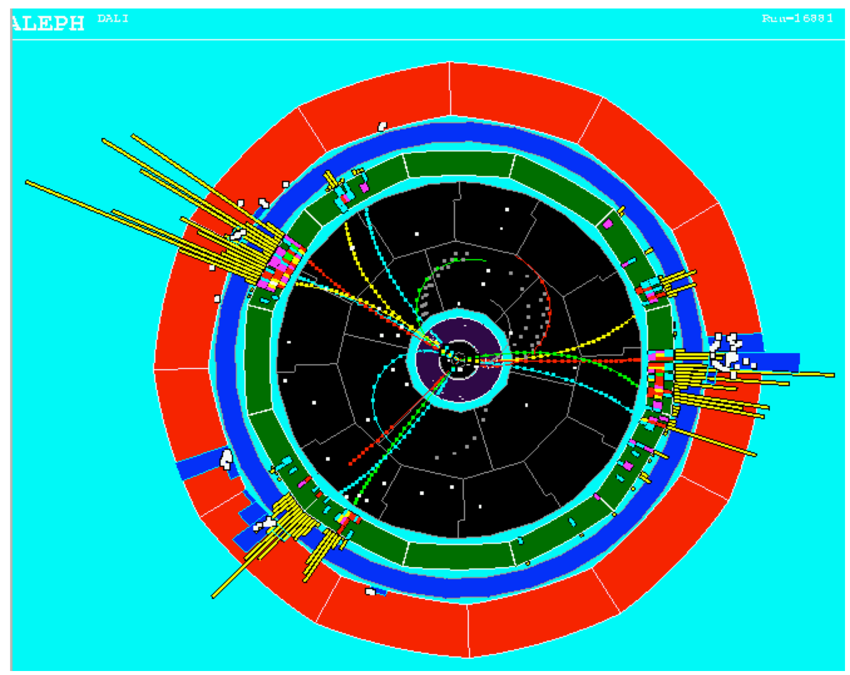


Final-state observables

Event-shape variables are a class of final-state observables, continuous measures of final-state energy-momentum flow

$$\tau \to 0 \quad \longleftrightarrow \quad \tau \to 1/3$$



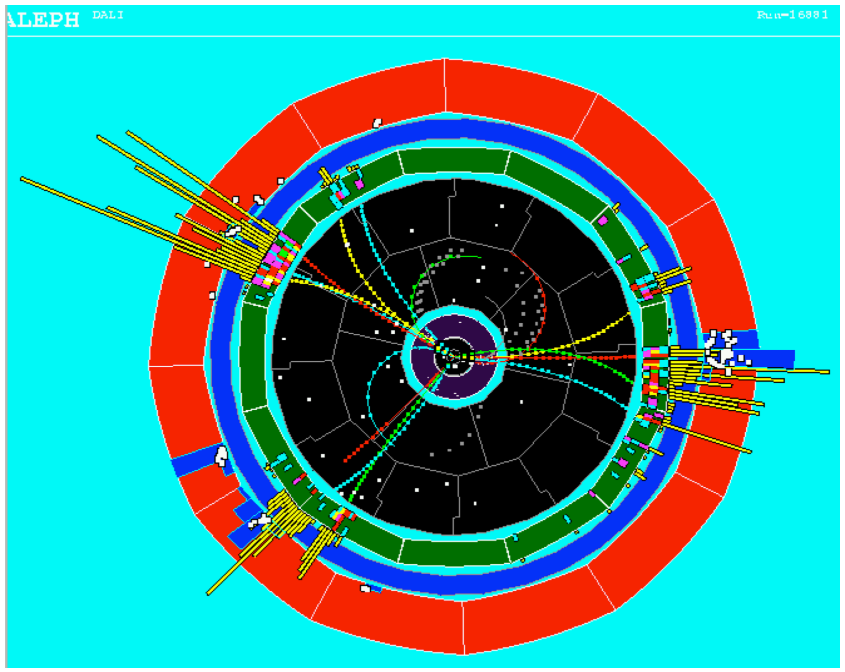


In the previous lecture we have encountered the heavy-jet mass, the maximum of the invariant masses of the two hemispheres in which the event is divided by the thrust axis

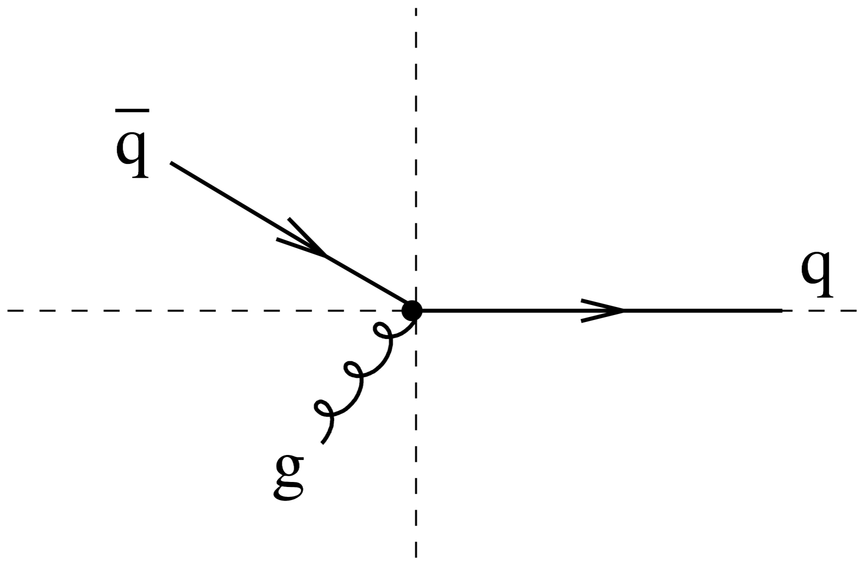
There are many other such observables, jet broadenings, jet-resolution parameters, etc.

Collinear and infrared safety

All final-state observables we consider are infrared and collinear (IRC) safe, so that we can safely compute their distributions using quark and gluon language



$\Rightarrow$



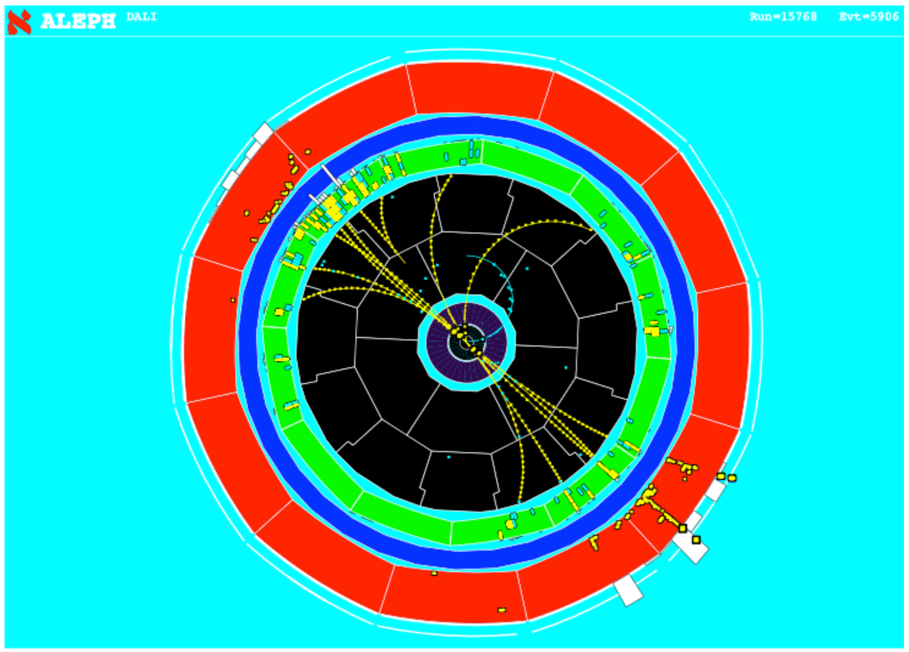
Example: jets from parton momenta are close to jets from hadons momenta if their momenta do not change after

- the addition of any number of soft partons (IR safety)
- any number of collinear splittings (collinear safety)

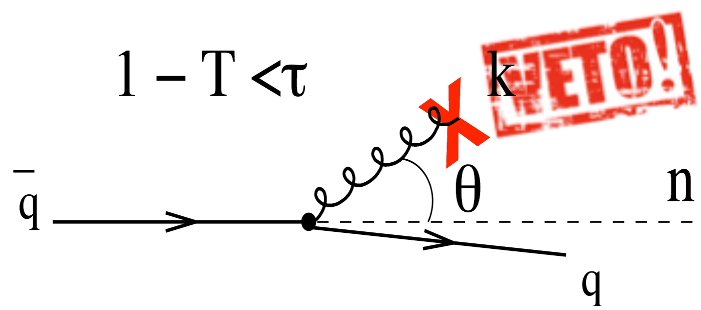
Large logarithms

Final-states have the property that for configurations close to the Born limit (e.g. a $q\bar{q}$ pair) their value is close to zero

Example: pencil like events are selected by requiring the thrust or the heavy-jet mass to be below a certain value



The restriction $1 - T < \tau$ acts as a veto on gluon emissions



To quantify the departure from the Born limit we consider the cumulative distribution $\Sigma(v)$,the probability that all events have $V(p_1, \ldots, p_n) < v$

Resummation for final-state observables

As we have seen in the example of the heavy-jet mass, close to the Born limit, $\Sigma(v)$ has up to two logarithms of v for each power of α_s

In the region $\alpha_s L \sim 1$, where $L = \ln(1/v)$, we wish the cumulative distribution of any final-state observable to be written in the form

$$\Sigma(v) \simeq e^{\underbrace{Lg_1(\alpha_s L)}_{\mathrm{LL}}} \times \left(\overset{1}{\underbrace{G_2(\alpha_s L)}_{\mathrm{NLL}}} \overset{+}{+} \overset{\alpha_s}{\underbrace{\alpha_s G_3(\alpha_s L)}_{\mathrm{NNLL}}} \overset{+\dots}{+\dots} \right)$$

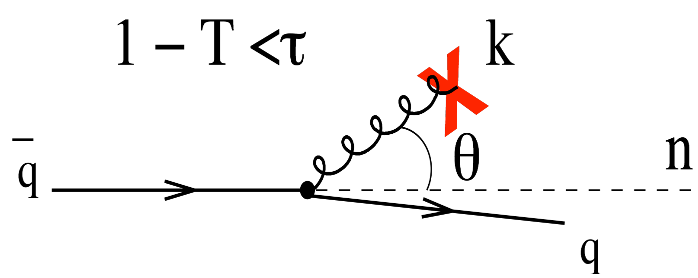
To achieve NLL accuracy we have to consider:

- Double logarithms $\alpha_s L^2$: they come from soft *and* collinear contributions, and have to exponentiate
- Single logarithms $\alpha_s L$: they come from soft and/or collinear contributions, and have to factorise from double logarithms

One gluon emission

We consider one gluon emission k and we compute the distribution $\Sigma(v)$

Example of kinematics: two lightlike momenta along the thrust axis



$$P_1 = \frac{Q}{2}(1,0,0,1)$$

$$P_2 = \frac{Q}{2}(1,0,0,-1)$$

Sudakov decomposition of k along P_1 and P_2

$$k = z_1 P_1 + z_2 P_2 + k_t \qquad 0 = k^2 = z_1 z_2 Q^2 - k_t^2$$

Phase space and matrix element in the soft-collinear limit

$$[dk] = Q^2 dz_1\, dz_2 \,\frac{d\phi}{2\pi}\, dk_t^2\, \delta(z_1 z_2 Q^2 - k_t^2) \qquad M^2(k) = \frac{\alpha_s C_F}{4\pi}\, \frac{z_1\, p_{gq}(z_1) \cdot z_2\, p_{gq}(z_2)}{k_t^2}$$

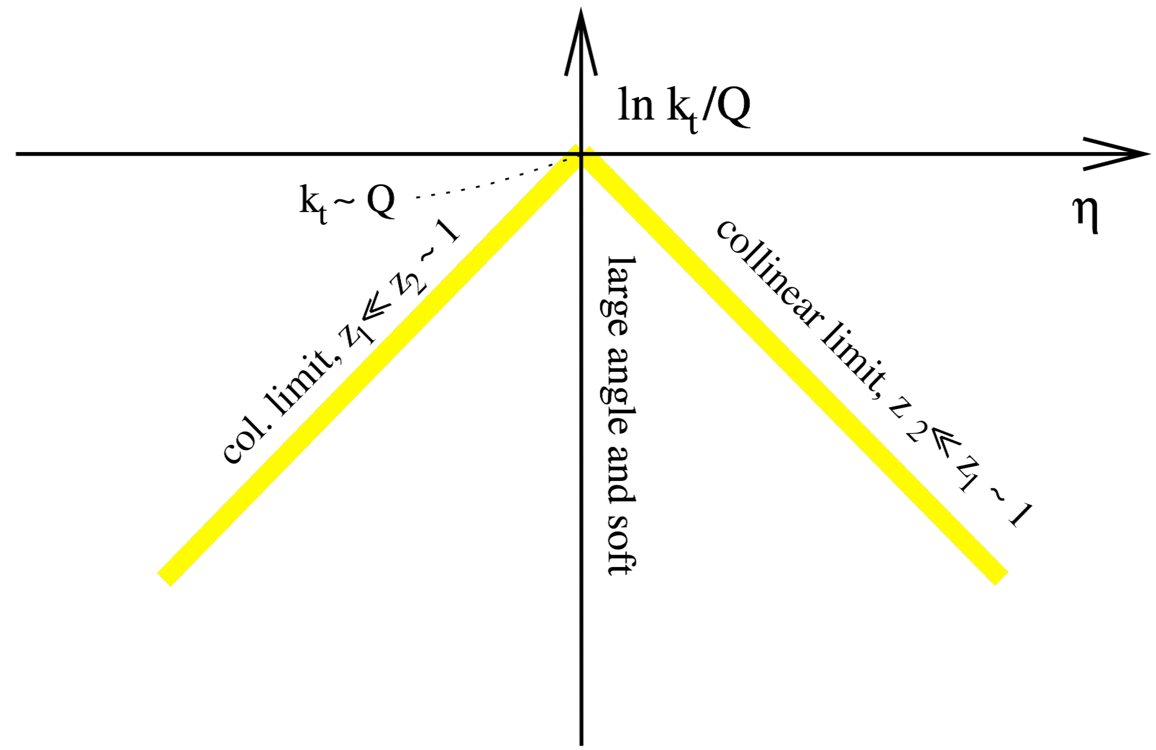
The Lund plane

For resummation purposes, it is extremely useful to visualise soft and collinear emissions in the Lund plane. We need to introduce the emission rapidity

$$\eta = \frac{1}{2}\ln\left(\frac{z_1}{z_2}\right) \qquad \Rightarrow \qquad [dk]M^2(k) \simeq 2C_F\frac{\alpha_s}{\pi}\frac{dk_t}{k_t}d\eta$$

The boundaries $z_1, z_2 < 1$ correspond the the collinear limits, giving the boundary

$$|\eta| < \ln\left(\frac{Q}{k_t}\right)$$



The Lund plane

For a single emission the cumulative distribution for any event shape is just

$$\Sigma(v) = 1 - \int [dk] M^2(k)\, \Theta(V(k) - v)$$

For a single emission, any "good " final state observable behaves as

- Soft and collinear to leg ℓ

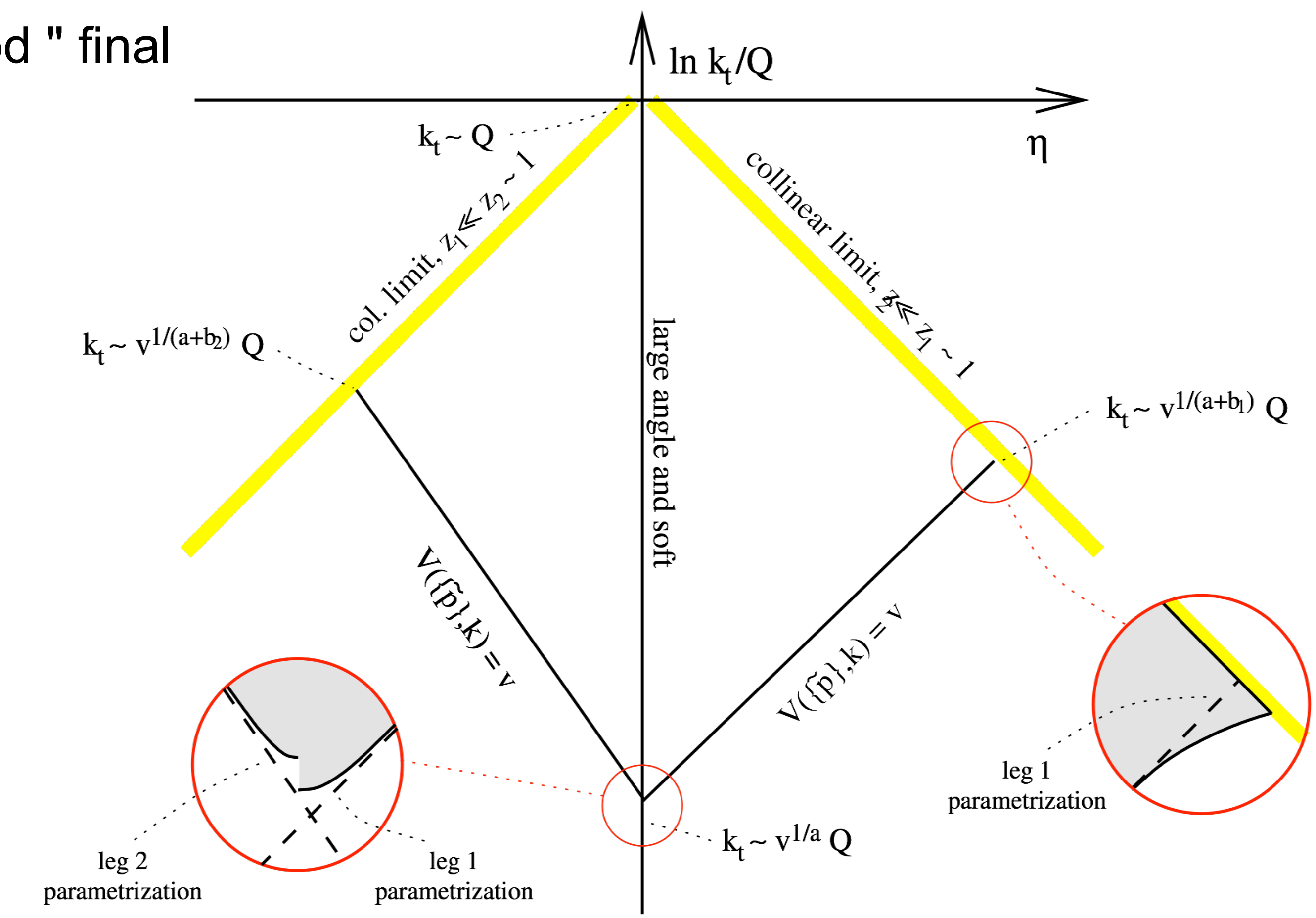
$$V(k) \simeq d_\ell \left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} g_\ell(\phi)$$

- Soft large-angle

$$V(k) \sim k_t^a$$

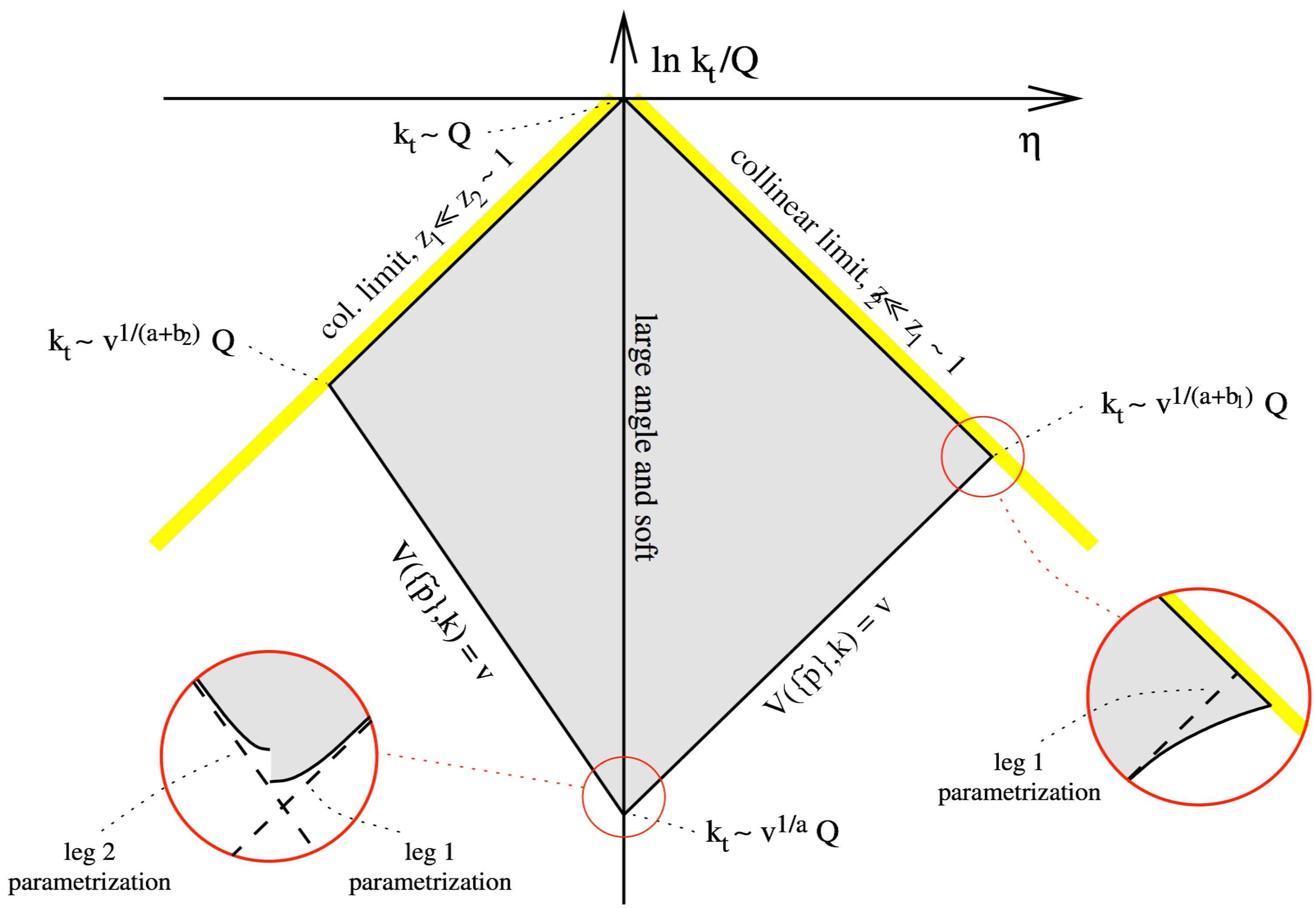
- Hard and collinear to leg ℓ

$$V(k) \sim k_t^{a+b_\ell}$$



The Lund plane

The first order contribution to $\Sigma(v)$ can be obtained by just computing the shaded area



Logarithms for thrust and heavy-jet mass

For one emission the thrust and the heavy-jet mass are the same observable

- Soft-collinear emission

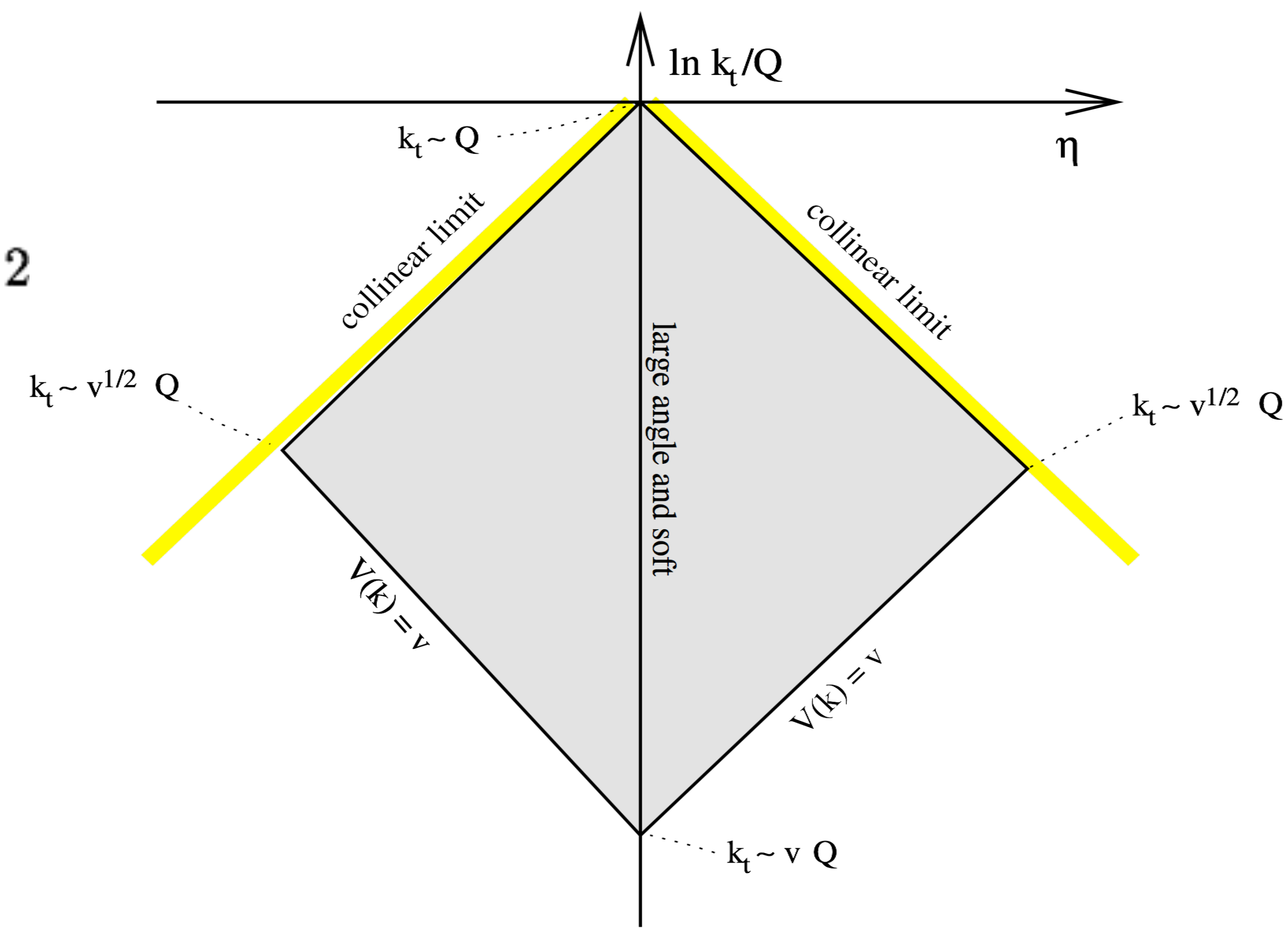
$$V(k) \simeq \left(\frac{k_t}{Q}\right) e^{-\eta^{(\ell)}} \qquad \ell = 1, 2$$

- Soft large-angle

$$V(k) \sim k_t$$

- Hard collinear

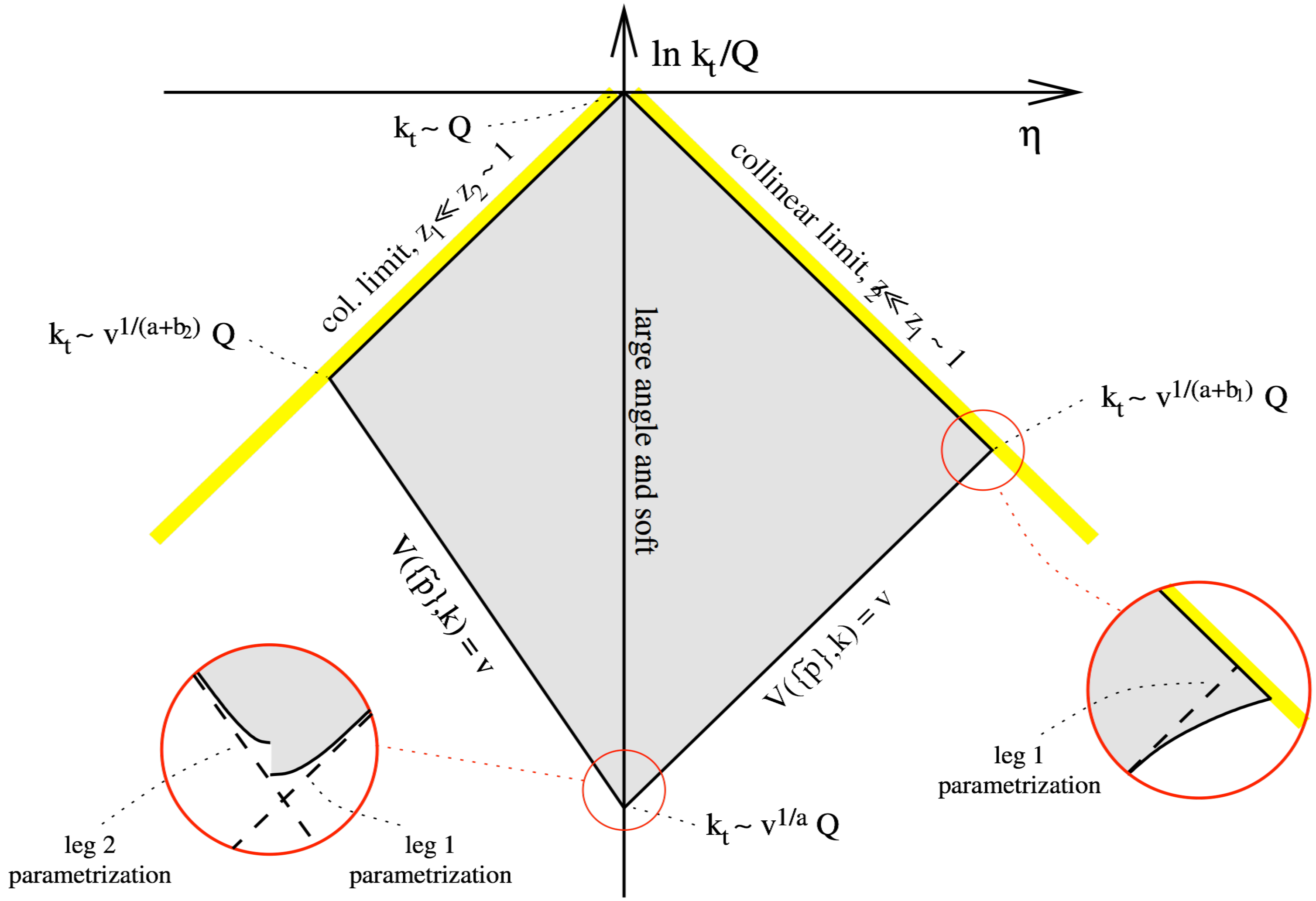
$$V(k) \sim k_t^2$$



Exercise. Compute the shaded area to derive $\Sigma(v) = 1 - C_F \frac{\alpha_s}{\pi} \ln^2\left(\frac{1}{v}\right)$

Double and single logarithms

In the Lund plane, areas correspond to double logarithms, lines to single logarithms and points to constant contributions, so the log counting can be visualised



Radiator at NLL accuracy

The one-gluon contribution is conveniently written in terms of a "radiator", when one includes also all inclusive splittings of the emitted gluon

$$\Sigma(v) = 1 - R(v) \quad \Rightarrow \quad R(v) = \int [dk] M^2(k) \Theta(V(k) - v)$$

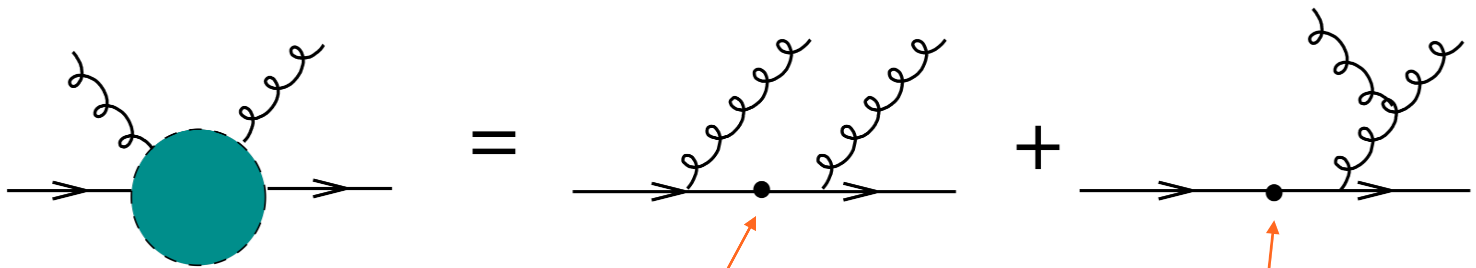
For two final-state emitting quark legs, as in e^+e^- annihilation

$$R(v) = \sum_{\ell=1,2} C_F \int^{Q^2} \frac{dk_t^2}{k_t^2} \frac{\alpha_s(k_t)}{2\pi} \int_0 d\eta \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} z p_{gq}(z) \Theta\left(d_\ell \left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta} g_\ell(\phi) - v\right)$$

Exercise. Derive the expressions in section 2.1.3 of hep-ph/0407286

Double soft-collinear emission

Considering the most singular case of two soft gluons with strongly ordered energies



independent emission

correlated emission, singular only when the gluons are close in angle

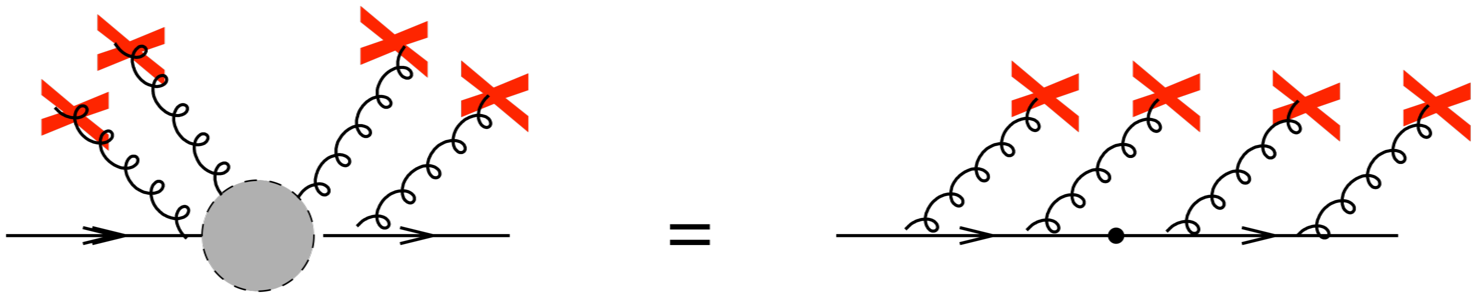
$$M^2(k_1,k_2) = M^2(k_1)M^2(k_2)$$
$$+\,C_A C_F \boxed{\left[\frac{(p_1p_2)}{(p_1k_1)(k_1k_2)(k_2p_2)} + \frac{(p_1p_2)}{(p_2k_1)(k_1k_2)(k_2p_1)} - \frac{(p_1p_2)}{(p_1k_1)(k_1p_2)}\frac{(p_1p_2)}{(p_1k_2)(k_2p_2)}\right]}$$

When the gluons are widely separated in angle, only the independent emission contribution survives

Multiple soft-collinear emissions

We first neglect correlated emission. Then the multi-gluon matrix element is simply

$$M^2(k_1, k_2, \dots, k_n) \simeq M^2(k_1)M^2(k_2)\dots M^2(k_n)$$



In this case the cumulative distribution of an event shape becomes

$$\Sigma(v) = e^{-\int [dk] M^2(k)} \sum_{n=0}^{\infty} \int \prod_i [dk_i] M^2(k_i)\, \Theta(v - V(k_1, \dots, k_n))$$

virtual corrections, ensure
that the inclusive sum of all
emissions gives 1

Leading logarithmic resummation

Suppose the emissions are strongly ordered, i.e. $V(k_1) \gg V(k_2) \gg \cdots \gg V(k_n)$

Assume also that the value of the observable is dominated by $V(k_1)$

$$\Theta(v - V(k_1, \ldots, k_n)) \simeq \prod_i \Theta(v - V(k_i))$$

$$\Sigma(v) = e^{-\int [dk] M^2(k)} \sum_{n=0}^{\infty} \int \prod_i [dk_i] M^2(k_i)\, \Theta(v - V(k_i)) = e^{-R(v)}$$

If the observable's value is dominated by the "hardest" emission, in the strongly ordered regime the cumulative distribution is obtained by the exponentiation of the contribution of a single gluon (Sudakov form factor)

Under these assumptions one obtains the exponentiation of leading logarithms

Failure of leading logarithmic exponentiation

In the case of the JADE jet algorithm double logarithms do not exponentiate

$$\Sigma(y_{23}) = 1 - \frac{C_F \alpha_s}{\pi} \ln^2\left(\frac{1}{y_{23}}\right) + \frac{1}{2!} \times \frac{5}{6} \times \left(\frac{C_F \alpha_s}{\pi} \ln^2\left(\frac{1}{y_{23}}\right)\right)^2$$

We need to identify what can go wrong

$$\Sigma(v) = e^{-R(v)} \times \left\{ e^{-\int^v [dk] M^2(k)} \sum_{n=0}^{\infty} \int \prod_i [dk_i] M^2(k_i)\, \Theta(v - V(k_1, \ldots, k_n)) \right\}$$

leading log exponentiation

corrections

Potential corrections spoiling leading logarithmic exponentiation may come from

- Emissions without strong ordering $V(k_1) \sim V(k_2) \cdots \sim V(k_n)$
- Strongly ordered emissions $V(k_1) \gg V(k_2) \gg \cdots \gg V(k_n)$

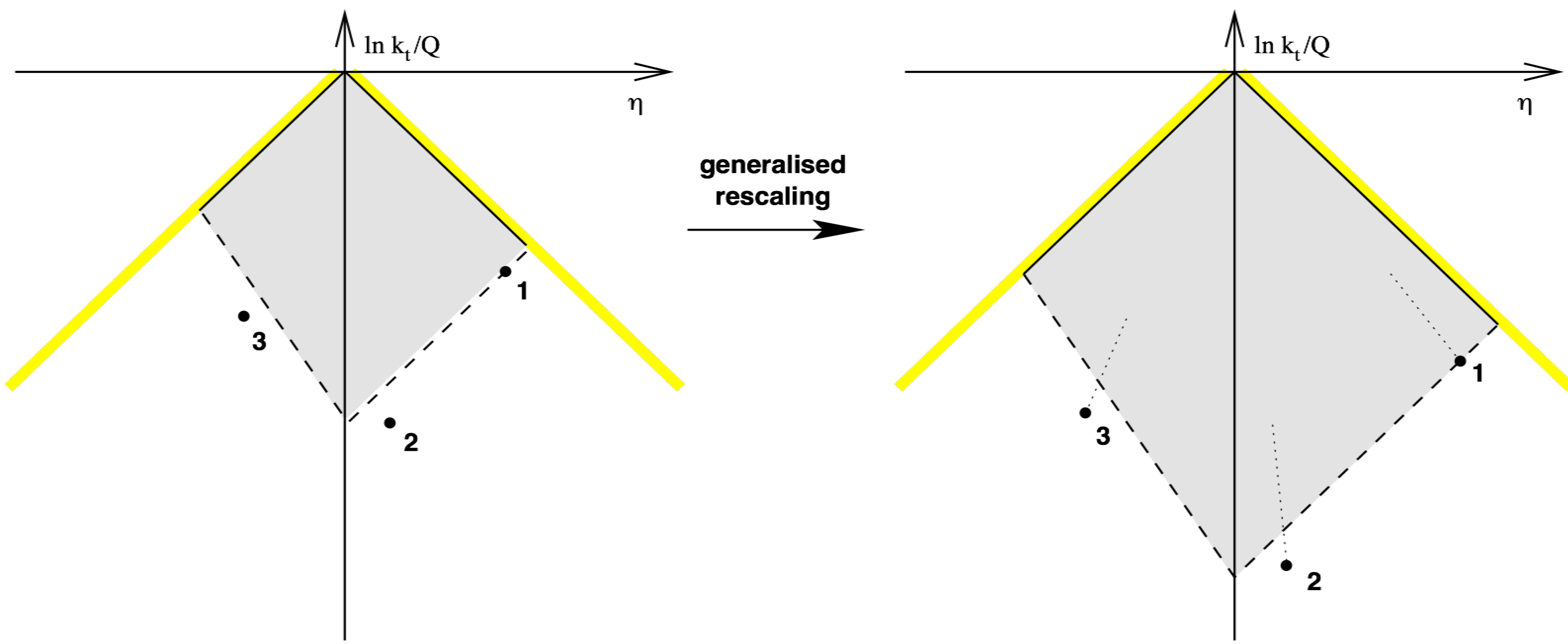
rIRC safety: condition 1

Recursive infrared and collinear (rIRC) safety conditions are safety condition we put on the observable, so that we have no surprises from multiple emissions

1. Let's scale all $V(k_i) = \bar{v}\,\zeta_i$, the observable should scale the same way, i.e.

$$\lim_{\bar{v}\to 0} \frac{V(k_1(\bar{v}\zeta_1),\ldots,k_n(\bar{v}\zeta_n))}{\bar{v}} = \text{finite, and non-zero}$$

This ensures that, when $V(k_1) \sim V(k_2) \cdots \sim V(k_n)$, we have $V(k_1,\ldots,k_n) \sim V(k_1)$



rIRC safety: condition 2

Recursive infrared and collinear (rIRC) safety conditions are safety condition we put on the observable, so that we have no surprises from multiple emissions

2. Let us make k_{n+1} softer than the others $\zeta_{n+1} \ll \zeta_1 \sim \zeta_2 \sim \cdots \sim \zeta_n$, the observable should not change its scaling

$$\lim_{\zeta_{n+1}} \lim_{\bar{v}\to 0} \frac{V(k_1(\bar{v}\zeta_1),\ldots,k_n(\bar{v}\zeta_n),k_{n+1}(\bar{v}\zeta_n))}{\bar{v}} = \lim_{\bar{v}\to 0} \frac{V(k_1(\bar{v}\zeta_1),\ldots,k_n(\bar{v}\zeta_n))}{\bar{v}}$$

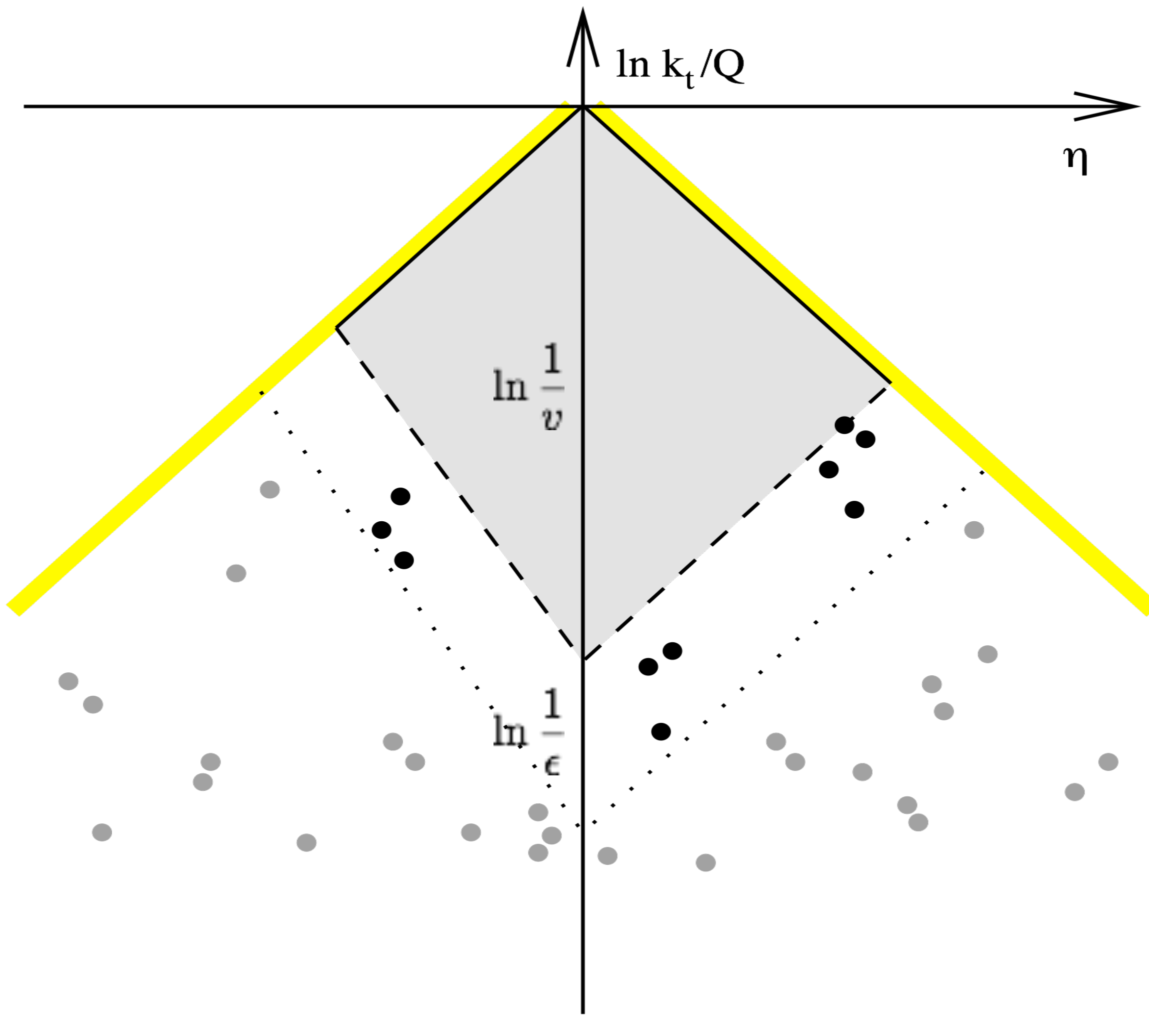
This ensures that, in the strongly ordered limit $V(k_1) \gg V(k_2) \gg \cdots \gg V(k_n)$, we still have $V(k_1,\ldots,k_n) \sim V(k_1)$

Exercise. An additive observable $V(k_1,\ldots,k_n) = V(k_1) + \cdots + V(k_n)$ is rIRC safe

Tough exercise. Show that the JADE three-jet resolution fails either of the two conditions

rIRC safety in the Lund plane

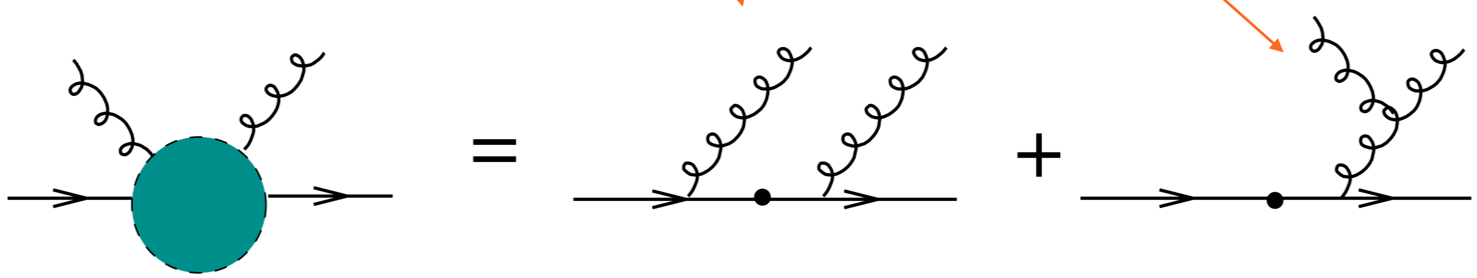
An immediate consequence of the rIRC safety conditions is that we can neglect all emissions with $V(k_i) < \epsilon v$, where $\ln(1/\epsilon) \ll \ln(1/v)$



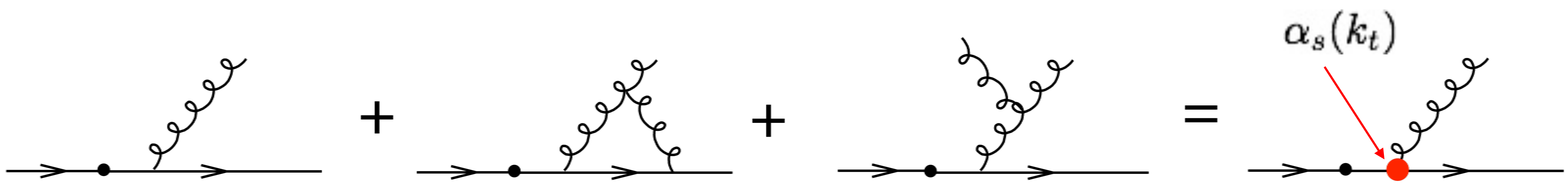
Two-gluon correlated emission

The two-gluon matrix element can be always written as the sum of an independent and correlated emission part

$$M^2(k_1, k_2) = M^2(k_1)M^2(k_2) + \tilde{M}^2(k_1, k_2)$$



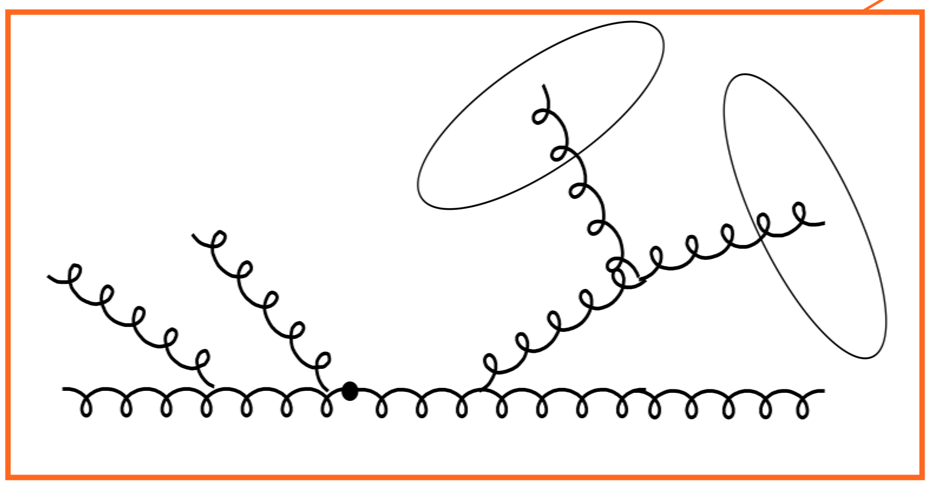
The correlated emission part, if integrated inclusively, is combined with the one-loop one-gluon matrix element to give the running coupling

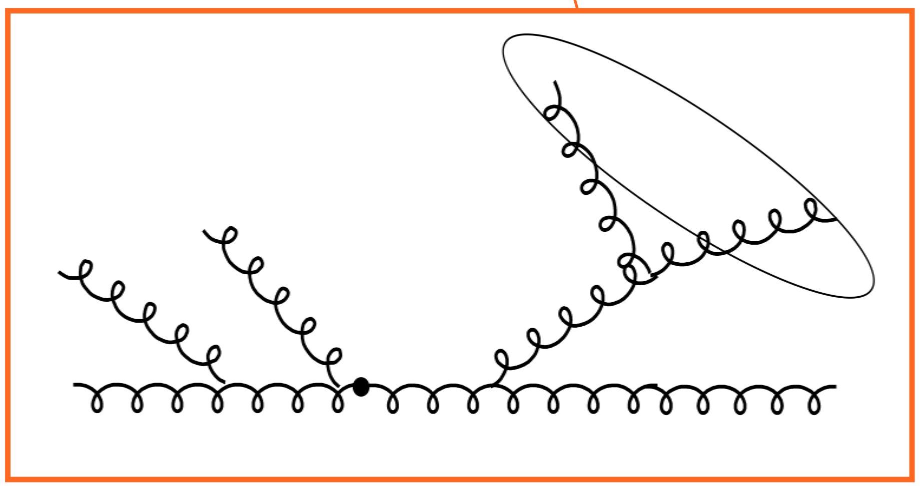


Two-gluon correlated emission

The remainder after extraction of the running coupling is

$$\int_{\epsilon v}[dk_1]\int_{\epsilon v}[dk_2]\tilde{M}^2(k_1,k_2)\,[\Theta(v-V(k_1,k_2))-\Theta(v-V(k_1+k_2))]$$





Example: in a jet-rate, two correlated emissions are clustered into different jets

The correlated matrix element has only a collinear singularity (the soft are cutoff by ϵv). Does this singularity produce an extra logarithm? If yes, this contribution can be potentially single logarithmic, we need to prevent this from happening.

rIRC safety condition 2.bis

We require a further condition on the observable with respect to collinear splittings

2.bis. Consider an emission k, splitting into k_a and k_b . We introduce a measure of the collinearity of the splitting as follows

$$k(\bar{v}\zeta) \to \{k_a, k_b\}(\bar{v}\zeta, \mu) \qquad \mu^2 = \frac{(k_a + k_b)^2}{k_t^2} \qquad \lim_{\mu \to 0}(k_a + k_b) = k$$

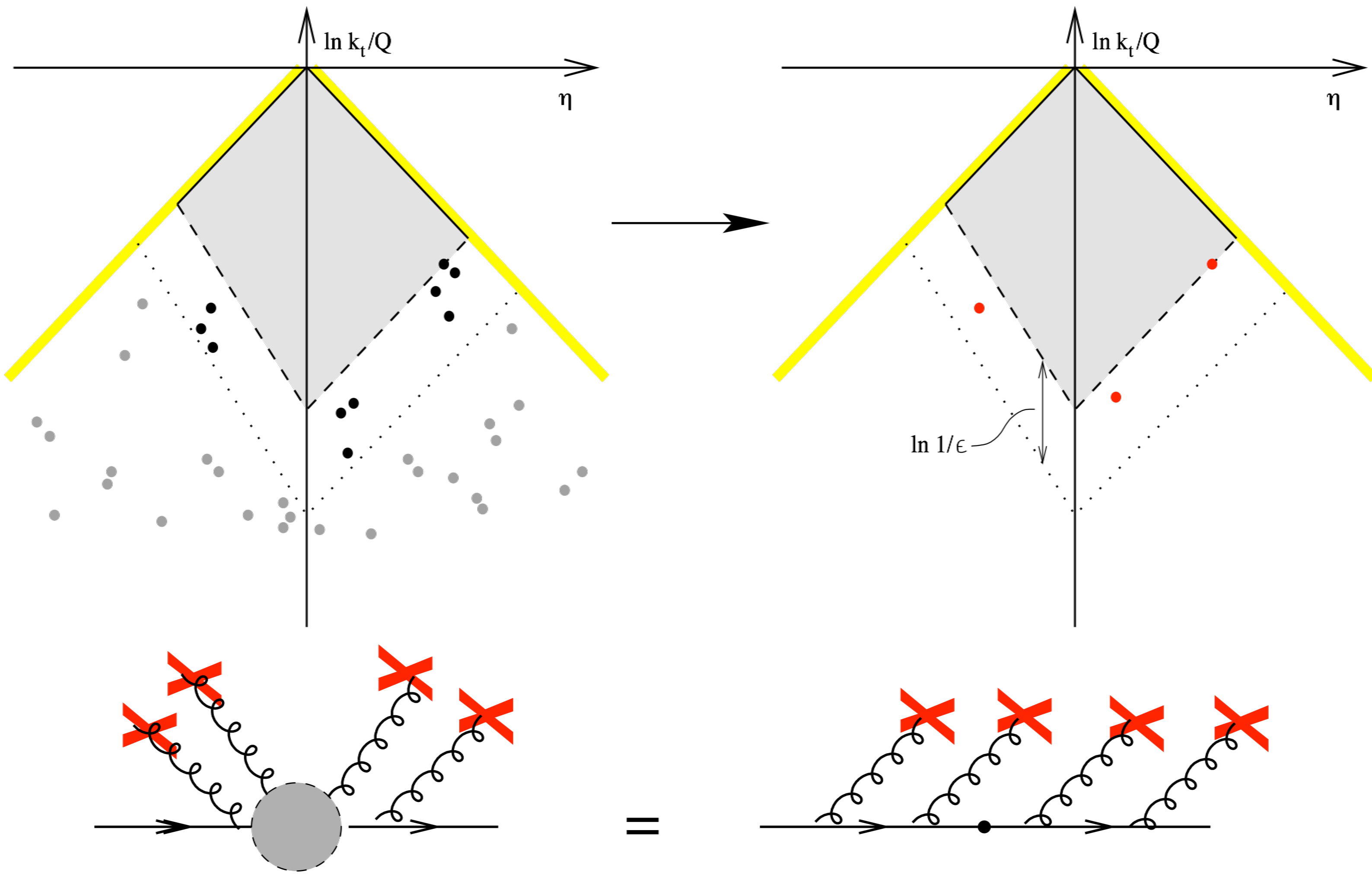
The scaling properties of the observable must not change after a collinear splitting

$$\lim_{\mu \to 0} \lim_{\bar{v} \to 0} \frac{V(\{k_a, k_b\}(\zeta\bar{v}, \mu), k_1(\bar{v}\zeta_1), \ldots, k_n(\bar{v}\zeta_n))}{v} = \lim_{\bar{v} \to 0} \frac{V(k(\zeta\bar{v}), k_1(\bar{v}\zeta_1), \ldots, k_n(\bar{v}\zeta_n))}{v}$$

Notice the order of the limits, first you rescale the observable, and then you take the collinear limit. If you take the limit in reverse order, the result is trivial because the observables we consider are all collinear safe.

rIRC safety 2.bis in the Lund plane

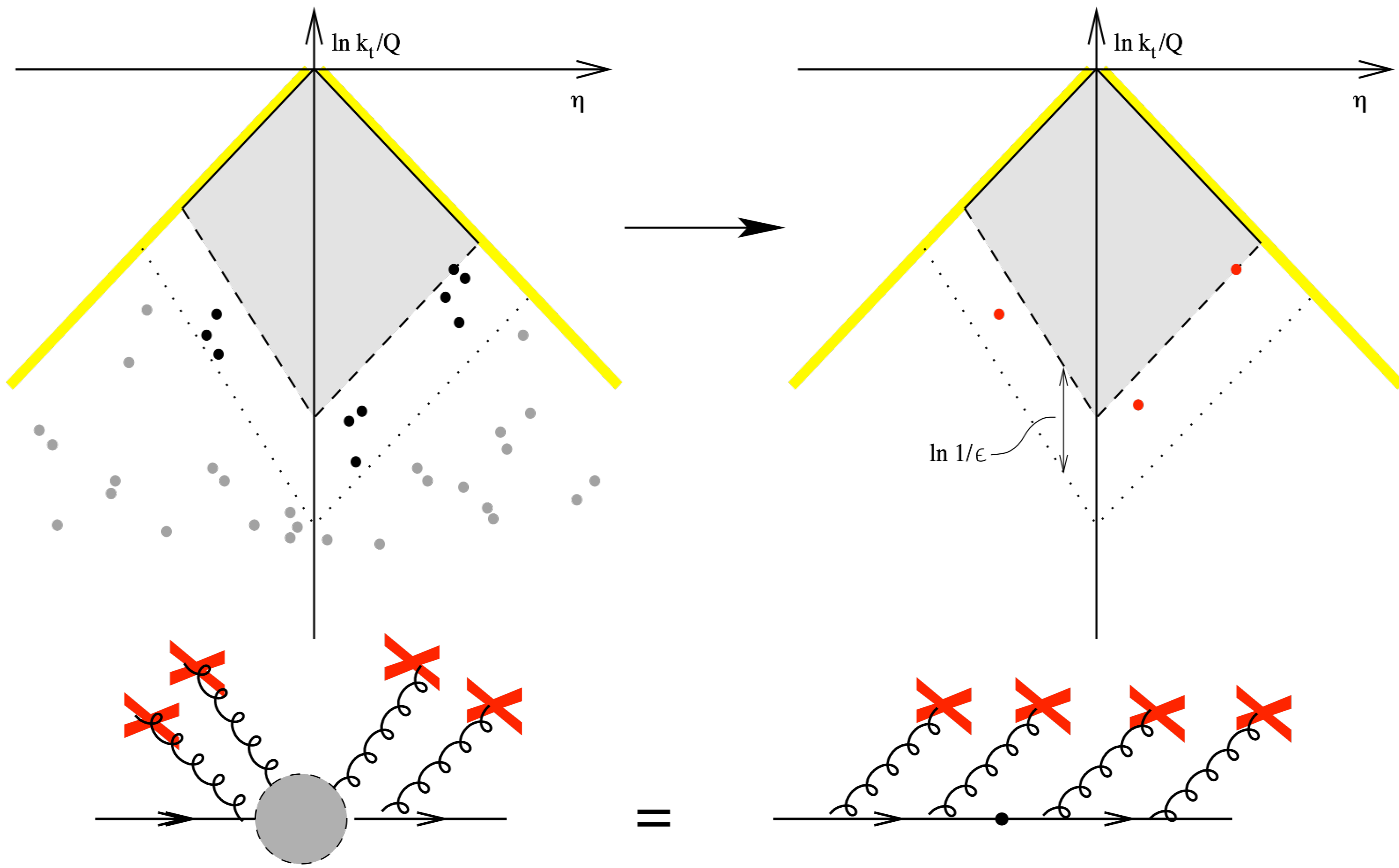
Clustering emissions close in rapidity does not produce extra logarithms



The relevant emissions are soft and collinear, widely separated in angle, and in a strip of size $\ln v \times \ln \epsilon$: this is a line, i.e. a single logarithmic contribution

NLL resummation

At last, we can write the NLL formula for a rIRC safe final-state observable



$$\Sigma(v) = e^{-R(v)} \underbrace{\left\{ e^{-\int_{\epsilon v}^{v} [dk] M^2(k)} \sum_{n=0}^{\infty} \int \prod_{i} [dk_i] M^2(k_i)\, \Theta(v - V(k_1, \ldots, k_n)) \right\}}_{\text{single-logarithmic correction}}$$

Multiple-emission correction

We now see explicitly that the multiple emission correction is single logarithmic

$$\int_{\epsilon v}^{v} [dk] M^2(k) = R(\epsilon v) - R(v) = \ln \frac{1}{\epsilon} R'(v) + \mathcal{O}(R''(v)) \qquad R'(v) = -v \frac{dR(v)}{dv}$$

We first parametrise the particle momenta collinear to leg ℓ in terms of $V(k_i)$

$$V(k_i) = v\zeta_i \qquad \Rightarrow \qquad k_i = k_i(\ell, v\zeta_i, \xi_i \eta_{\max}, \phi_i)$$

For simplicity we consider event shapes only, whose value does not depend on particle's rapidities. We can then integrate freely on the rapidity to get

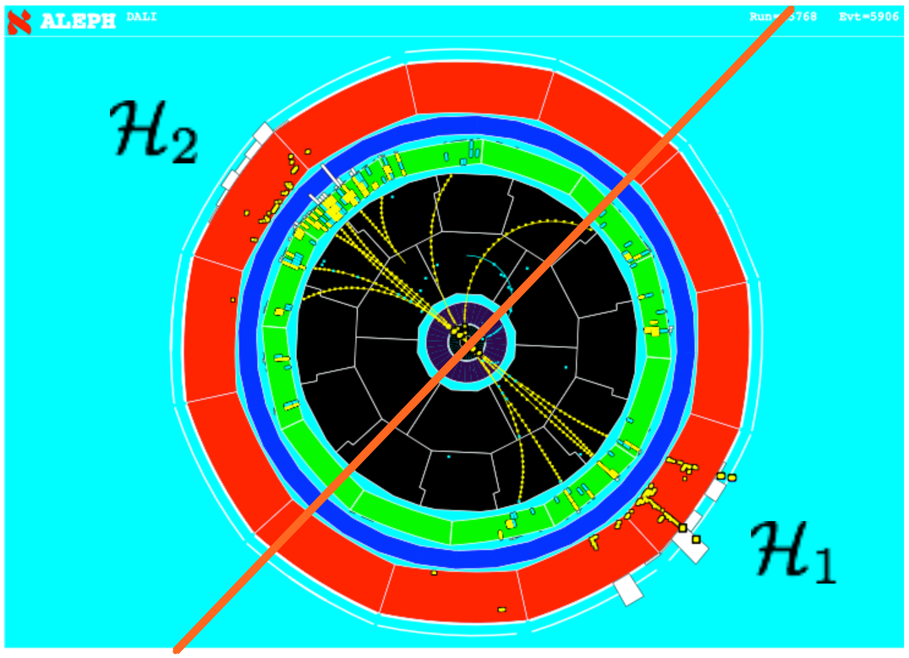
$$[dk]\, M^2(k)\, \Theta(V(k) - \epsilon v) \simeq \frac{R'(v)}{2} \sum_{\ell=1,2} \frac{d\zeta_i}{\zeta_i} \Theta(\zeta_i - \epsilon) \frac{d\phi_i^{(\ell)}}{2\pi} + \mathcal{O}(R'')$$

At last, the single logarithmic correction from multiple emissions

$$\mathcal{F}(R') = \epsilon^{R'} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_i \sum_{\ell=1,2} \frac{R'}{2} \int_{\epsilon}^{\infty} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i^{(\ell)}}{2\pi} \right) \Theta\left(1 - \lim_{v\to 0} \frac{V(k_1, \ldots, k_n)}{v}\right)$$

Exercise: the thrust at NLL accuracy

For soft and/or collinear emissions, the thrust can be written as



$$1 - T(k_1, \dots, k_n) = \sum_i \frac{k_{ti}}{Q} e^{-|\eta_i|} + \sum_{\ell=1,2} \frac{1}{Q^2} \frac{\left|\sum_{i\in\mathcal{H}_\ell} \vec{k}_{ti}\right|^2}{1 - \sum_{i\in\mathcal{H}_\ell} z_i}$$

1. Determine the scaling behaviour of the thrust in the soft-collinear, hard collinear and soft large-angle region

2. Prove that the thrust is recursively infrared and collinear safe

3. Show that the multiple emission correction is given by $\mathcal{F}(R') = \dfrac{e^{-\gamma_E R'}}{\Gamma(1+R')}$

Summary

In this lecture we have learnt:

- how to compute double logarithms fast using the Lund diagrams
- there are rIRC safety conditions you have to impose on your observable so that leading logarithms exponentiate
- for rIRC safe observables, NLL accuracy forces all real emissions to be soft and collinear, and widely separated in rapidity
- multiple emission contributions are encoded in a single-logarithmic function