# What do we know about NNLO computations in QCD?

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## The goal

The goal of this talk is to tell you about technology for NNLO computations that is being developed currently and to speculate about what we can realistically hope to achieve using it for the LHC physics in the next several years. I will do so assuming that we all agree that achieving NNLO QCD accuracy for hard collider processes is an important goal to pursue and I will not try to justify the importance of NNLO and its place in the grand scheme of hadron collider physics.

I want to start with a few historic remarks. The first NNLO QCD computation for hadron collider processes was the Drell-Yan NNLO paper by van Neerven. This happened in 1990, i.e. almost quarter of a century ago! The second NNLO computation -- the Higgs production in gluon fusion by Harlander and Kilgore -- appeared eleven years after that, in 2001.

The above results refer to total cross-sections which are not measurable. First results for fiducial volume cross-sections at NNLO appeared even later -- Higgs production with decay to two photons, fully differentially, was obtained in 2004.

First results with strongly interacting particles in the final state (top pairs, jets etc.) -- which (in a certain sense) indicate complete understanding of a NNLO technology -- appeared only in 2013, i.e. just about a year ago. Therefore, things that I will tell you about are a) relatively new and b) took a while to understand, as the historical sketch shows.

#### What I will not talk about

There is more than one method for NNLO computations, at different stages of developments. Unfortunately, I can not talk about all of them because there is no time and because I am not an expert on technical details for many of these them.

I will focus on a method that, conceptually, is a generalization of the Frixione-Kunszt-Signer NLO subtraction framework to NNLO and I will try to explain how it works in detail. However, before I go there, I would like to mention a number of recent result related to the development of a generic method for NNLO computations to indicate that this is a rather active field.

I) Czakon, Fiedler, Mitov: top quark pair production at NNLO;

2) J. Currie, T. Gehrmann, N.. Glover, A. Gehrmann - de Ridder, J. Pires : dijet production at NNLO;
3) G. Abeloff, A. Gehrmann - de Ridder, P. Maierhofer, S. Pozzorini: top quark pair production at NNLO;

4) R. Boughezal, F. Caola, K. Melnikov, F. Petriello and M. Schulze: H+jet at NNLO;

5) M. Bruscherseifer, F. Caola, K. Melnikov : top decay at NNLO;

6) M. Bruscherseifer, F. Caola, K. Melnikov: t-channel single top production at NNLO (large N);

7) F. Cascioli, T. Gehrmann, M. Grazzini, et al. : ZZ production at NNLO;

8) C.Anastasiou, A. Lazopoulos, F. Herzog, R.Mueller, Higgs production in bottom fusion;

An alternative NNLO computational scheme is being developed by P. Bolzoni, V. Del Duca, G. Somogyi and Z. Trosczanyi.

### Perturbation theory for quark-gluon S-matrix

To describe collisions of quarks and gluons, we use conventional perturbation theory where a small parameter is the QCD coupling constant. Our goal is to compute jet cross-sections for fixed number of jets and arbitrary number of colorless particles (Z,W,H, etc.) or massive colored particles.

We start with identifying each jet with a single parton; this defines the leading order approximation. We improve on it by adding both elastic (loops) and inelastic (additional gluons) corrections to the leading order approximation.

The need to combine elastic and inelastic contributions is related to the absence of mass gap in pQCD and the ensuing infra-red and collinear divergences in processes with fixed parton multiplicities. Kinoshita-Lee-Nauenberg theorem ensures their cancellation for properly-defined observables (infra-red and collinear safety).



#### Next-to-next-to-leading order computations

There are three contributions to NNLO cross-sections that differ by the number of partons in the final state; it is customary to refer to these contributions as double-virtual, real-virtual and double-real.

Since ``parton multiplicity" is not an infra-red safe concept, these three contributions are unphysical, when taken separately. For physical results defined in terms of, say, energy flows, the three contributions need to be combined since higher-multiplicity contributions become indistinguishable from lower-multiplicity contributions if additional partons become either soft or collinear or both. Moreover, when this happens, higher-multiplicity matrix elements become singular and can not be integrated over the phase-space necessitating introduction of dimensional ( or other) regularization to isolate the divergencies.



A necessary ingredient of any NNLO computation are the two-loop virtual corrections. As with any virtual corrections computation, the complexity increases with larger number of external particles and with larger number of kinematic invariants (masses included).

The technology that is currently used for these computations involves three steps:

- a) diagrammatic analysis;
- b) reduction of integrals using integration-by-parts identities;
- c) computation of master integrals.

To give you an idea of how these computations are done, I will consider the production of a pair of W-bosons (on- or off-shell) in the collisions of an up quark and and up antiquark

$$\mathcal{M} = \mathcal{A}_{\mu\nu}(p_1, p_2, p_3, p_4)\epsilon_3^{\mu}\epsilon_4^{\mu}$$
$$A_{\mu\nu}^{(a)} = \mathcal{A}_{\mu\nu}^{(a)} + n_g \mathcal{A}_{\mu\nu}^{(b)} + A_{\mu\nu}^{(z)}$$



We will focus on the first term that describes coupling of W-bosons to continuous (up-down-up) quark line.

The problem with two-loop computations is that no algebraic framework exists for expressing tensor integrals through Lorentz scalar integrals. This is in variance with the Passarino-Veltman procedure at one loop.

At two-loops a similar task is accomplished by the integration-by-parts technique which requires that integrands contain enough ``propagators'' to express any scalar product of loop momenta, and any scalar product of any loop momenta and any external vector through them.

All Feynman diagrams should be written as linear combinations of these and similar integrals; this can be done but this is not the form in which Feynman integrals appear in Feynman diagrams.

To put Feynman diagrams into a required form, we need to do some algebra. For example, we can write an amplitude in terms of form factors and construct projection operators. This procedure is straightforward but it becomes increasingly cumbersome for larger multiplicities.

We can express the amplitude in a compact form using spinor-helicity notations

$$\mathcal{M}^{(a)} = -F_1 \langle 57 \rangle [86] \langle 2\hat{3}1] + F_2 \langle 15 \rangle \langle 17 \rangle [16] [18] \langle 2\hat{3}1] + F_3 \langle 15 \rangle \langle 27 \rangle [16] [28] \langle 2\hat{3}1] \\ + F_5 \langle 17 \rangle \langle 25 \rangle [18] [26] \langle 2\hat{3}1] + F_6 \langle 25 \rangle \langle 27 \rangle [26] [28] \langle 2\hat{3}1] + F_{14} \langle 15 \rangle \langle 27 \rangle [16] [18] \\ + F_{11} \langle 25 \rangle \langle 17 \rangle [16] [18] + F_{12} \langle 25 \rangle \langle 27 \rangle [16] [28] + F_{15} \langle 25 \rangle \langle 27 \rangle [26] [18]$$

$$F_{1} = -2T_{1}, \quad F_{2} = T_{2} - \alpha_{3}\alpha_{4}T_{10} - \alpha_{3}T_{8} + \alpha_{4}T_{4},$$

$$F_{3} = T_{3} - \frac{4T_{17}}{s} - \alpha_{3}\beta_{4}T_{10} - \alpha - 3T_{9} + \beta_{4}T_{4}$$

$$\cdots$$

$$F_{15} = 2T_{15} - 2\beta_{3}T_{16}$$

To compute the relevant form factors we construct projection operators

$$\mathcal{A}_{\mu\nu} = \bar{v}_{p_2} \hat{\Gamma}_{\mu\nu} u_{p_1}. \qquad \sum \mathcal{A}_{\mu\nu} \times \bar{u}_{p_1} \hat{O} v_{p_2} = \operatorname{Tr} \left[ \hat{p}_2 \Gamma_{\mu\nu} \hat{p}_1 \hat{O} \right]$$

$$G_{1} = -\frac{\operatorname{Tr}\left[\hat{p}_{2}\Gamma_{\mu\nu}\hat{p}_{1}\hat{p}_{\perp}\right]}{4p_{\perp}^{2}(p_{1}\cdot p_{2})^{3}} \times p_{1}^{\mu}p_{1}^{\nu}, \quad G_{1} = T_{6}.$$
$$G_{2} = -\frac{\operatorname{Tr}\left[\hat{p}_{2}\Gamma_{\mu\nu}\hat{p}_{1}\hat{p}_{\perp}\right]}{4p_{\perp}^{2}(p_{1}\cdot p_{2})^{3}} \times p_{2}^{\mu}p_{2}^{\nu}, \quad G_{2} = T_{2}.$$

$$G_{17} = -\frac{\operatorname{Tr}\left[\hat{p}_{2}\Gamma_{\mu\nu}\hat{p}_{1}\left(\gamma^{\nu}\hat{p}_{\perp}\gamma^{\mu}-\mu\leftrightarrow\nu\right)\right]}{8p_{\perp}^{2}(p_{1}p_{2})}, \quad G_{17} = -(2d^{2}-14d+20)T_{17}+(p_{1}p_{2})T_{5}-(p_{1}p_{2})T_{3}.$$

$$T_1 = \frac{G_{10} - G_9 - G_4 - G_3}{d - 3}, \quad T_2 = G_2, \quad \dots, \quad T_{17} = -\frac{G_4 - G_3 + G_{17}}{2(d - 3)(d - 4)}.$$

Finally, we can combine equations for T's to obtain the form factors F; these form factors are expressed in terms of two-loop four-point integrals of the type shown earlier These integrals satisfy many linear equations that originate from integration-by-parts identities that allow one to map all the integrals that are needed on a small set of integrals called ``master integrals''.

Procedures for the reduction of two-loop integrals to master integrals are non-trivial. Large number of interesting ideas that facilitate this process appeared in the past O(10) years. In addition, large effort went into an automation of integration-by-parts (IBP) procedure; now public programs (FIRE, REDUZE) and their more powerful private versions exist. The complexity of the IBP process increases with the number of kinematic invariants and masses that are present in the problem. At the moment, 2->2 processes can be dealt with; anything beyond that has never been tried.

Calculations of master integrals is a much less straightforward procedure; it was traditionally done on a case-by-case basis. An interesting recent development is related to Henn's conjecture that postulates that it is always possible to choose a set of master integrals in such a way that they satisfy differential equations of the following type

$$\partial_x \vec{f} = \epsilon \hat{A}_x(x, y, z...) \vec{f}$$

The important point is that on the right-hand side, the dimensional regularization parameter appears explicitly, and only as a multiplicative pre-factor. It is then possible to solve these equations iteratively order by order in (d-4).

While differential equations were used to find master integrals for a long time starting from papers by Kotikov and Remiddi in early 1990s, the idea by Henn streamlines and simplifies such computations significantly. This already lead to very impressive advances (e.g. master integrals for Bhabha, VI V2 production) that may have interesting consequences for phenomenology.

To summarize the situation with the two-loop virtual corrections, let me say that

I) they are needed since they are always part of any NNLO computation;

2) they can be computed in many ways (direct Feynman parameter integration, numerics, Mellin-Barnes, differential equations) but their computation is always difficult;

3) recent advances seem to streamline computations of master integrals so that one can expect significant progress in computing two-loop virtual corrections to various 2 -> 2 processes;

4) larger number of kinematic invariants (multi-leg, masses etc). makes such computations increasingly complicated and, at the moment, we do not know if two-loop computations for 2->2 amplitudes with large number of kinematic invariants or 2->3 processes are feasible within this framework;

5) There are interesting attempts to understand if two-loop computations can be done using unitarity techniques that turned out to be so powerful at one-loop. While there was an impressive progress in this field related to classification of integrand residuals based on some algebraic geometry concepts, there are still many outstanding issues. Currently, the main problem there seems to be the lack of understanding of how to avoid the use of integration-by-parts.

For the purpose of the following discussion, we will assume that the relevant two-loop computation exists. If so, we will find the result whose essential features are easily expressed through Catani's formula.

$$\mathcal{M} = \left(\frac{\alpha_s}{2\pi}\right)^r \left(\mathcal{M}^{(0)} + \frac{\alpha_s}{2\pi}\mathcal{M}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathcal{M}^{(2)}\right)$$
$$\mathcal{M}^{(1)} = \hat{I}_1 \mathcal{M}^{(0)} + \mathcal{M}^{(1)}_{\text{fin}} \qquad \mathcal{M}^{(2)} = \hat{I}_2 \mathcal{M}^{(0)} + \hat{I}_1 \mathcal{M}^{(1)} + \mathcal{M}^{(2)}_{\text{fin}}$$
$$\hat{I}_1 = \frac{1}{2} \frac{e^{\epsilon \gamma_E}}{\Gamma(1-\epsilon)} \sum_i \frac{V_i}{\vec{T}_i^2} \sum_{j \neq i} \vec{T}_i \cdot \vec{T}_j \left(\frac{\mu^2}{-2p_i \cdot p_j - i0}\right)^\epsilon \qquad V_i = \frac{\vec{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}.$$
$$\hat{I}_2 = -\frac{1}{2} \hat{I}_1 \left(\hat{I}_1 + 4\pi\beta_0 \frac{1}{\epsilon}\right) + \frac{e^{-\epsilon \gamma_E} \Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{2\pi\beta_0}{\epsilon} + K\right) \hat{I}^{(1)}_{\epsilon \to 2\epsilon} + \frac{H^{(2)}}{\epsilon}.$$

To compute cross-sections, we need to take the d -> 4 limit; to cancel singularities that appear in the above formula for the virtual corrections we need to consider inelastic processes where additional partons are emitted into the final state.

#### Differences between virtual and real

Explicit (d-4) poles in virtual corrections should be contrasted with higher-multiplicity partonic processes where such poles are implicit.

Indeed, consider as an example, the computation of NNLO QCD corrections to the production of the Higgs boson in association with a jet. The leading order is computed using the matrix element for 0->gggH. The next-to-next-to-leading order correction requires matrix element 0->gggggH, that needs to be integrated over the phase-space of the Higgs boson and three final-state gluons.

$$\sigma \sim \int \mathrm{dLips}_{gg \to Hggg} |\mathcal{M}(0 \to ggggH)|^2 \mathcal{F}_J.$$

However, the integrand of the above expression does not contain any poles in the dimensional regularization parameter. Indeed, depending on the measurement function, it can describe production of the Higgs boson in association with 3 jets at tree-level ( in which case everything is finite); production of the Higgs boson in association with 2 jets at next-to-leading order ( in which case we should produce at most two poles in (d-4) upon integration over the phase-space) or production of the Higgs boson in association with 1 jet at NNLO ( in which case four poles in (d-4) should be produced to cancel singularities of the elastic cross-section.

#### Differences between virtual and real

A properly-organized NNLO computation should be able to describe all of the above processes (Born, Born + jet, Born + 2 jets) and it should be able to describe all of them at a fully differential level.

Hence, we arrive at a somewhat paradoxical situation where , on one hand, the need to keep everything at a fully-differential level prevents us from integrating over phase-spaces for processes with additional emissions and, on the other hand, the need to explicitly extract singularities in (d-4) forces us to perform such integration.

 $\int \frac{\mathrm{d}x}{x^{1+n\epsilon}} F(x)$ 

It is straightforward to understand how one can resolve this issue, as a matter of principle. Suppose we managed to reduce the problem to an integral of the type

Assuming that the function F(x) is non-singular at x = 0, we can use the expansion in plus-distributions to extract singularities without integration.

$$\frac{1}{x^{1+n\epsilon}} = -\frac{1}{n\epsilon}\delta(x) + \left[\frac{1}{x}\right]_{+} + \dots$$

We can apply this general idea to real-emission singularities that need to be dealt with in pQCD computations. We will first discuss the NLO since all the main steps can be illustrated already at that level.

Consider the production of the Higgs boson in association with a jet as an example. I will look at the gluon channel only; all quark-initiated contributions will be ignored.

$$\sigma \sim \int dLips_{g_1+g_2 \to H+g_3+g_4} |\mathcal{M}(g_1+g_2 \to H+g_3+g_4)|^2 F_J(g_1,g_2,g_3,g_4)$$

The immediate problem that we face with the above integral is that singularities can appear in different ways. Indeed, they appear when either g3 or g4 become soft or when either g3 or g4 become collinear to either g1 or g2 or to each other.

As the first step we would like to separate the different singular regions in the phasespace. To do this, we introduce partition of unity.







It remains to isolate collinear singularities associated with gluon g4. In this case, the phase-space partitioning involves scalar products of unit vectors that define directions of `` hard'' gluons g1, g2 and g3 and the unit vector that defines direction of g4. We again introduce partition of unity, this time using the relative angles of gluons.

$$1 = \Delta_{\theta}^{(41)} + \Delta_{\theta}^{(42)} + \Delta_{\theta}^{(43)}. \qquad \Delta_{\theta}^{(4i)} = \frac{\rho^{j4}\rho^{k4}}{\rho^{13}\rho^{24} + \rho^{14}\rho^{34} + \rho^{24}\rho^{34}}$$
$$\rho^{ij} = 1 - \vec{n}_j \cdot \vec{n}_j = 1 - \cos\theta_{ij}$$
$$\frac{1}{2!} dLips_{g_1g_2 \to Hg_3g_4} |\mathcal{M}|^2 = \sum_{\alpha=1}^3 dLips_{g_1g_2 \to Hg_3g_4} \Delta_{p_{\perp}}^{(4)} |\mathcal{M}|^2 \Delta_{\theta}^{4\alpha}$$



The above formula defines three sectors for the phase-space partitioning. In each of these sectors, kinematic configurations that lead to singularities are unique. Indeed, in sector I, gluon g4 can be soft and/or collinear to incoming g1; in sector 2, gluon g4 can be soft and/or collinear to incoming g2; in sector 3, gluon g4 can be soft and/or collinear to outgoing g3.

To extract singularities, in each of these sectors, we choose the reference frame which is most appropriate for this purpose, i.e. the z-axis is aligned with g1 in the 1st sector, with g2 in the 2nd and with g3 in the 3rd.

In more detail, phase-space is parametrized by splitting it into regular and singular parts.

$$\mathrm{dLips}_{g_1g_2 \to Hg_3g_4} \Delta_{p_\perp}^{(4)} \Delta_{\theta}^{4\alpha} = \Delta_{p_\perp}^{(4)} \Delta_{\theta}^{4\alpha} \mathrm{dLips}_{Q(12) \to 3H} \times [\mathrm{d}g_4]^{(4\alpha)}.$$

$$dLips_{Q(12)\to g_3H} = \frac{dx_4 dx_5}{8\pi} \frac{2E_{g_3}}{Q_0 - \vec{Q} \cdot n_3} \left(\frac{E_{g_3}^2 \sin^2 \theta_3}{p_{\perp,H}^2}\right)^{-\epsilon}$$

 $Q = p_1 + p_2 - p_4, \quad p_3 = E_{g_3} (1, \vec{n}_3) \quad E_{g_3} = \frac{Q^2 - m_H^2}{2(Q_0 - \vec{Q} \cdot \vec{n}_3)}, \quad \vec{n}_3 = (\sin \theta_3 \cos \phi_3, \sin \theta_3 \sin \phi_3, \cos \theta_3)$ 

$$\cos\theta_3 = 1 - 2x_4, \quad \phi_3 = 2\pi x_5$$

Parametrization of the singular phase-space depends on the sector; as we explained earlier, choice of the z-axis is important. Consider for definiteness the first sector.

$$p_4 = E_{g_4} \left( 1, \sin \theta_4 \cos \phi_4, \sin \theta_4 \sin \phi_4, \cos \theta_4 \right) \qquad p_1 \cdot p_4 \sim \left( 1 - \cos \theta_4 \right) \Rightarrow \cos \theta_4 = 1 - 2x_2$$
$$E_{g_4} = E_{\max} x_1 \qquad \qquad d \operatorname{Lips}_{12 \to 34H}^{(41)} = \operatorname{Norm} \times \operatorname{PS}_{w} \times \operatorname{PS}^{-\epsilon} \frac{d x_1 d x_2 d x_3 d x_4 d x_5}{x_1^{1+2\epsilon} x_2^{1+\epsilon}} \times [x_1^2 x_2].$$

Expression for the phase-space is such that all the implicit factors there, are non-singular in the limit  $x I \rightarrow 0$  (gluon g4 soft) or  $x 2 \rightarrow 0$  (gluon g4 collinear g1) -- the only two singular kinematic configurations that are relevant for sector 1.

We then need to perform an integration over the phase-space of Sector 1.

$$dLips_{12\to34H}^{(41)} = Norm \times PS_{w} \times PS^{-\epsilon} \frac{dx_{1}dx_{2}dx_{3}dx_{4}dx_{5}}{x_{1}^{1+2\epsilon}x_{2}^{1+\epsilon}} \times [x_{1}^{2}x_{2}].$$
  
$$\sigma \sim \int dLips_{12\to34H}^{(41)} |\mathcal{M}|^{2} \Rightarrow \int_{0}^{1} \frac{dx_{1}}{x_{1}^{1+2\epsilon}} \frac{dx_{2}}{x_{2}^{1+\epsilon}} F(x_{1}, x_{2}), \quad F(x_{1}, x_{2}) = [x_{1}^{2}x_{2}]|\mathcal{M}|^{2}$$

We can easily extract singularities from this expression provided that the function F(x|,x2) is finite in the limit  $x| \rightarrow 0$  and  $x2 \rightarrow 0$ . Is this so?

Recall that the physical meaning of x1 > 0 is that the gluon g4 becomes soft and the physical meaning of x2 > 0 is that the gluon g4 becomes collinear to gluon g1. We can use universal factorization properties of scattering amplitudes to show that the function F(x1,x2) is finite in both of these limits. We will do this on the next slide.

For now, let us assume that the function F(x I, x2) is indeed smooth at the origin. If so, the divergencies can be easily extracted using the plus-distribution prescription that we already described.

$$\int_{0}^{1} \frac{\mathrm{d}x_{1}}{x_{1}^{1+2\epsilon}} \frac{\mathrm{d}x_{2}}{x_{2}^{1+\epsilon}} F(x_{1}, x_{2}) = \int_{0}^{1} \mathrm{d}x_{1} \mathrm{d}x_{2} \left[ \frac{1}{2\epsilon^{2}} F(0, 0) - \frac{1}{2\epsilon} \left( F(0, x_{2}) - F(0, 0) \right) - \frac{1}{\epsilon} \left( F(x_{1}, 0) - F(0, 0) \right) + \dots \right]$$

The last equation has large number of ``subtraction terms'' that are generated automatically; we will show now that all such terms can be obtained from universal limits of scattering amplitudes .

The soft  $\times I \rightarrow 0$  limit of the amplitude gives a factorized expression that consists of eikonal factors and the reduced scattering amplitude

$$\lim_{p_4 \to 0} |\mathcal{M}_H(g_1, g_2, g_3, g_4)|^2 \to C_A \sum_{i \neq j} \frac{p_i p_j}{(p_i p_4)(p_j p_4)} \times |\mathcal{M}_H(g_1, g_2, g_3)|^2.$$
$$\lim_{x_1 \to 0} x_1^2 \times \sum_{ij} \frac{p_i p_j}{(p_i p_4)(p_j p_4)} \to \sum_{i \neq j} \frac{1 - \vec{n}_i \vec{n}_j}{(1 - \vec{n}_i n_4)(1 - \vec{n}_j \vec{n}_4)}. \qquad \Delta_{\theta}^{(4i)} = \frac{\rho^{j4} \rho^{k4}}{\rho^{13} \rho^{24} + \rho^{14} \rho^{34} + \rho^{24} \rho^{34}}.$$

The apparent collinear singularities in the last expression are removed by the damping factors that we introduced when phase-space partitioning was discussed.

The  $x^2 \rightarrow 0$  limit is also straightforward. Indeed,  $x^2 \rightarrow 0$  corresponds to the collinear singularity of the matrix element. Therefore, the limiting behavior of the amplitude is given by the gluon splitting function (note spin correlations).

$$|\mathcal{M}_{H}(g_{1},g_{2},g_{3},g_{4})|^{2} \to \frac{C_{A}g_{s}^{2}}{s_{14}}P_{\mu\nu}(z)\mathcal{M}_{H}^{\mu}(g_{1},g_{2},g_{3})\mathcal{M}_{H}^{\nu,*}(g_{1},g_{2},g_{3}) \qquad 1-\cos\theta_{14} = 2x_{2}$$
$$P_{gg}^{\mu\nu} = 2C_{A}\left[-g^{\mu\nu}\left(\frac{z}{1-z} + \frac{1-z}{z}\right) - 2(1-\epsilon)z(1-z)\frac{k_{\perp}^{\mu}k_{\perp}^{\nu}}{k_{\perp}^{2}}\right] \qquad \frac{x_{2}}{s_{14}} = \frac{1}{4E_{g_{1}}E_{g_{4}}}.$$

The important point is that this factorization does not depend on the details of the process; it is universal. Hence, the relevant limits can be computed once and for all.

I) The above discussion applies to any NLO computation since generalization of phase-space partitioning to higher-multiplicity final states is straightforward.

2) The building blocks of this method are

- a) phase-space partitioning,
- b) scattering amplitudes,
- c) universal soft-collinear limits
- d) virtual corrections.

3) Symmetries are very helpful in reducing the number of independent sectors that one has to deal with.

4) In the NLO community, the procedure that I just described is known as Frixione-Kunszt-Signer (FKS) method that was suggested about 20 years ago and then neglected for about 10 years in favor of Catani-Seymour dipole subtraction method.

5) The FKS method made an impressive comeback and it is used in POWHEG and MadLoop interfaces of fixed order computations and parton showers.

6) Given that the FKS method was always much less popular for NLO computations than the CS method, it is ironic that generalization of the FKS method to NNLO happened in a seamless fashion.

#### Next-to-next-to-leading order (FKS@NNLO)

If we understand how NLO computations can be done within this framework, it is also straightforward to understand extensions of the method to NNLO. For definiteness, I will consider Higgs boson production in an association with a jet, gluons only.

Consider the following process g(1) + g(2) -> H + g(3) + g(4) + g(5). To partition the phase-space, we start by ordering gluons in their hardness using symmetries of phase-space matrix elements and measurement functions. Similar to NLO, we can use transverse momenta of final state gluons as a measure of hardness and introduce partition of unity



After the ordering, gluon g(3) becomes a designated jet and gluons g(4) and g(5) can be soft or collinear to g(3) or to the incoming beam direction.

The next step is to perform collinear partitioning. Having identified a ``jet", we obtain three collinear directions, that correspond to gluons g(1), g(2) and g(3). The gluons g(4) and g(5) can develop collinear singularities if their momenta are aligned with one of the three collinear directions or if momentum of g(4) is collinear to g(5).

$$\begin{split} \rho^{ij} &= 1 - \vec{n}_i \cdot \vec{n}_j \qquad d_{i \in [4,5]} = \sum_{j=1}^3 \rho_{ij}, \quad d_{i \in [4,5]k} = \sum_{j=1, j \neq k}^3 \rho_{ij}, \quad d_{45ij} = \rho_{45} + \rho_{4i} + \rho_{5j}. \\ w_{4i;5j}|_{i \neq j} &= \frac{\rho_{4k}\rho_{4n}\rho_{5l}\rho_{5m}}{d_4d_5} \left(\frac{1}{d_{4k}} + \frac{1}{d_{4n}}\right) \left(\frac{1}{d_{5l}} + \frac{1}{d_{5m}}\right) \frac{\rho_{45}}{d_{45ij}}. \\ w_{4i;5j}|_{i=j} &= \frac{\rho_{4k}\rho_{4n}\rho_{5k}\rho_{5n}}{d_4d_5} \left[ \left(\frac{1}{d_{4k}} + \frac{1}{d_{4n}}\right) \left(\frac{1}{d_{5k}} + \frac{1}{d_{5n}}\right) \\ &+ \left(\frac{1}{d_{4i}} + \frac{1}{d_{4k}}\right) \left(\frac{1}{d_{5k}} + \frac{1}{d_{5n}}\right) \frac{\rho_{4i}}{d_{45ni}} + \left(\frac{1}{d_{4i}} + \frac{1}{d_{4n}}\right) \left(\frac{1}{d_{5k}} + \frac{1}{d_{5n}}\right) \frac{\rho_{4i}}{d_{45ki}} \\ &+ \left(\frac{1}{d_{4k}} + \frac{1}{d_{4n}}\right) \left(\frac{1}{d_{5i}} + \frac{1}{d_{5k}}\right) \frac{\rho_{5i}}{d_{45in}} + \left(\frac{1}{d_{4k}} + \frac{1}{d_{4n}}\right) \left(\frac{1}{d_{5i}} + \frac{1}{d_{5n}}\right) \frac{\rho_{5i}}{d_{45ik}} \right], \end{split}$$

As the result of this procedure, we have nine sectors to consider. In each of the sectors, the structure of singularities is clearly exposed.

$$\frac{1}{3!} dLips_{12 \to H345} = \sum_{\alpha \in S} dLips_{12 \to H345}^{(\alpha)}, \quad dLips_{12 \to H345}^{(\alpha)} = dLips_{12 \to H345} \Delta_{p_{\perp}}^{(45)} \theta(E_{g_4} - E_{g_5}) w_{\alpha}$$
$$S = [(41, 51), (42, 52), (43, 53), (41, 52), (42, 51), (41, 53), (43, 51), (42, 53), (43, 52)].$$

In each of the nine sectors, singularities can appear if gluons g(4) and g(5) become soft and/or if they become collinear to a well-defined direction(s). In each sector we choose the phase-space parametrization that reflects this fact.

Consider the sector S(14,15) (``triple collinear initial-initial, singularities appear when g4||g1 and g5|| g1"). Choose the momentum parametrization to make collinear singularities simple and find the ensuing parametrization of the ``unresolved" part of the phase-space.

$$p_{1,2} = \frac{\sqrt{s}}{2} (1,0,0,\pm 1;0) ,$$

$$p_3 = E_{g_3} (1,\sin\theta_3\cos\tilde{\varphi}_3,\sin\theta_3\sin\tilde{\varphi}_3,\cos\theta_3;0) ,$$

$$p_4 = E_{g_4} (1,\sin\theta_4,0,\cos\theta_4;0) ,$$

$$p_5 = E_{g_5} (1,\sin\theta_5\cos\varphi_5,\sin\theta_5\sin\varphi_5\cos\alpha,\cos\theta_5;\sin\theta_5\sin\varphi_5\sin\alpha) .$$

$$[dg_4][dg_5]\theta(E_{g_4} - E_{g_5}) = \frac{d\Omega^{(d-3)}d\Omega^{(d-4)}}{2^{4+2\epsilon}(2\pi)^{2d-2}}d\varphi_4 \left[\sin^2(\varphi_4 - \varphi_3)\right]^{-\epsilon}d\cos\alpha \left[\sin^2\alpha\right]^{-1-\epsilon}$$

$$\times [\xi_1\xi_2]^{1-2\epsilon} [\eta_4(1-\eta_4)]^{-\epsilon} [\eta_5(1-\eta_5)]^{-\epsilon} [\lambda(1-\lambda)]^{-1/2-\epsilon} (\eta_4 - \eta_5|^{1-2\epsilon} ) D^{1-2\epsilon} ) D^{1-2\epsilon} ,$$

$$(2E_{\max})^{4-4\epsilon} \theta(\xi_1 - \xi_2) \theta(\xi_{\max} - \xi_2) d\xi_1 d\xi_2 d\eta_4 d\eta_5 d\lambda .$$

$$F_{g_4,g_5} = E_{\max}\xi_{1,2}, \quad \xi_{\max} = \min\left[1, \frac{1-\xi_1}{1-(1-m_h^2/s)\xi_1\eta_{45}}\right], \quad Czakon$$

$$\eta_{45} = \frac{|\eta_4 - \eta_5|^2}{D}, \quad \sin^2\varphi_5 = 4\lambda(1-\lambda)\frac{|\eta_4 - \eta_5|^2}{D^2}, \quad \eta_{4,5} = 1 - \vec{n}_1 \cdot \vec{n}_{4,5}$$

$$D = \eta_4 + \eta_5 - 2\eta_4\eta_5 + 2(2\lambda - 1)\sqrt{\eta_4\eta_5(1-\eta_4)(1-\eta_5)}.$$

All singularities are extracted if each sector is further split into five sectors using appropriate changes of variables. The required changes are shown below.

$$dLips^{(41;51)} = \sum_{i=1}^{5} dLips^{(41;51,i)}.$$
  

$$Sc^{(41;51,1)} : \xi_{1} = x_{1}, \xi_{2} = x_{1}x_{\max}x_{2}, \eta_{4} = x_{3}, \eta_{5} = \frac{x_{3}x_{4}}{2},$$
  

$$Sc^{(41;51,2)} : \xi_{1} = x_{1}, \xi_{2} = x_{1}x_{\max}x_{2}, \eta_{4} = x_{3}, \eta_{5} = x_{3}\left(1 - \frac{x_{4}}{2}\right),$$
  

$$Sc^{(41;51,3)} : \xi_{1} = x_{1}, \xi_{2} = x_{1}x_{\max}x_{2}x_{4}, \eta_{4} = \frac{x_{3}x_{4}}{2}, \eta_{5} = x_{3},$$
  

$$Sc^{(41;51,4)} : \xi_{1} = x_{1}, \xi_{2} = x_{1}x_{\max}x_{2}, \eta_{4} = \frac{x_{3}x_{4}x_{2}}{2}, \eta_{5} = x_{3},$$
  

$$Sc^{(41;51,5)} : \xi_{1} = x_{1}, \xi_{2} = x_{1}x_{\max}x_{2}, \eta_{4} = x_{3}\left(1 - \frac{x_{4}}{2}\right), \eta_{5} = x_{3}.$$

With these changes, the phase-space for each sector becomes

$$dLips_{41;51}^{(i)} \sim Norm \times PS_{w,i}PS_i^{-\epsilon} \times \frac{(-\epsilon)}{x_9^{1+\epsilon}} \prod_{k=5}^9 dx_k \times \prod_{j=1}^4 \frac{dx_j}{x_j^{1+a_j^{(i)}\epsilon}} \times \left[ x_1^{b_1^{(i)}} x_2^{b_2^{(i)}} x_3^{b_3^{(i)}} x_4^{b_4^{(i)}} \right]$$

It is apparent that the form of the phase-space is similar to what we had to deal with at next-to-leading order except that the number of singular integrations is larger.

$$\int_{0}^{1} \frac{\mathrm{d}x_{1}}{x_{1}^{1+a_{1}\epsilon}} \frac{\mathrm{d}x_{2}}{x_{2}^{1+a_{2}\epsilon}} \dots F^{(i)}(x_{1}, x_{2}, \dots) \qquad F^{(i)}(x_{1}, x_{2}, \dots) = \left[x_{1}^{b_{1}} x_{2}^{b_{2}} x_{3}^{b_{3}} x_{4}^{b_{4}}\right] |\mathcal{M}_{gg \to Hggg}|^{2}.$$

Nevertheless, expanding the above expression in plus-distributions produces integrable expressions provided that F(0,0,....) can be calculated and that it is free of singularities.

It is easy to see that also for the function F(0,0,...) the one-loop story essentially repeats itself. Indeed, all the subtraction terms required for the integration can be obtained without a reference to the underlying process thanks to the universality of soft and collinear limits.

If any of the four x-variables (or their combinations) vanishes we obtain a kinematic configuration where gluons g(4) and g(5) are either soft or collinear or both. All singular limits are known since circa 2000 and can be directly borrowed from the relevant papers. Although we only discussed the tree-level case, the same applies to one-loop real-virtual NNLO contributions.



Collinear factorization at one-loop (Kosower, Uwer)

Related work on singular limits by Campbell, Glover, Berends, Giele, Bern, Del Duca, Kilgore, Schmidt

#### A glimpse into technical details: D-dim angles

One interesting aspect of this construction is that integrations over unresolved phase-spaces should involve gluons with momenta extended to d-dimensions. The need for this can be understood in many ways including the spin correlations in the collinear limit as shown below.



In next-to-leading order computations, integration over unresolved phase-space is performed analytically; hence the discussion of d-dimensional momenta and their numerical implementation never appears.

In our approach to NNLO, all integrations (resolved or unresolved) are done numerically; therefore appropriate construction for (d-4) components of unresolved momenta is required.

# Will (d-4) components of parton momenta appear in observables and scattering amplitudes?

The answer is that they should not. To have this properly implemented in the computation, suppose we always choose the softest gluon g5 to have (d-4) components. Then, in all but one singular configurations, gluon with (d-4) dimensional momentum decouples from the hard matrix element. The only configuration where this does not happen is when the harder gluon g4 is collinear to one of the hard gluons. But if this happens, one independent spatial direction disappears (e.g. g3||g4) and one can rotate all momenta removing the (d-4) directional components of g5 from the matrix element.

D-dim momenta always decouple from reduced matrix elements and observables, at any stage of the computation. Their presence is essential in eikonal and splitting functions and the unresolved phase-space. At NLO, these terms are integrated out analytically, so the presence of such terms is usually not emphasized. But for NNLO numerical integrations over unresolved phase-spaces their presence is important.

#### The NNLO technology

The above discussion summarizes main ideas behind the recent development of techniques for NNLO QCD computations that combine sector decomposition and phase-space partitioning. Since there is no time for details, let me add a few things:

I) Within this framework, the necessary subtraction terms are generated locally and automatically; similar to the original FKS, the new framework is very robust.

2) All of the subtraction terms are related to universal limits of scattering amplitudes making the whole procedure scalable in the right way (need no diagrams, need amplitudes, all limits are hard-coded once and for all);

- 3) Can work with helicity states for external resolved particles;
- 4) All spin-correlations in amplitudes are subtracted locally;
- 5) No need for (d-4) terms in amplitudes squared, except in their collinear limits;
- 6) No special treatment of massive particles is required ;
- 7) Decay kinematics can be treated along the same lines;
- 8) Yet another place where fast (e.g. compact) NLO amplitudes for multi-parton processes can be used and, in fact, are essential for the efficiency of NNLO.

#### t-channel single-top production at NNLO

I would like to show you a few examples of how this procedure works. I will start with a t-channel single-top production in hadron collisions.

This process occurs due to an exchange of a W-boson in the t-channel. As the result, there is no color transfer from light-quark line to heavy-quark line at LO and NLO. It appears for the first time at NNLO where it is color-suppressed. We will neglect these contributions in our NNLO computation.



The relevant two-loop amplitudes are shown below; they involve one-loop corrections applied to heavyand light-quark lines separately and the two-loop corrections to either heavy- or light-quark lines. The last diagram is the color-suppressed interference effect and we do not consider it ( color suppression).



#### Ingredients for single-top NNLO computation

I) Two-loop form factors for heavy- (tWb) and light-quark (qWq') weak transitions are needed and they are known. Bonciani, Ferroglia; Bell; Astarian, Greub and Pecjak;

Beneke, Huber and Li; Huber

2) Amplitudes for 0-> tbW(II')gg and 0->tbW(II')qq and 0->qq'W(II)gg etc. Such amplitudes are either available or can be computed in a straightforward way;

3) Collinear limits ( known);

R.K.Ellis and J. Campbell

4) Soft limits for tree-level amplitudes (known) (eikonal factors are slightly more difficult for massive particles).

5) Soft limits for one-loop scattering amplitude that include top quarks are less well-known; they require the soft-current at one loop for the massive fermion.

Bierenbaum, Czakon and Mitov

6) One-loop amplitudes for bW -> t g are known in a compact form and can be borrowed from e.g. MCFM; J. Campbell and F. Tramontano

With these ingredients at place, one needs to perform phase-space partitioning (simple for heavy-quark line since no final state singularities), calculate the relevant limits, remove remaining singularities by performing renormalization (PDFs including). All of this has to be done for a multitude of partonic channels (quark-quark, quark-gluon etc.) -- a bit of a logistic nightmare.

#### t-channel single-top production at NNLO

Since all calculations are done numerically, cancellation of singular contributions to the final result are also not exact. In fact, the degree of cancellation provides a useful check on the correctness of the implementation of various contributions.



#### t-channel single top production at NNLO

We obtain the following results for the cross-sections at leading, next-to-leading and next-to-next-to-leading order in perturbative QCD at 8 TeV LHC.

8 TeV LHC, MSTW2008, m<sub>t</sub> = 173.2 GeV

 $\sigma_{\rm LO} = 53.8^{+3.0}_{-4.3} \text{ pb}$   $\sigma_{\rm NLO} = 55.1^{+1.6}_{-0.9} \text{ pb}$ 

$$\sigma_{\rm NNLO} = 54.2^{+0.5}_{-0.2} \text{ pb}$$

- $\mu_R = \mu_F = \{m_t/2, m_t, 2 m_t\}$
- next-to-leading order corrections at central scale are very small, much smaller than their natural O(10%) size; this is a consequence of significant cancellations between different channels.
- Delicate interplay/cancellations between different channels -> important to consistently compute corrections to all of them;
- The NNLO result is very close to the NLO result (-1.6%), reduced µ dependence -> good theoretical control

#### t-channel single top production at NNLO

$p_{\perp}$	$\sigma_{ m LO},{ m pb}$	$\sigma_{\rm NLO},{\rm pb}$	$\delta_{ m NLO}$	$\sigma_{\rm NNLO},{\rm pb}$	$\delta_{ m NNLO}$
0 GeV	$53.8^{+3.0}_{-4.3}$	$55.1^{+1.6}_{-0.9}$	+2.4%	$54.2^{+0.5}_{-0.2}$	-1.6%
$20 \mathrm{GeV}$	$46.6^{+2.5}_{-3.7}$	$48.9^{+1.2}_{-0.5}$	+4.9%	$48.3^{+0.3}_{-0.02}$	-1.2%
$40 \mathrm{GeV}$	$33.4^{+1.7}_{-2.5}$	$36.5^{+0.6}_{-0.03}$	+9.3%	$36.5^{+0.1}_{+0.1}$	-0.1%
$60 \mathrm{GeV}$	$22.0^{+1.0}_{-1.5}$	$25.0^{+0.2}_{+0.3}$	+13.6%	$25.4_{\pm 0.2}^{-0.1}$	+1.6%



- Contrary to next-to-leading order, NNLO corrections are similarly small for all values of pt;
- Scale dependence typically improves;

#### t-channel single top production at NNLO

8 TeV LHC, MSTW2008,  $m_t = 173.2 \text{ GeV}$ 



The results for the ratio appear very stable, at least for the choice of PDFs indicated above. Note strong PDF dependence -- should eventually give a useful constraint on quark/anti-quark PDF ratios. Note that scale variation errors at LO and NLO are not good indicators of higher orders, as it is often the case with ratios.

#### H+jet at NNLO

5) all soft limits are known, for both tree- and one-loop amplitudes;

6) highly symmetric situation for ``gluons only" contribution; less symmetric (more sectors, more bookeeping) for quark and gluon.





Bougezhal, Caola, K.M., Petriello, Schulze

Fairly large corrections, O(30%) at NNLO

#### H+jet at NNLO

At the same time -- very small scale uncertainty (gluons only). Using existing framework one can -- in principle -- produce kinematic distributions. As an example, the right pane shows a cumulative histogram for Higgs + jet production cross-section as a function of the minimal jet transverse momentum.



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#### Summary

The story of NNLO QCD computations is interesting. Many pieces that need to be known for a NNLO computation have been known for a very long time but we did not know how to put the various pieces together in a consistent way.

The technology for NNLO QCD computations that I described solves this problem. It is based on FKS phase-space partitioning and sector decomposition and allows one to compute NNLO QCD corrections for various 2 -> 2 processes and, perhaps, beyond.

At the very least, the technology is based on proper ingredients -- scattering amplitudes, universal soft and collinear limits etc. and therefore probably scales in an optimal way with increased number of particles.

There are other ideas about how generic NNLO computations should be organized and there are several other techniques currently in the making.

The technology for computing two-loop integrals -- essential ingredients for these computations -- is also rapidly developing and may surpass the 2->2 threshold. An interesting new development here is an attempt to extend unitarity methods to two-loops but it is too early to say how successful these extensions are going to be.

#### What to expect from NNLO in the coming years ?

I) Progress with NNLO computations in the past two years was very impressive. There is no doubt that the new NNLO technology will keep being applied to broader classes of processes. As many of you know, there is currently a NNLO wishlist created as part of the Snowmass community planning exercise in US that happened last year. We can expect that, within a year or two, results for main 2->2 processes from that wishlist will become available.

2) Inclusion of massive particles and their decays in the narrow width approximation will be pursued and should be expected to be straightforward since many existing NNLO computations are amplitudes-based (from this perspective, a decay amplitude for the decay of heavy particle is a particular choice of its polarization vector);

3) Attempts to extend NNLO to 2->3 processes (a few of such processes are included in the wishlist) will require further progress with virtual corrections.

4) Attempts to interface NNLO computations with parton showers or resummations, at least for relatively simple processes. I believe that conceptually this is a straightforward thing to do ( especially since at NLO the FKS-based implementations exist) -- and I am sure we will hear a lot about this topic during the workshop. I think the central question here is whether or not existing NNLO technology can deliver NNLO computations that are sufficiently efficient to be used with parton showers on the fly.