Progress on multiplet bases

- Dealing with exact color summed calculations
- Introduction to color space
- Multiplet bases
- Decomposing Feynman diagrams into multiplet bases
- Gluon emission in multiplet bases, parton showers and recursion
- Gluon exchange in multiplet bases, NLO and resummation
- Conclusions and outlook

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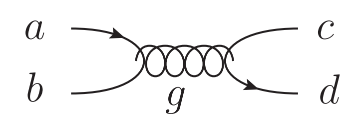
Dealing with color space

- We never observe individual colors
  $\rightarrow$ we are only interested in color summed/averaged quantities
- For given external partons, the color space is a finite dimensional vector space equipped with a scalar product

$$\langle A, B \rangle = \sum_{a,b,c,\ldots} (A_{a,b,c,\ldots})^* B_{a,b,c,\ldots}$$

Example: If

$$A = \sum_g (t^g)^a{}_b (t^g)^c{}_d =$$ 

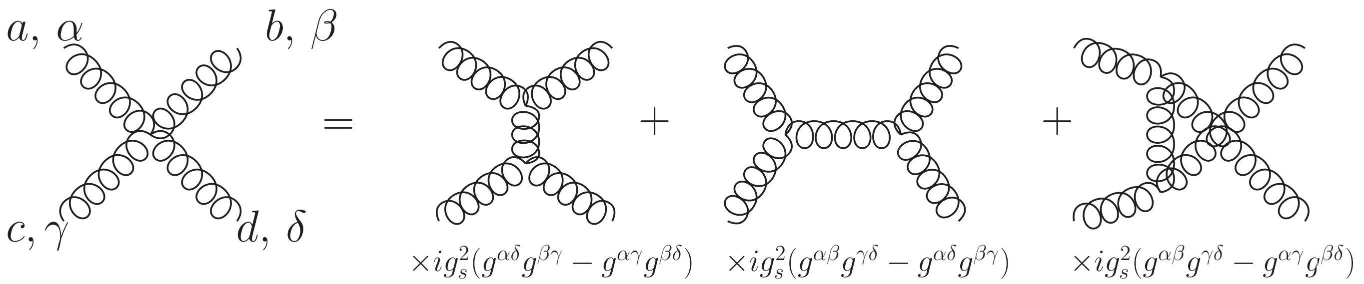
,

then $\langle A|A\rangle = \sum_{a,b,c,d,g,h} (t^h)^b{}_a (t^h)^d{}_c (t^g)^a{}_b (t^g)^c{}_d$

- One way of dealing with color space is to just square the amplitudes one by one as they are encountered
- Alternatively, we may use any basis (spanning set)

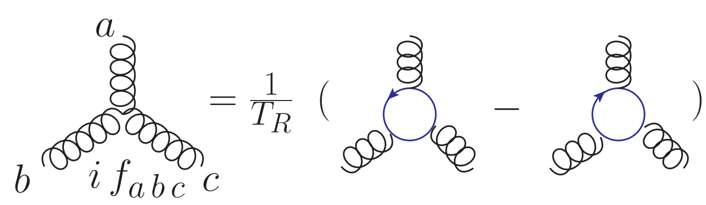
The standard treatment: Trace bases

- Every 4g vertex can be replaced by 3g vertices:

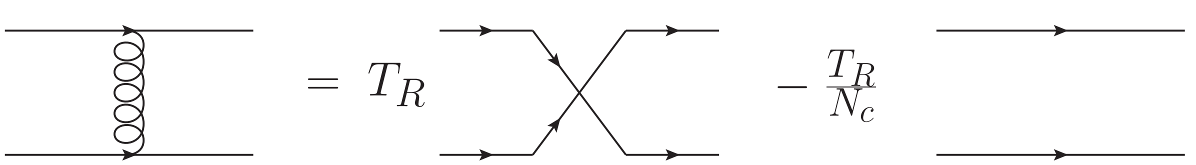


(read counter-clockwise)

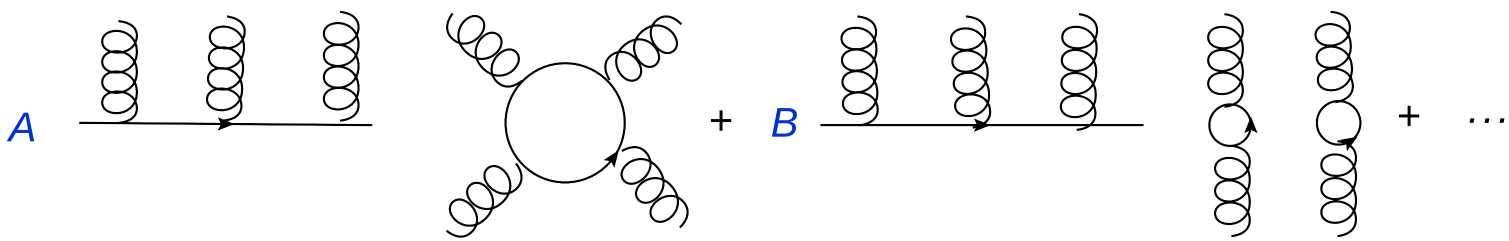
- Every 3g vertex can be replaced using:



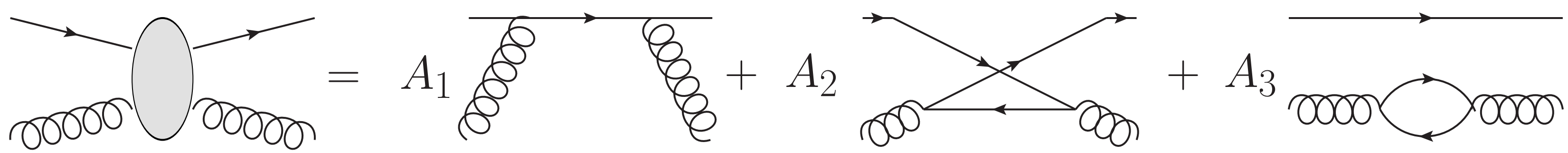
- After this every internal gluon can be removed using:



- This can be applied to any QCD amplitude, tree-level or beyond
- In general an amplitude can be written as linear combination of different color structures, like



- For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair

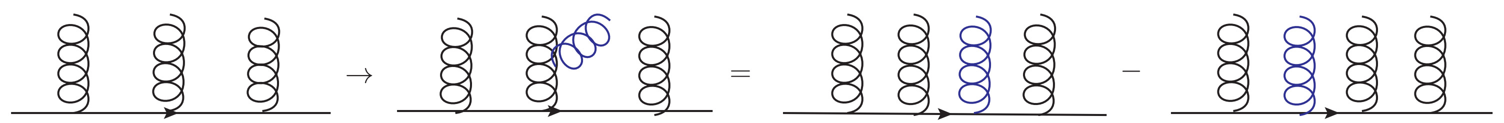


(an incoming quark is the same as an outgoing anti-quark)

The above type of color structures can be used as a spanning set, a trace basis. (Technically it's in general overcomplete, so it is rather a spanning set.)

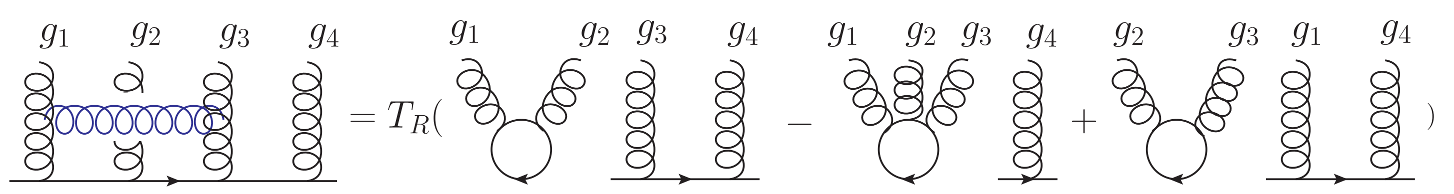
These bases have some nice properties

- The effect of gluon emission is easily described:



Convention: + when inserting after, minus when inserting before.

- So is the effect of gluon exchange:



Convention: + when inserting after, - when inserting before

However...

- The trace "bases" are non-orthogonal and overcomplete
  (for more than N_c gluons plus $q\bar{q}$-pairs)
- ... and the number of spanning vectors grows as a factorial in $N_g + N_{q\bar{q}}$
  $\rightarrow$ when squaring amplitudes we run into a factorial square scaling
- Hard to go beyond ~ 8 gluons plus $q\bar{q}$-pairs

Color flow bases

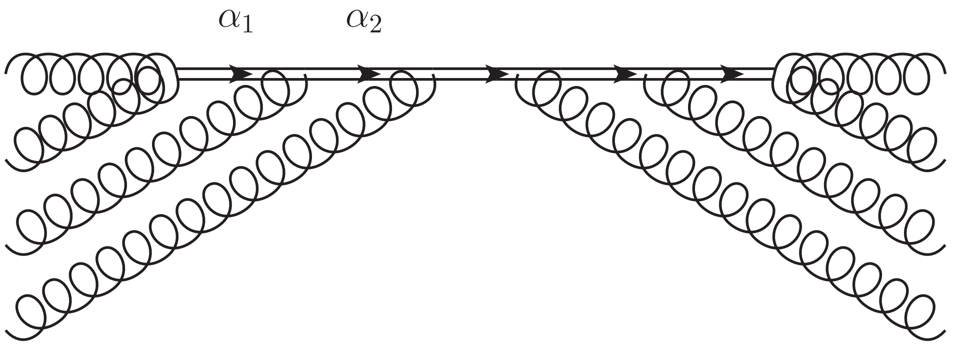
- One way out is to rewrite all vertices in terms of color flows (Maltoni, Stelzer, Willenbrock)
- Explicit colors (r, g, or b) are then assigned to the lines, and one may run a Monte Carlo sum over colors to sample color space
- This is not exact but much quicker

Orthogonal multiplet bases

In collaboration with Stefan Keppeler (Tübingen)

- QCD is based on SU(3) $\rightarrow$ the color space may be decomposed into irreducible representations
- Basis vectors corresponding to irreducible representations may be constructed
- The construction of the corresponding basis vectors is non-trivial, and a general strategy was only presented recently JHEP09(2012)124, arXiv:1207.0609
- With general, I mean general: general number of quarks and gluons, general order in α_s and general N_c
- In this presentation I will – for comparison – talk about processes with gluons only, however, processes with quarks can be treated similarly

- The gluon basis vectors are of form



and can thus be characterized by a chain of representations $\alpha_1, \alpha_2, \ldots$ (In principle we have to differentiate between different vertices as well)

For many partons the size of the vector space is much smaller for $N_c = 3$ (exponential), compared to for $N_c \to \infty$ (factorial)

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$	LO Vectors $N_c \to \infty$
4	8	9	3!=6
5	32	44	4!=24
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880

Number of basis vectors for N_g gluons *without* imposing vectors to appear in charge conjugation invariant combinations

... but the real advantage comes when squaring as the multiplet bases are orthogonal and the trace bases are not

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$	LO Vectors $N_c \to \infty$
4	8	$(9)^2$	$(6)^2$
5	32	$(44)^2$	$(24)^2$
6	145	$(265)^2$	$(120)^2$
7	702	$(1\ 854)^2$	$(720)^2$
8	3 598	$(14\ 833)^2$	$(5\ 040)^2$
9	19 280	$(133\ 496)^2 \sim 10^{10}$	$(40\ 320)^2 \sim 10^{9}$
10	107 160	$(1\ 334\ 961)^2 \sim 10^{12}$	$(362\ 880)^2 \sim 10^{11}$

Number of terms from color when squaring for N_g gluons *without* imposing charge conjugation invariant combinations

- Multiplet bases can potentially speed up exact calculations in color space very significantly, as squaring amplitudes becomes much quicker
- Before squaring, amplitudes must be decomposed in color bases
- How quickly can amplitudes be decomposed in multiplet bases?
- ... using Feynman diagrams?
- ... using parton showers?
- ... using tree-level gluon recursion relations?
- ... at higher order? (gluon exchange)

Decomposing Feynman diagrams

In collaboration with Johan Thorén, work in progress

- One way of decomposing color structure into multiplet bases would be to simply evaluate the scalar product between each possible Feynman diagram and each possible vector
- The problem is that this scales very badly, a factorial from the number of diagrams, an exponential from the number of color structures and another (growing) factor from each single scalar product evaluation

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- The problem is that this scales very badly, a factorial from the number of diagrams, an exponential from the number of color structures and another (growing) factor from each single scalar product evaluation
- $\rightarrow$ no way
- We need a better strategy

- Luckily there is one:

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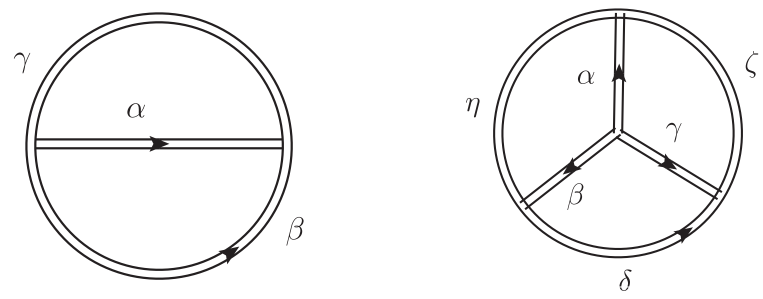
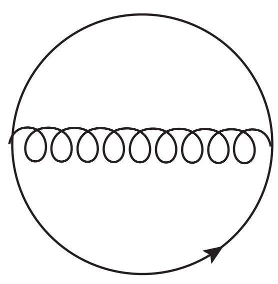
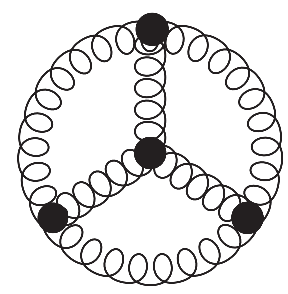


- For example



$=T_R(N_c^2-1)$



$=2\,T_R^2 N_c^2(N_c^2-1)$

Using standard normalization of vertices

- Using the multiplet basis we can evaluate the needed 3j and 6j coefficients for higher representations

- Furthermore, only a small number of such symbols are needed

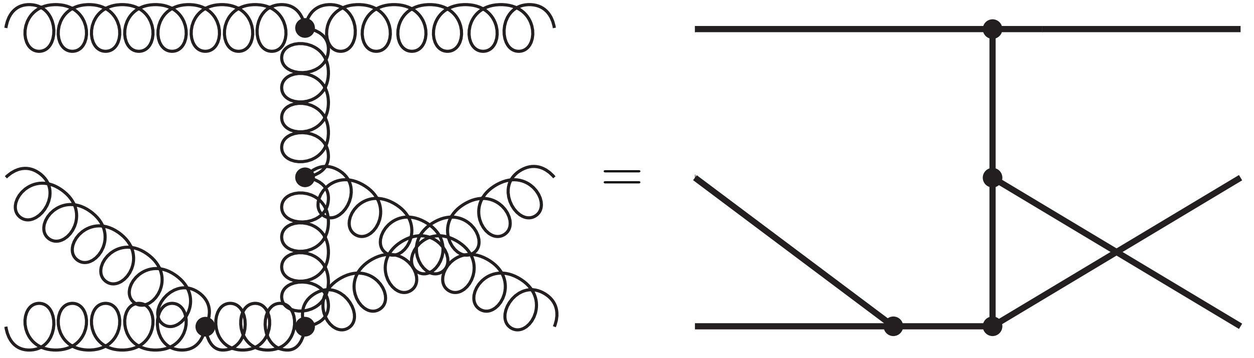
N_g	4	6	8	10	12
$N_c \geq N_g$	52	396	2126	9059	32702
$N_c = 3$	38	130	277	479	736

  and they can be evaluated once and for all
  (Numbers could be slightly reduced by additional symmetries, and smart choice of 3 rep. vertices)

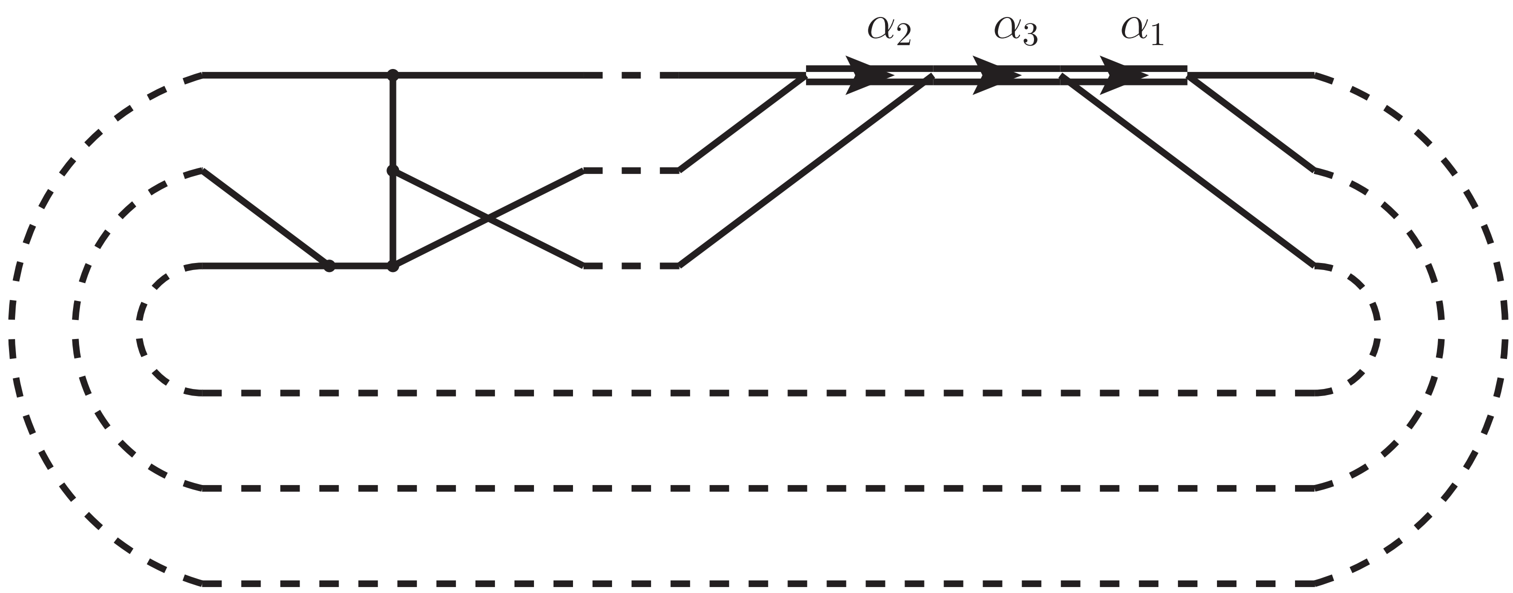
- As a test case, all 6j symbols needed for evaluation of processes with up to 6 gluons have been explicitly calculated
  (Master thesis of Johan Thorén, with aid of ColorMath, Eur. Phys. J. C 73:2310 (2013), arXiv:1211.2099)

Decomposing color with 6j and 3j coefficients

As an example consider the color structure of the Feynman diagram:

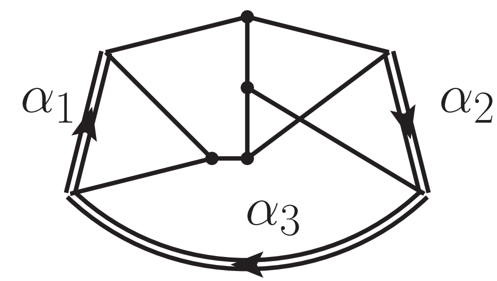


The scalar product between the color structure and a basis vector is given by:



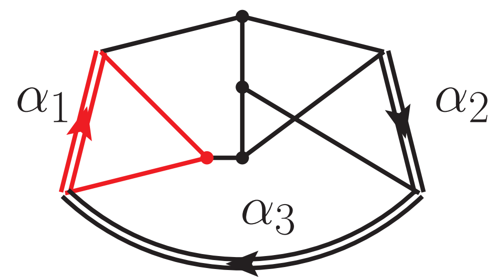
In a more compact form:

$$A(\alpha_1, \alpha_3, \alpha_2) \quad =$$

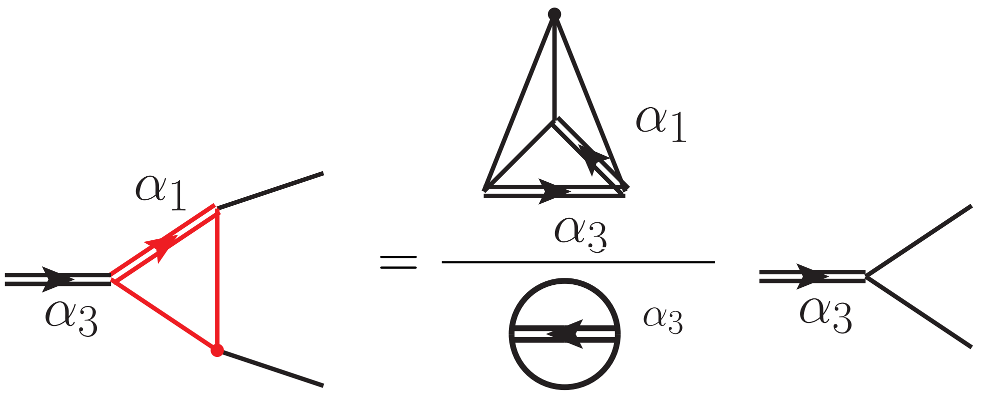


Here we note that we have a vertex correction:

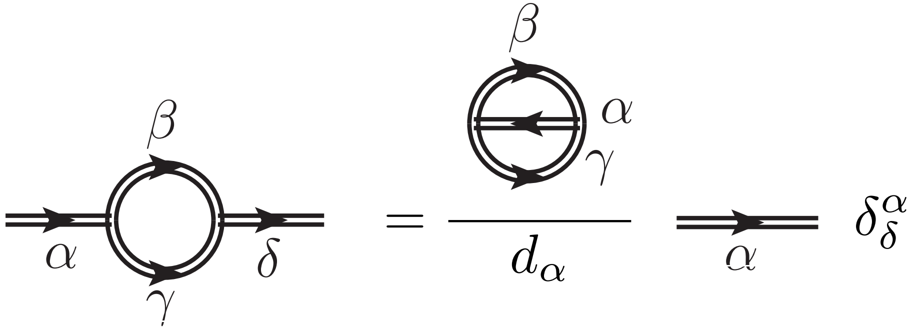
$$A(\alpha_1, \alpha_3, \alpha_2) \quad =$$



A vertex correction only gives a factor (expressed in 6j and 3j coefficients):

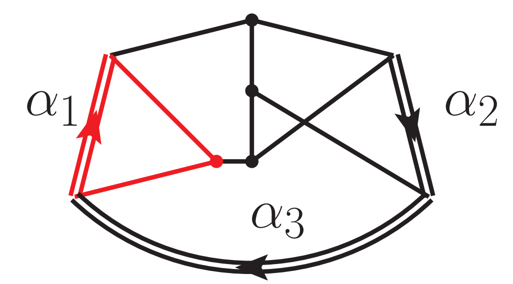


Two vertex loops are also easy to deal with:

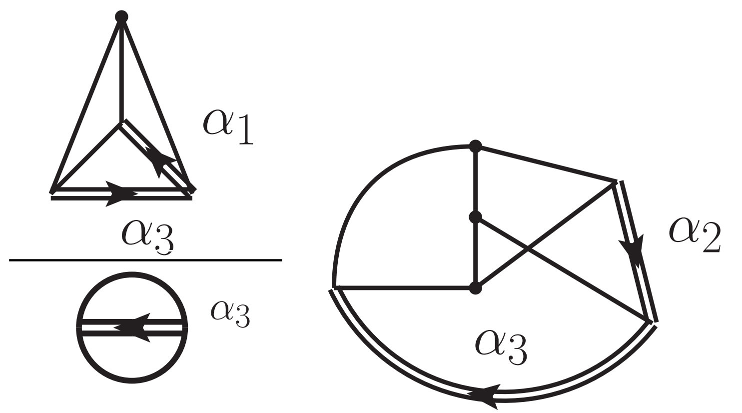


Using the vertex correction results in:

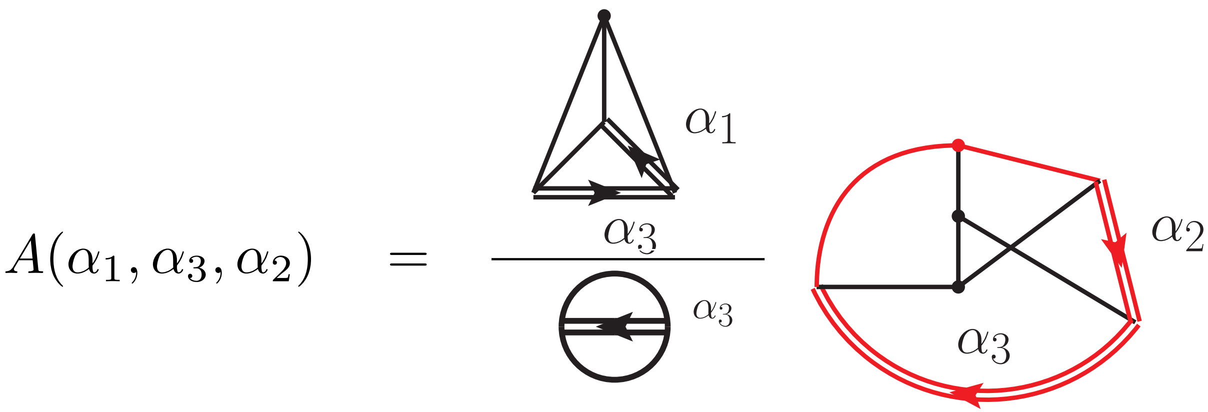
$$A(\alpha_1, \alpha_3, \alpha_2) \qquad =$$



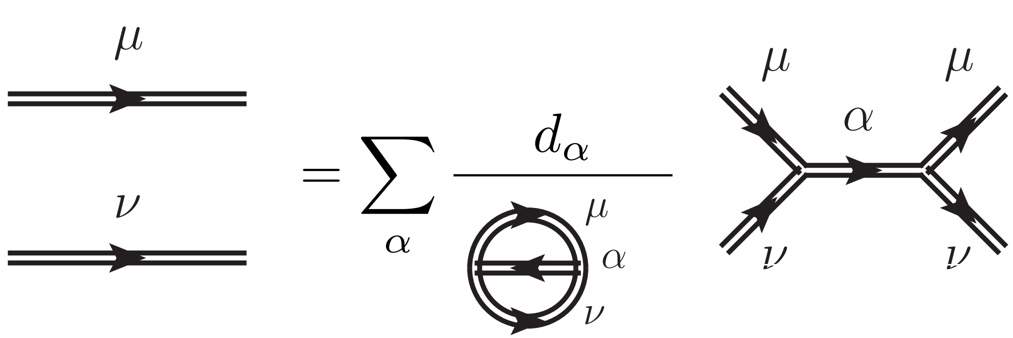
$$=$$



Now there is no trivial color structure, but we can pick any loop...

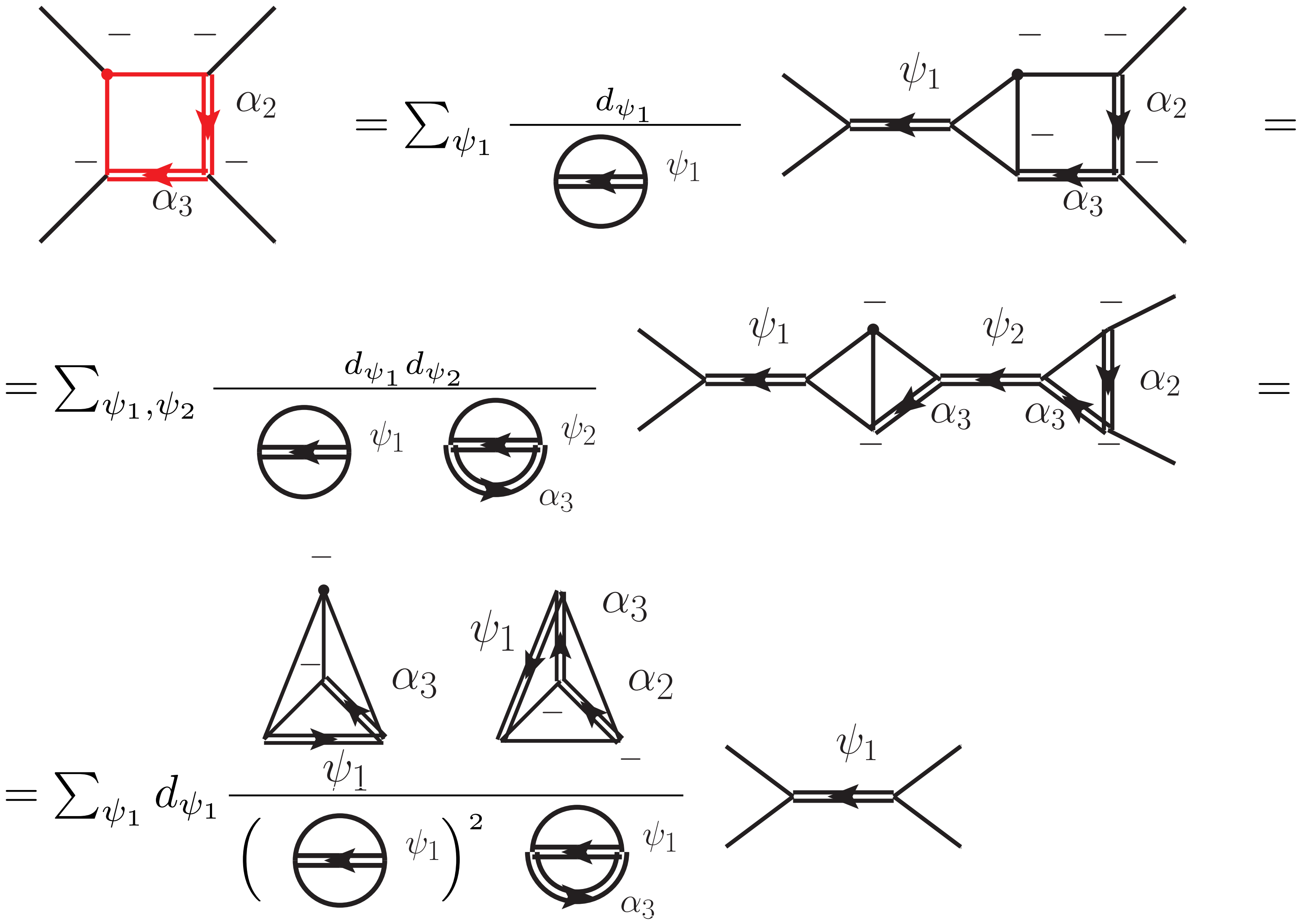


and use the completeness relation

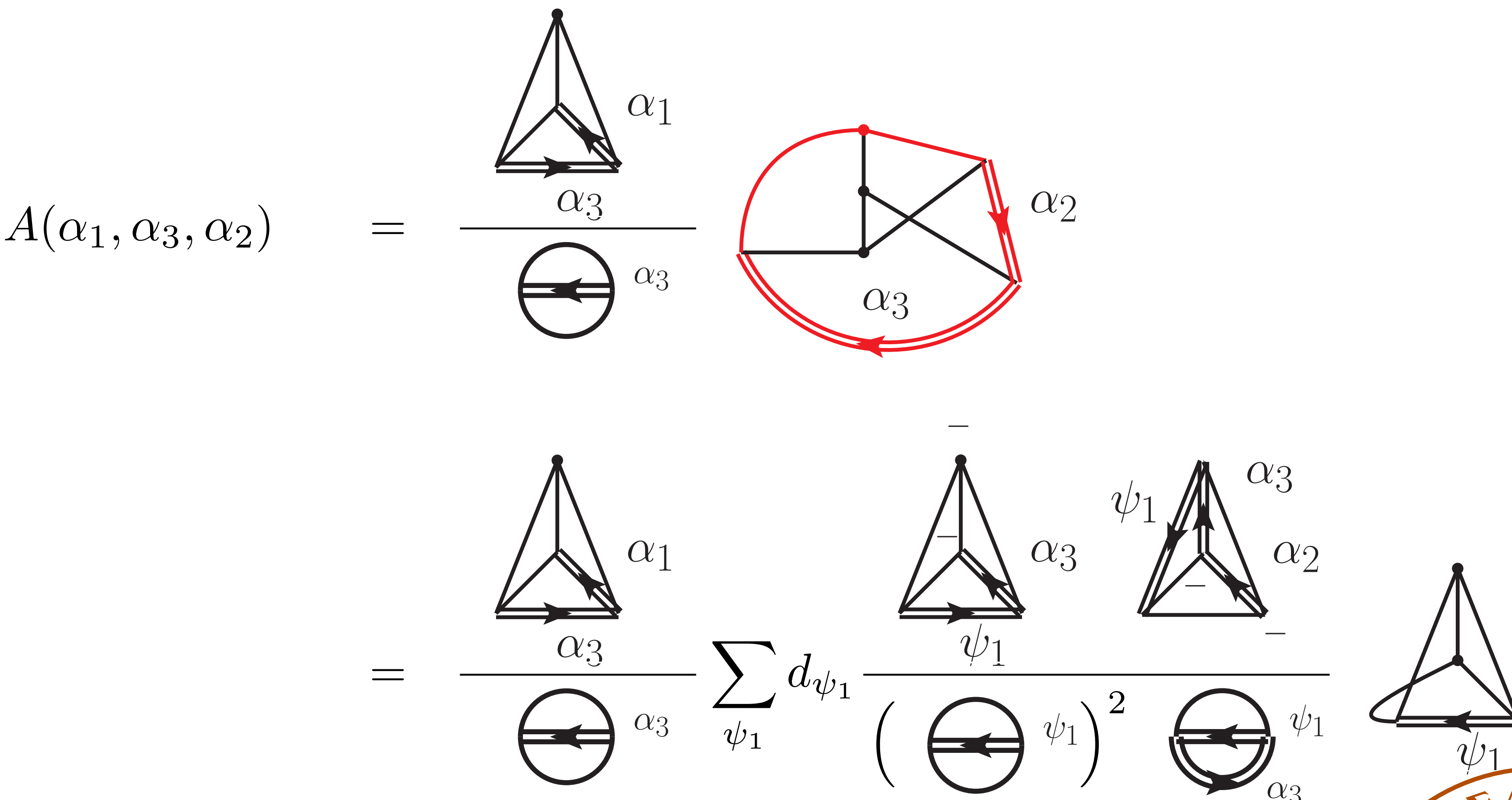


to remove it

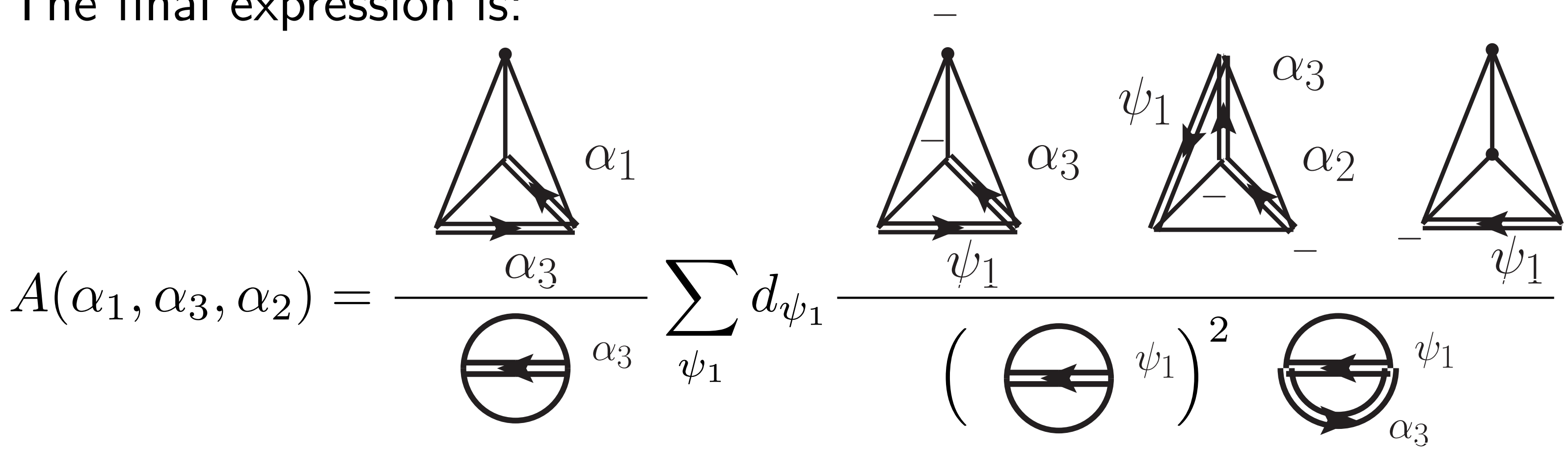
Applying the completeness relation and removing vertex corrections:



Removing the 4-vertex loop we get:



The final expression is:

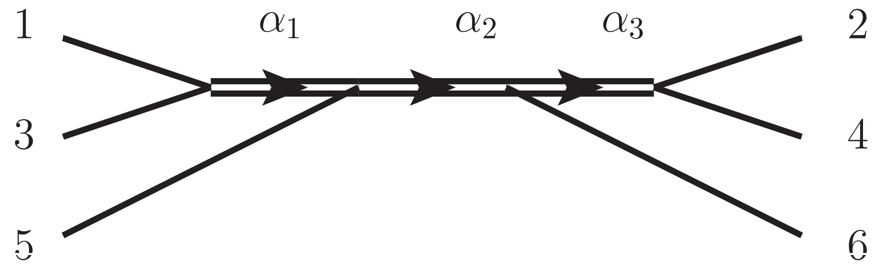


- Knowing the 3j and 6j Wigner coefficients we can immediately write down the scalar product with any basis vector!
- This only has to be done once for each Feynman diagram, not once for each Feynman diagram *and* each basis vector
- We only need to care about non-zero projections, we could list the non-zero 6j-coefficients
- Each sum contains at most 8 terms for SU(3), at most $N_c^2 - 1$ for SU(N_c)

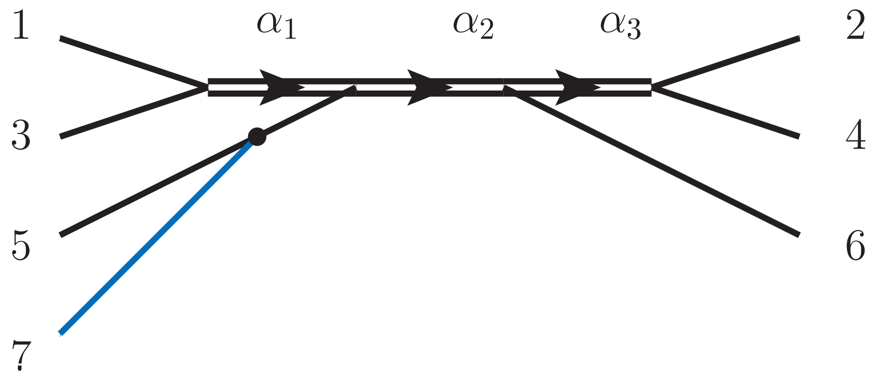
A parton shower perspective

- In a parton shower we start with some amplitude which we can assume that we have decomposed in the multiplet basis

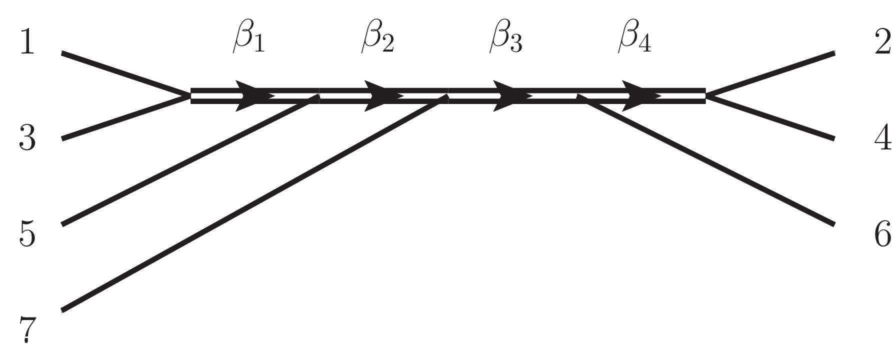
$$\text{Amp} = \sum_{\alpha_1,\alpha_2,\alpha_3} c_{\alpha_1,\alpha_2,\alpha_3}$$



- Knowing the decomposition for $N_g - 1$ gluons, how can we decompose the N_g gluon amplitude?

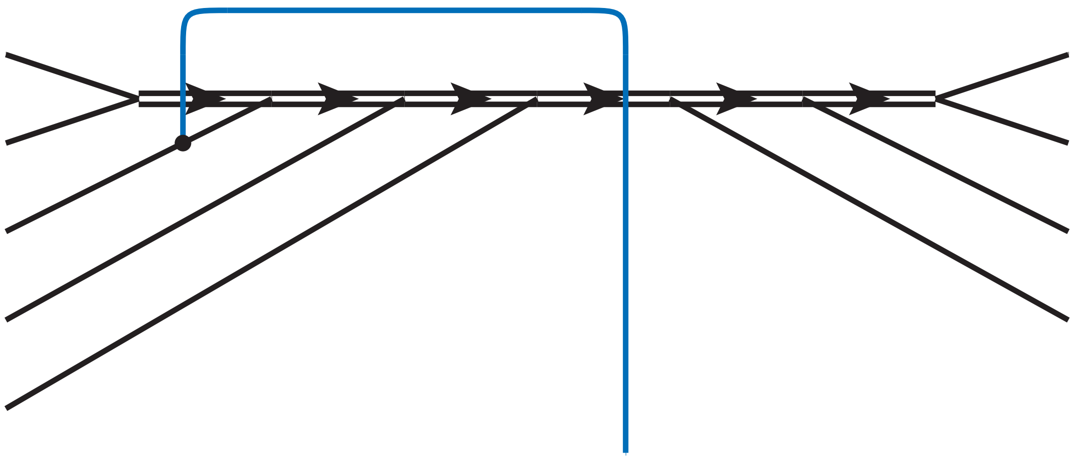


$$= \sum_{\beta_1,\beta_2,\ldots} \tilde{c}_{\beta_1,\beta_2,\ldots}$$

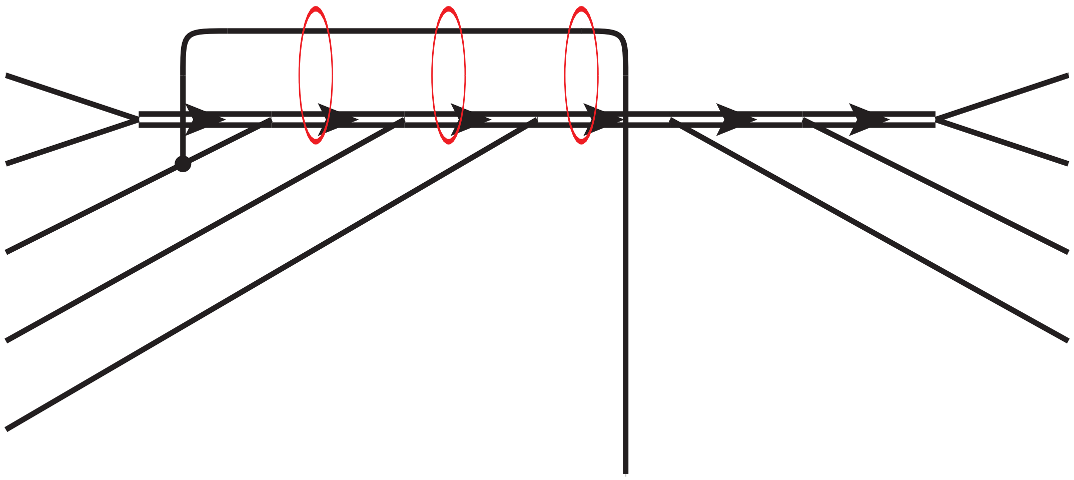


- Scalar products? Too slow!

Let one of the gluons emit a new gluon:

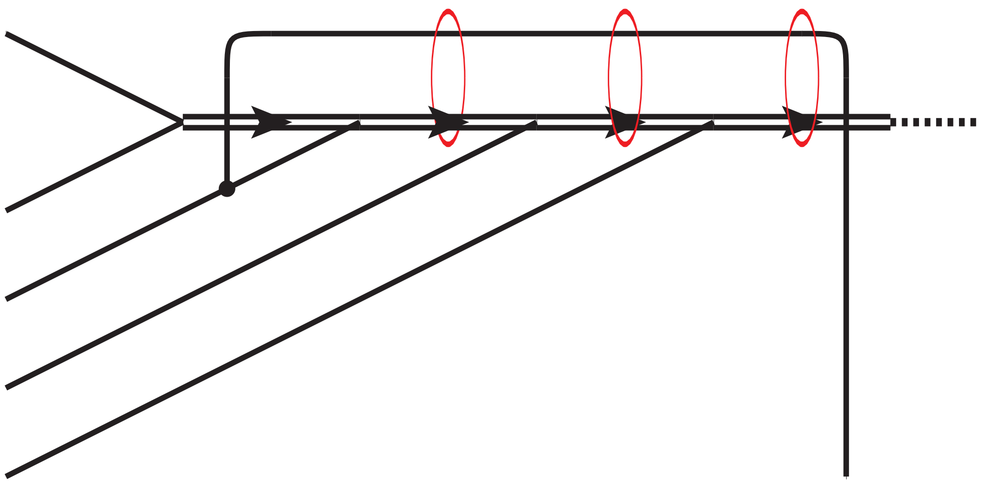


To decompose the affected side, we may insert the completeness relation, repeatedly:

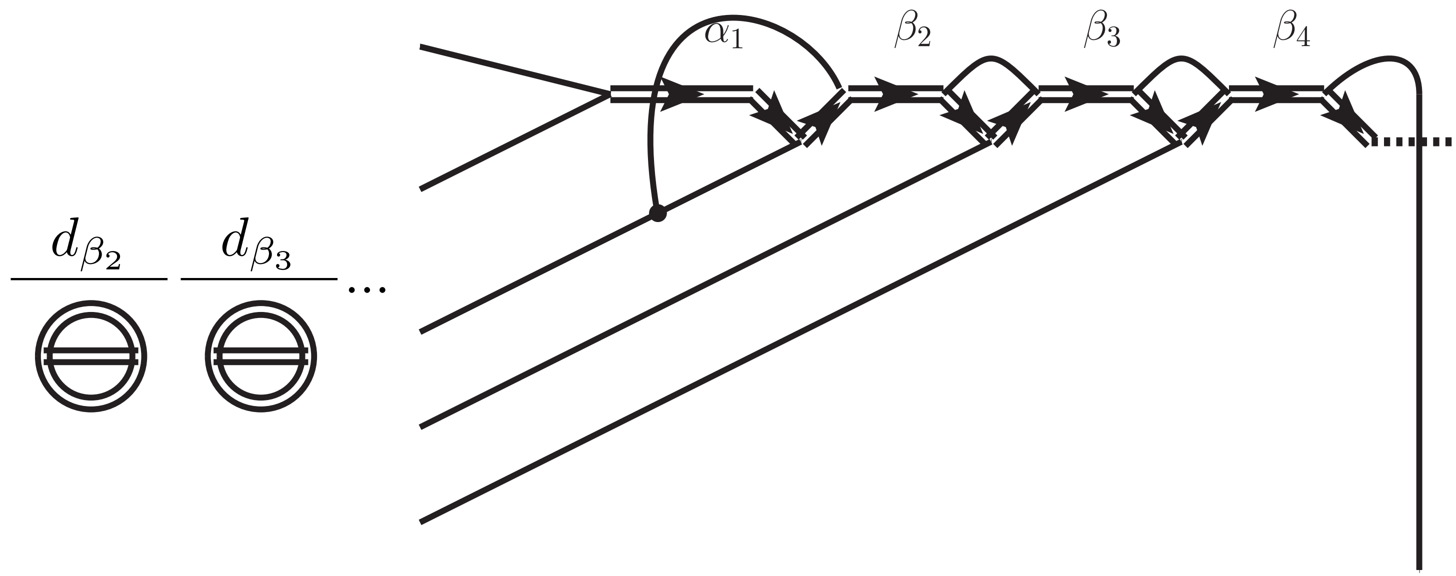


The representations on the other side (here right) don't change

Consider the affected side:

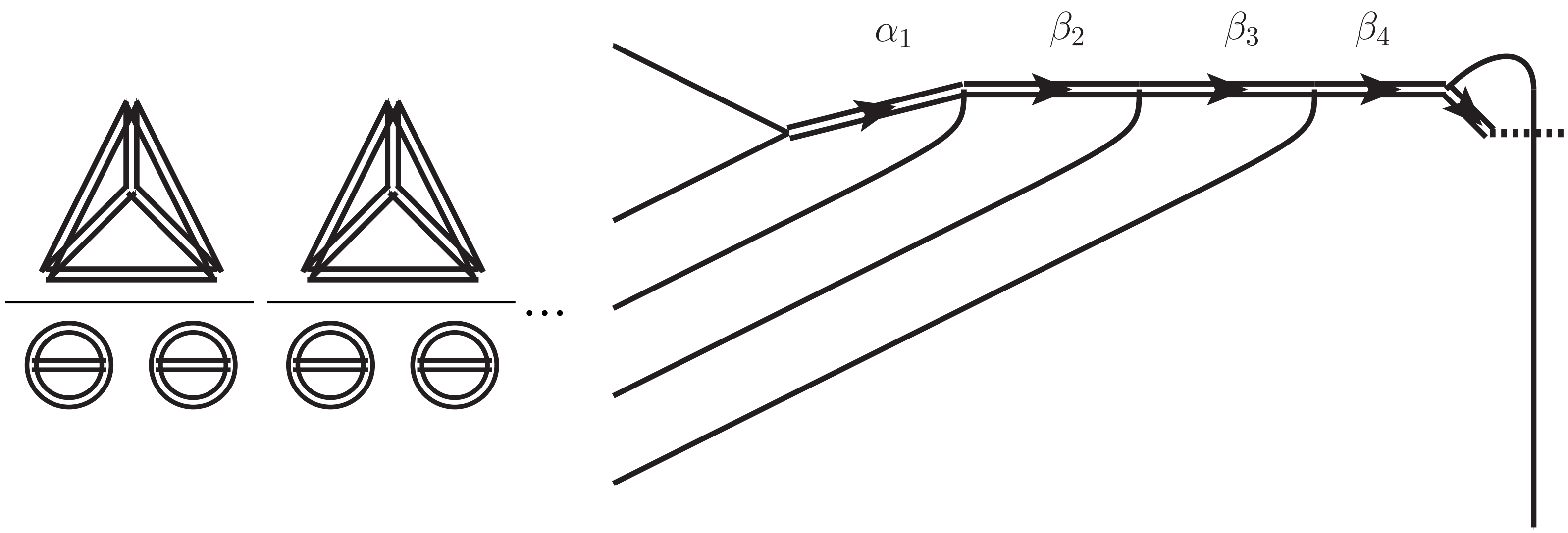


Inserting completeness relations we get a sum of terms of form:



What we have here are just vertex corrections which can be rewritten in terms of 3j and 6j coefficients

Giving us a sum of terms of form:



i.e., knowing the 3j and 6j symbols we can write down the resulting vectors

- By inserting the new gluon "in the middle" in the basis we guarantee that the emitted gluon need never "be transported" across more than $\sim$ half of the reps
- Typically we get only a small fraction of all basis vectors in the larger basis: (preliminary)

N_g	5→6	6→7	7→8	8→9	9→10
$N_c = 3$	0.094	0.027	0.012	0.0032	0.0014
$N_c \geq N_g$	0.071	0.014	0.0054	0.00092	0.00032

Total number of terms, all emissions

Consider the sum of all terms from all emissions (all emitters and all vectors) and compare to the number encountered when squaring a tree-level amplitude (preliminary)

N_g	Fraction ($N_c = 3$)	All terms ($N_c = 3$)	(# tree vectors)^2 (any N_c)
5→6	0.094	2 184	$(120)^2$
6→7	0.027	16 372	$(720)^2$
7→8	0.012	212 914	$(5\,040)^2$
8→9	0.0032	1 758 620	$(40\,320)^2 \sim 10^9$
9→10	0.0014	25 407 328	$(362\,880)^2 \sim 10^{11}$

Numbers will be somewhat reduced by clever vertex choices, and non-general linear combinations

Amplitudes using recursion

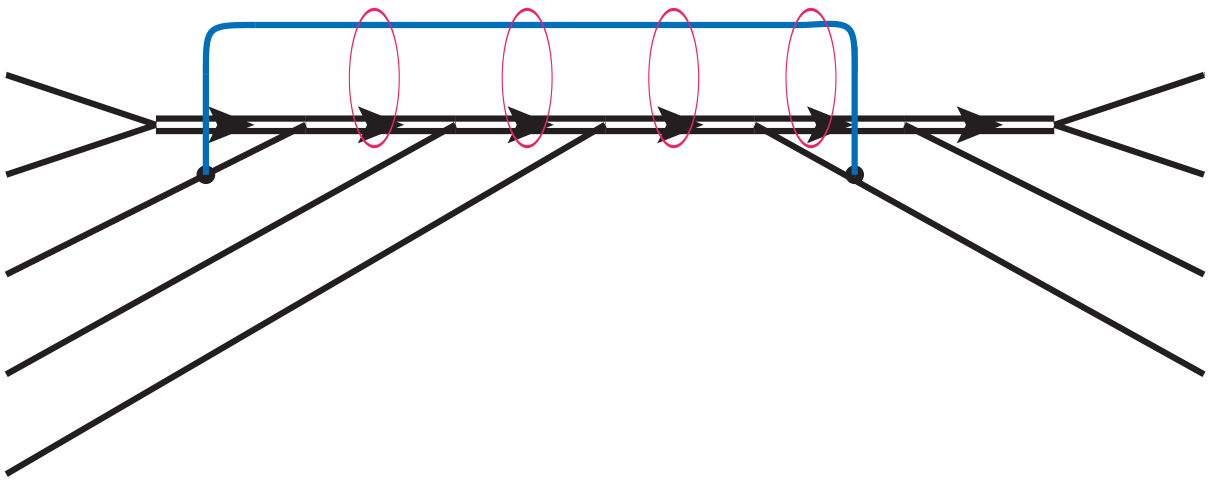
In collaboration with Yi-Jian Du and Johan Thorén, work in progress

- Contemporary techniques for evaluating amplitudes with many external partons are based on recursion relations, rather than Feynman diagrams
- In trace bases and color flow bases, where all gluons enter on equal footing recursion in the number of external legs works nicely (the problem comes when squaring...)
- In multiplet bases the gluons do not enter on equal footing $\rightarrow$ amplitudes are not simply related by relabeling of indices (in most cases)

- How can we do recursion?
- BCFW recursion works "just as normally", but the contributions to different color structures have to be calculated separately
- The color structure turns out to work just as for the parton showers!
- Can get amplitudes for all helicity configurations
- As a proof of concept we have considered tree-level gluon amplitudes and recalculated the amplitudes for up to six gluons

Gluon exchange

- For higher order calculations or for resummation we need to describe the effect of gluon exchange on the color structure
- Gluon exchange may be treated similar to emission



- Here we get a linear combination of basis vectors where only the intermediate representations can have changed

- For each starting color vector we may immediately write down the linear combination of basis vectors after the gluon exchange in terms of 3j and 6j coefficients
- As only the in-between multiplets can be affected, the result is typically a linear combination of a small fraction of all basis vectors
- $\rightarrow$ The soft anomalous dimension matrices may be written down directly, and they are relatively sparse...
- but probably the main gain for all order resummation is that the basis is minimal

Conclusions

- One way of dealing with color space is to use multiplet bases (JHEP09(2012)124, arXiv:1207.0609)
- Color structure can be decomposed elegantly into multiplet bases using the Wigner 3j and 6j coefficients
  - Feynman diagrams
  - parton showers and recursion
  - resummation
- Only a relatively small number of these coefficients are needed
- They can be calculated knowing the multiplet bases
- As a proof of concept all necessary 6j (3j) symbols have been calculated for up to 6 gluons

Outlook

- The secrets of the group theory description of QCD color space is just starting to unravel
- I think we have a lot to learn
- We have just considered one type of multiplet bases
- Probably there are even smarter ones
- So far I have spoken about exact color structure treatment, but what about Monte Carlo sums?

Backup: Number of projection operators and basis vectors

In general, for many partons the size of the vector space is much smaller for $N_c = 3$, compared to for $N_c \to \infty$

Case	Projectors $N_c = 3$	Projectors $N_c = \infty$	Vectors $N_c = 3$	Vectors $N_c = \infty$
2g $\to$ 2g	6	7	8	9
3g $\to$ 3g	29	51	145	265
4g $\to$ 4g	166	513	3 598	14 833
5g $\to$ 5g	1 002	6 345	107 160	1 334 961

Number of projection operators and basis vectors for $N_g \to N_g$ gluons *without* imposing projection operators and vectors to appear in charge conjugation invariant combinations

- The size of the vector spaces asymptotically grows as an exponential in the number of gluons/$q\bar{q}$-pairs for finite N_c
- For general N_c the basis size grows as a factorial

$$N_{\text{vec}}[n_q, N_g] = N_{\text{vec}}[n_q, N_g - 1](N_g - 1 + n_q) + N_{\text{vec}}[n_q, N_g - 2](N_g - 1)$$

where

$$\begin{aligned} N_{\text{vec}}[n_q, 0] &= n_q! \\ N_{\text{vec}}[n_q, 1] &= n_q n_q! \end{aligned}$$

- For general N_c and gluon only amplitudes (to all order) the size is given by Subfactorial(N_g)
- For tree-level gluons amplitudes traces may be used as spanning vectors giving $(N_g - 1)!$ spanning vectors

- Counting all contributions from all emitters and all basis vectors to all new basis vectors and comparing to the squaring step in the trace basis (preliminary)

N_g	Terms $N_c = 3$	Terms $N_c \geq 2N_g$	(# tree vectors)^2
5→6	2 184	4 136	$(120)^2$
6→7	16 372	42 094	$(720)^2$
7→8	212 914	1 039 456	$(5\ 040)^2$
8→9	1 758 620	14 544 744	$(133\ 496)^2 \sim 10^{10}$
9→10	25 407 328	515 182 440	$(362\ 880)^2 \sim 10^{11}$
10→11			

Backup: ColorMath

- Calculations are done using my Mathematica package, ColorMath, Eur. Phys. J. C 73:2310 (2013), arXiv:1211.2099
- ColorMath is an easy to use Mathematica package for color summed calculations in QCD, SU(N_c)
- Repeated indices are implicitly summed

```
In[2]:= Amplitude = I f[g1, g2, g] t[{g}, q1, q2]
Out[2]= i t^{g}q1_q2 f^{g1,g2,g}

In[3]:= CSimplify[Amplitude Conjugate[Amplitude /. g → h]]
Out[3]= 2 Nc (-1 + Nc^2) TR^2
```

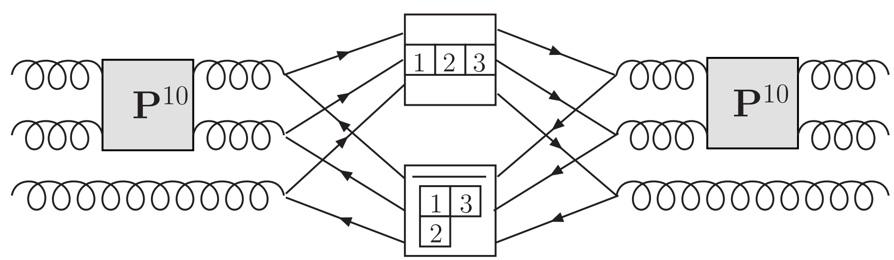
- The package and tutorial can be downloaded from http://library.wolfram.com/infocenter/MathSource/8442/ or www.thep.lu.se/~malin/ColorMath.html

Backup: 2 gluon projectors

- Problem first solved for two gluons by MacFarlane, Sudbery, and Weisz 1968, however only for $N_c = 3$
- General N_c solution for two gluons by Butera, Cicuta and Enriotti 1979
- General N_c solution for two gluons by Cvitanović, in group theory books, 1984 and 2008, using polynomial equations
- General N_c solution for two gluons by Dokshitzer and Marchesini 2006, using symmetries and intelligent guesswork

Backup: Key observation

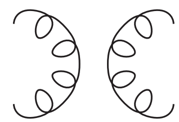
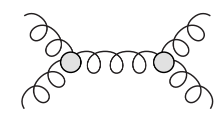
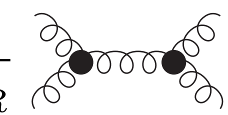
- Starting in a given multiplet, corresponding to some $q\overline{q}$ symmetries, such as 10, from $\boxed{1}\boxed{2} \otimes \overline{\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}}$, it turns out that for each way of attaching a quark box to $\boxed{1}\boxed{2}$ and an anti-quark box to $\overline{\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}}$, to there is at most one new multiplet! For example, the projector $\mathbf{P}^{10,35}$ can be seen as coming from

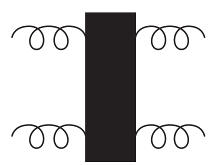
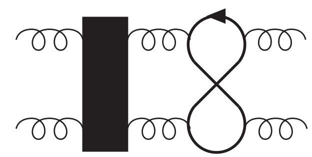


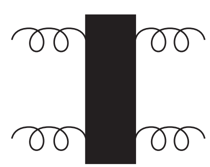
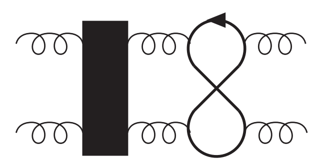
after having projected out "old" multiplets

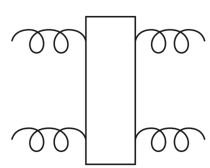
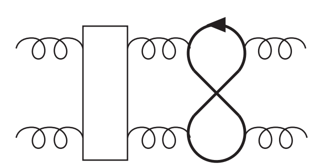
- In fact, for large enough N_c, there is precisely one new multiplet for each set of $q\overline{q}$ symmetries

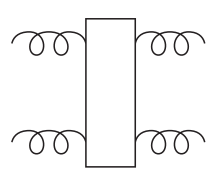
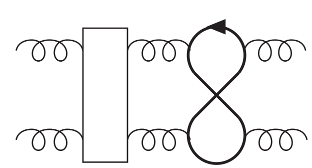
Backup: 2 gluon projectors

$\mathbf{P}^1 = \frac{1}{N_c^2-1}$  , $\mathbf{P}^{8s} = \frac{N_c}{2T_R(N_c^2-4)}$  , $\mathbf{P}^{8a} = \frac{1}{2N_cT_R}$  ,

$\mathbf{P}^{10} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{\overline{10}} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{27} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{N_c-2}{2N_c}\mathbf{P}^{8s} - \frac{N_c-1}{2N_c}\mathbf{P}^1$

$\mathbf{P}^{0} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{N_c+2}{2N_c}\mathbf{P}^{8s} - \frac{N_c+1}{2N_c}\mathbf{P}^1$

Backup: Some 3g example projectors

$$
\begin{aligned}
\mathbf{P}^{8a,8a}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^2}\frac{1}{4N_c^2} if_{g_1\,g_2\,i_1} if_{i_1\,g_3\,i_2} if_{g_4\,g_5\,i_3} if_{i_3\,g_6\,i_2} \\
\mathbf{P}^{8s,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R}\frac{N_c}{2(N_c^2-4)} d_{g_1\,g_2\,i_1} \mathbf{P}^{27}_{i_1\,g_3\,i_2\,g_6} d_{i_2\,g_4\,g_5} \\
\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{4(N_c+1)}{N_c^2(N_c+3)} \mathbf{P}^{27}_{g_1\,g_2\,i_1\,g_3} \mathbf{P}^{27}_{i_1\,g_6\,g_4\,g_5} \\
\mathbf{P}^{27,64=c111c111}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\mathbf{T}^{27,64}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c^2}{162(N_c+1)(N_c+2)} \mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
&- \frac{N_c^2-N_c-2}{81N_c\,(N_c+2)} \mathbf{P}^{27,27s}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}
\end{aligned}
$$

Backup: Three gluon multiplets

SU(3) dim	1	8	10	$\overline{10}$	27	0
Multiplet	c0c0	c1c1	c11c2	c2c11	c11c11	c2c2
	$((45)^{8s}6)^{1}$	$2 \times ((45)^{8s}6)^{8s \text{ or } a}$	$((45)^{8s}6)^{10}$	$((45)^{8s}6)^{\overline{10}}$	$((45)^{8s}6)^{27}$	$((45)^{8s}6)^{0}$
	$((45)^{8a}6)^{1}$	$2 \times ((45)^{8a}6)^{8s \text{ or } a}$	$((45)^{8a}6)^{10}$	$((45)^{8a}6)^{\overline{10}}$	$((45)^{8a}6)^{27}$	$((45)^{8a}6)^{0}$
		$((45)^{10}6)^{8}$	$((45)^{10}6)^{10}$	$((45)^{\overline{10}}6)^{\overline{10}}$	$((45)^{10}6)^{27}$	$((45)^{10}6)^{0}$
		$((45)^{\overline{10}}6)^{8}$	$((45)^{10}6)^{10}$	$((45)^{\overline{10}}6)^{\overline{10}}$	$((45)^{\overline{10}}6)^{27}$	$((45)^{\overline{10}}6)^{0}$
		$((45)^{27}6)^{8}$	$((45)^{27}6)^{10}$	$((45)^{27}6)^{\overline{10}}$	$((45)^{27}6)^{27}$	$((45)^{0}6)^{0}$
		$((45)^{0}6)^{8}$	$((45)^{0}6)^{10}$	$((45)^{0}6)^{\overline{10}}$	$((45)^{27}6)^{27}$	$((45)^{0}6)^{0}$

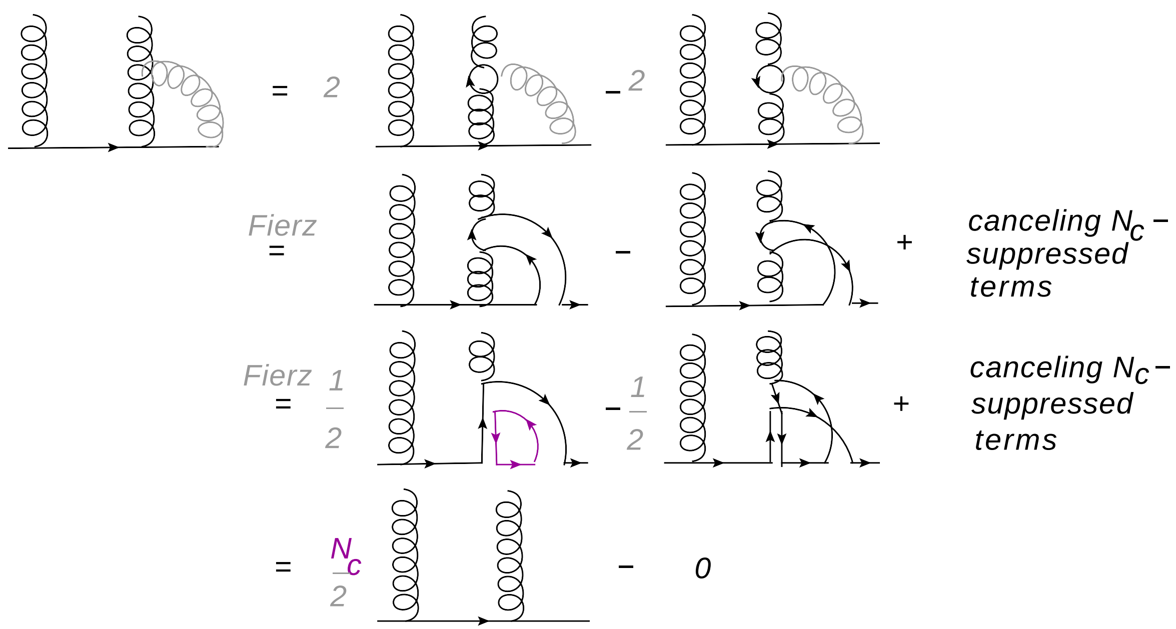
SU(3) dim	64	35	$\overline{35}$	0
Multiplet	c111c111	c111c21	c21c111	c21c21
	$((45)^{27}6)^{64}$	$((45)^{10}6)^{35}$	$((45)^{\overline{10}}6)^{\overline{35}}$	$((45)^{10}6)^{c21c21}$
		$((45)^{27}6)^{35}$	$((45)^{27}6)^{\overline{35}}$	$((45)^{\overline{10}}6)^{c21c21}$
				$((45)^{27}6)^{c21c21}$
				$((45)^{0}6)^{c21c21}$

SU(3) dim	0	0	0	0	0
Multiplet	c111c3	c3c111	c21c3	c3c21	c3c3
	$((45)^{10}6)^{c111c3}$	$((45)^{\overline{10}}6)^{c3c111}$	$((45)^{10}6)^{c21c3}$	$((45)^{\overline{10}}6)^{c3c21}$	$((45)^{0}6)^{c3c3}$
			$((45)^{0}6)^{c21c3}$	$((45)^{0}6)^{c3c21}$	

Multiplets for $g_4 \otimes g_5 \otimes g_6$

Backup: Gluon exchange in trace bases

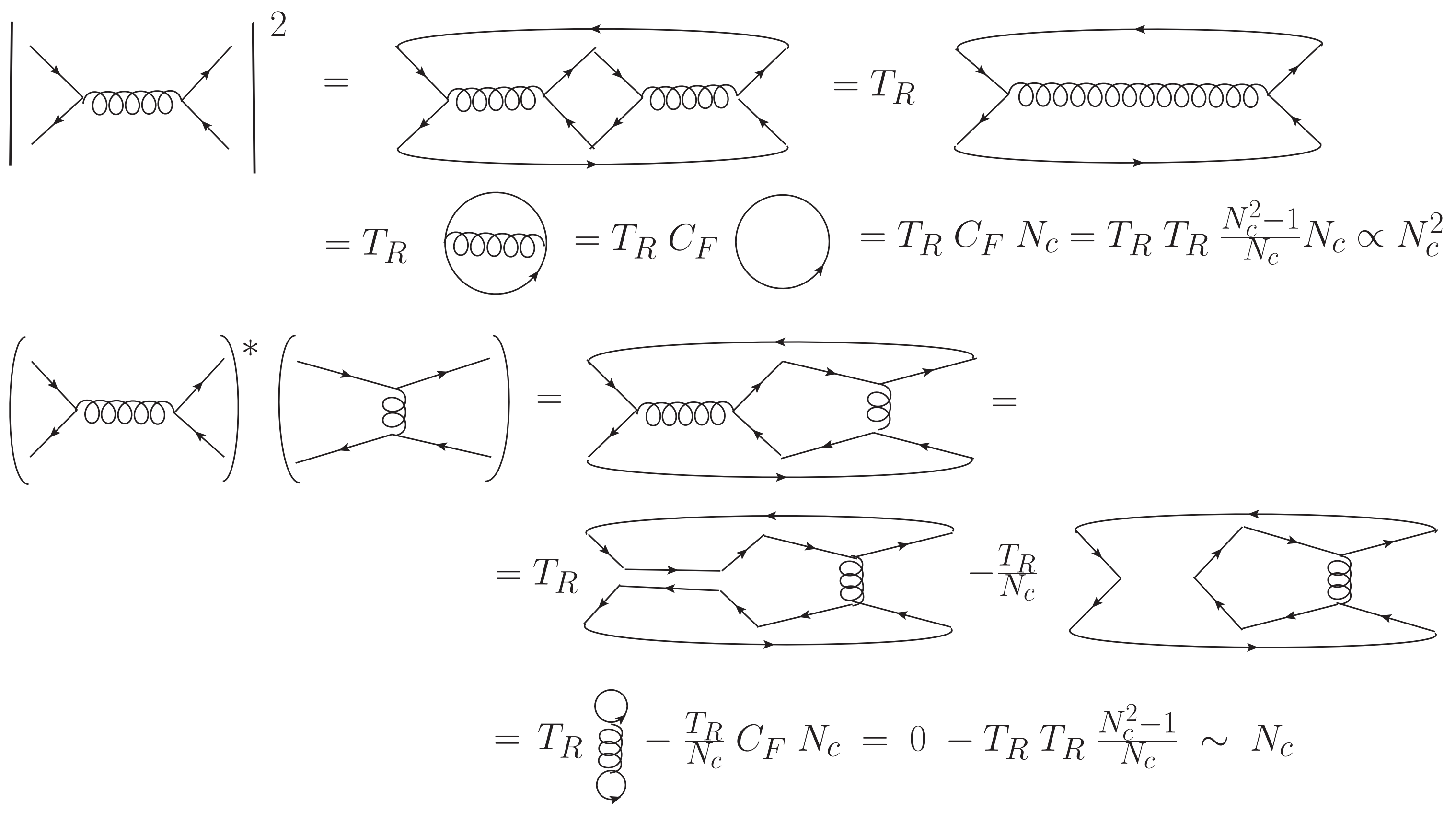
A gluon exchange in this basis "directly" i.e. without using scalar products gives back a linear combination of (at most 4) basis tensors



- N_c-enhancement possible only for near by partons
  $\rightarrow$ only "color neighbors" radiate in the $N_c \rightarrow \infty$ limit

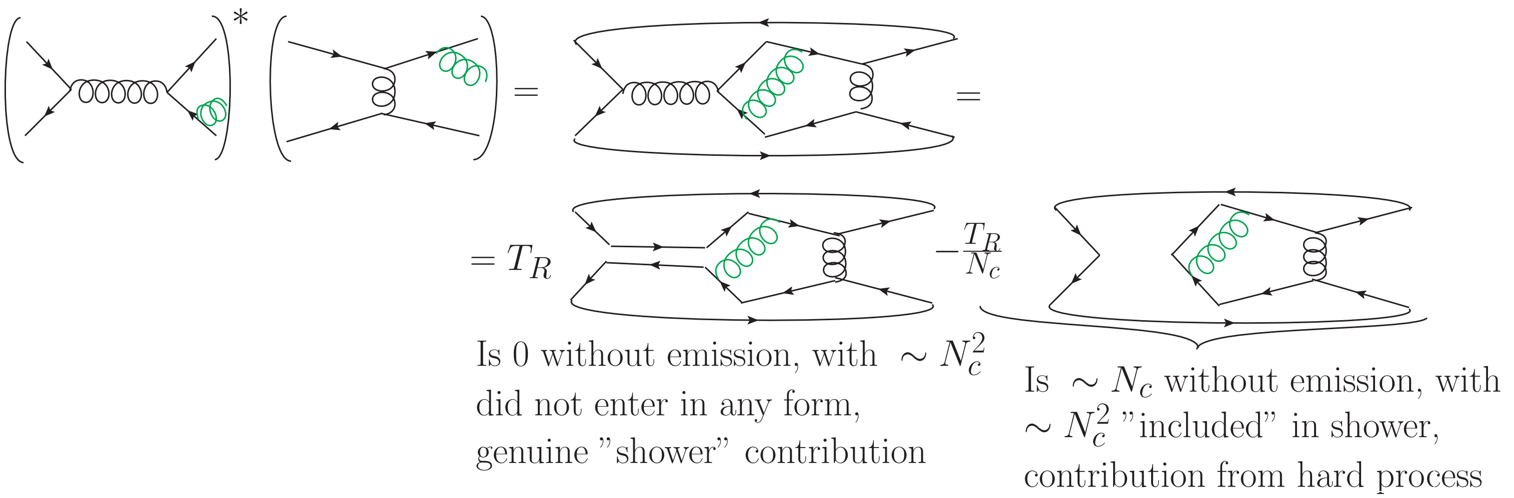
Backup: N_c-suppressed terms

That non-leading color terms are suppressed by $1/N_c^2$, is guaranteed only for same order α_s diagrams with only gluons ('t Hooft 1973)



Backup: N_c-suppressed terms

For a parton shower there may also be terms which only are suppressed by one power of N_c



The leading N_c contribution scales as N_c^2 before emission and N_c^3 after