Summing large-N towers in colour flow evolution.

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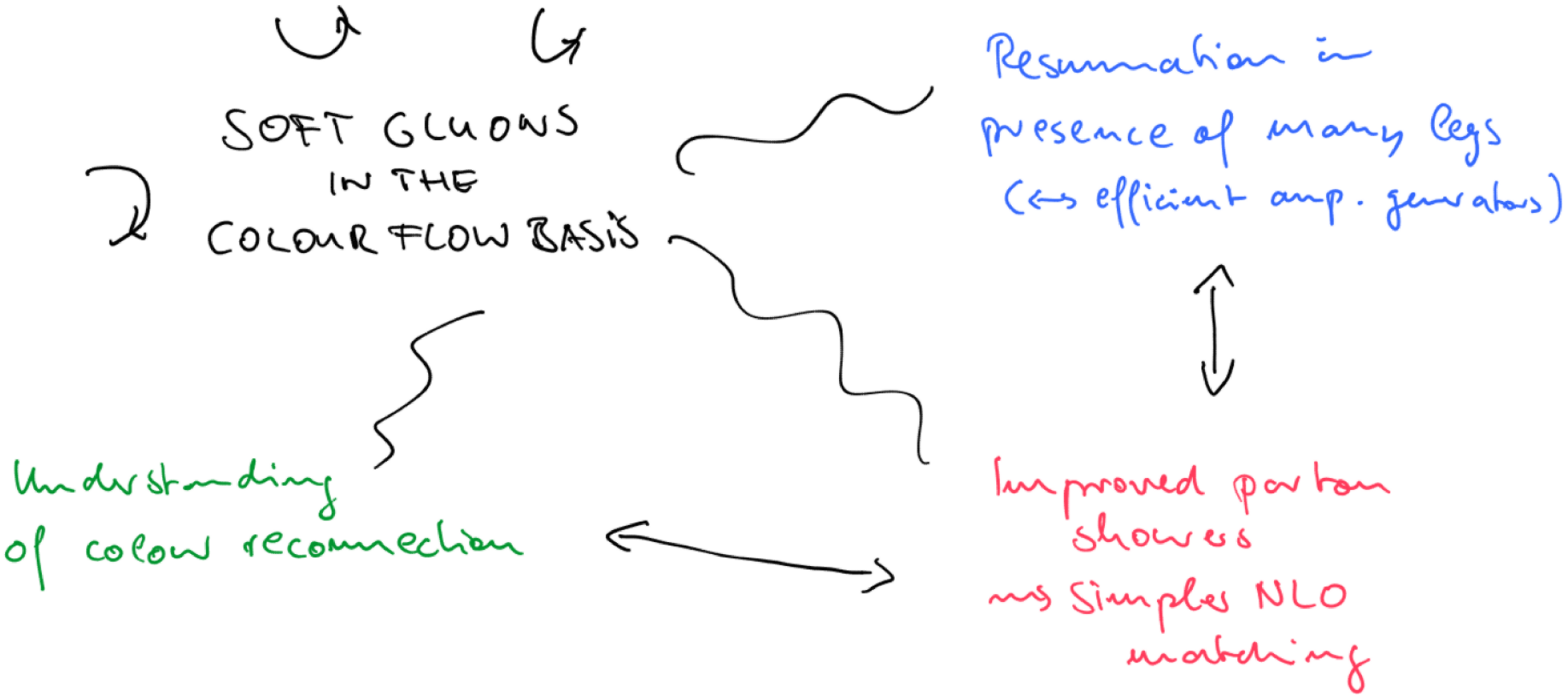
DESY

based on SP – arXiv:1312.2448

Putting it into context.



Outline.

- Colour flow bases.
- Soft anomalous dimensions.
- Summing large-N towers.
- Conclusions & outlook.

Colour flow bases.

Physically transparent QCD amplitudes:
Track flow of colour charge.

Not tied to large-N limit.

Basis for highly efficient, recursive amplitude evaluation.

Comix, OpenLoops, Weinzierl et al.

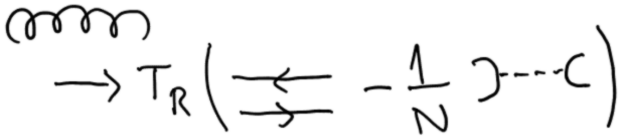
Very close connection to parton showers.
Initial conditions driven by large-N flows.

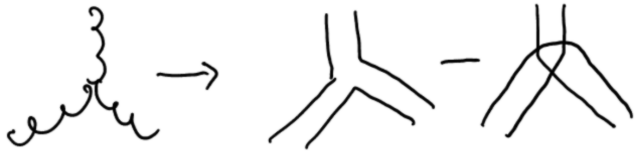
Rarely used in the context of soft-gluon evolution.

- Properties of soft anomalous dimensions?
- Insight into colour evolution dynamics?

$$A_a^\mu \to A_a^\mu (t^a)^i{}_{\bar{i}} \equiv \mathcal{A}^{\mu i}{}_{\bar{j}}$$

$$(t^a)^i{}_{\bar{i}} (t^a)^j{}_{\bar{j}} = T_R \left(\delta^i_{\bar{j}} \delta^j_{\bar{i}} - \frac{1}{N} \delta^i_{\bar{i}} \delta^j_{\bar{j}} \right)$$





Note: 'Singlet' gluon decouples from pure-gluon vertices.

Full set of Feynman rules e.g. in

[Maltoni et al. – Phys.Rev.D 67, 014026 (2003)]

Colour flow bases.

Amplitude decomposition:

$$|\mathcal{M}_n\rangle = \sum_\sigma \mathcal{M}_{n,\sigma}|\sigma\rangle$$

'Normal ordered' numbering of (outgoing) legs α:

$$1_{\mathbf{N}}, 2_{\bar{\mathbf{N}}}, \ldots, n_{q,\bar{\mathbf{N}}}, (n_q+1)_{\mathbf{A}}, \ldots, (n_q+n_g)_{\mathbf{A}}$$

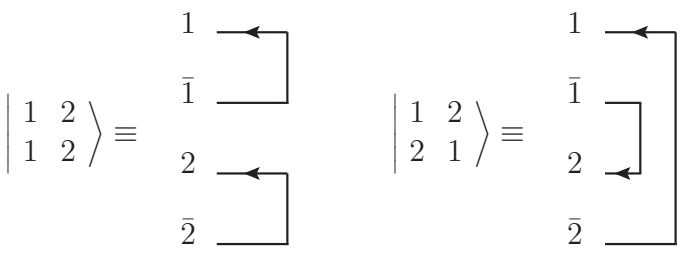
translated to (anti-)fundamental indices:

$$\begin{aligned} k &\leftrightarrow \alpha = k_{\mathbf{N}} \\ \overline{k-1} &\leftrightarrow \alpha = k_{\bar{\mathbf{N}}} \\ \left.\begin{matrix}\overline{k-n_q/2} \\ k-n_q/2\end{matrix}\right\} &\leftrightarrow \alpha = k_{\mathbf{A}} \end{aligned}$$

Basis tensors labelled by $\sigma \in S_m$, $m = n_q/2 + n_g$:

$$|\sigma\rangle = \delta^{i_1}_{i_{\overline{\sigma(1)}}} \cdots \delta^{i_m}_{i_{\overline{\sigma(m)}}}$$

Cannot restrict the basis to the tree-level subspace, even when only considering evolution of tree-level amplitudes only!



Partial amplitudes in this basis used to caluclate weights for large-N colour flow assignment.

Soft anomalous dimensions.

$$|\mathcal{M}_n'\rangle = e^{\mathbf{\Gamma}}|\mathcal{M}_n\rangle \qquad \mathbf{\Gamma} = \sum_{\alpha\neq\beta} \Gamma^{\alpha\beta}\, \mathbf{T}_\alpha \cdot \mathbf{T}_\beta$$

Translate colour correlations: Legs $\alpha \leftrightarrow$ (anti-)colour indices $\bar{i}, j$.

$$\begin{aligned}
\mathbf{T}_\alpha \cdot \mathbf{T}_\beta &= \mathbf{T}_k \cdot \mathbf{T}_l & \alpha &= k_{\mathbf{N}}, \beta = l_{\mathbf{N}} \\
\mathbf{T}_\alpha \cdot \mathbf{T}_\beta &= \mathbf{T}_k \cdot \mathbf{T}_{\bar{l}} & \alpha &= k_{\mathbf{N}}, \beta = (l+1)_{\bar{\mathbf{N}}} \\
\mathbf{T}_\alpha \cdot \mathbf{T}_\beta &= \mathbf{T}_k \cdot \mathbf{T}_{\bar{l}} + \mathbf{T}_k \cdot \mathbf{T}_l & \alpha &= k_{\mathbf{N}}, \beta = (l+n_q/2)_{\mathbf{A}} \\
\mathbf{T}_\alpha \cdot \mathbf{T}_\beta &= \mathbf{T}_k \cdot \mathbf{T}_{\bar{l}} + \mathbf{T}_k \cdot \mathbf{T}_l & \alpha &= (k+n_q/2)_{\mathbf{A}}, \\
&+\mathbf{T}_l \cdot \mathbf{T}_{\bar{k}} + \mathbf{T}_{\bar{k}} \cdot \mathbf{T}_{\bar{l}} & \beta &= (l+n_q/2)_{\mathbf{A}}
\end{aligned}$$

Express colour correlators by Fierz identities:

$$\mathbf{T}_i \cdot \mathbf{T}_j = +\frac{1}{2}\left(\delta_j^{i'}\delta_i^{j'} - \frac{1}{N}\delta_i^{i'}\delta_j^{j'}\right) \qquad \mathbf{T}_i \cdot \mathbf{T}_{\bar{j}} = -\frac{1}{2}\left(\delta^{i'}{}_{\bar{j}'}\delta^{\bar{j}}{}_i - \frac{1}{N}\delta^{i'}{}_i\delta^{\bar{j}}{}_{\bar{j}'}\right)$$

Soft anomalous dimensions.

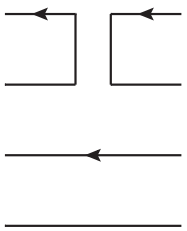
$$\mathbf{\Gamma} = \sum_{i<j} (\gamma_{ij} \mathbf{T}_i \cdot \mathbf{T}_j + \gamma_{\bar{i}\bar{j}} \mathbf{T}_{\bar{i}} \cdot \mathbf{T}_{\bar{j}}) + \sum_{i,j} \gamma_{i\bar{j}} \mathbf{T}_i \cdot \mathbf{T}_{\bar{j}}$$

Anomalous dimension in the flow basis:

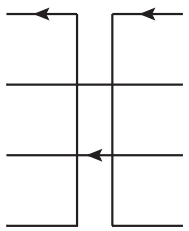
$$\mathbf{T}_i \cdot \mathbf{T}_j = \frac{1}{2} \left(\delta_j^{i'} \delta_i^{j'} - \frac{1}{N} \delta_i^{i'} \delta_j^{j'} \right)$$

$$[\tau|\mathbf{\Gamma}|\sigma] = \left(-N\Gamma_\sigma + \frac{1}{N}\rho \right) \delta_{\tau\sigma} + \Sigma_{\tau\sigma}$$

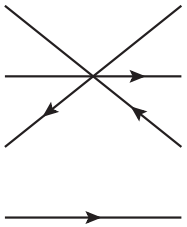
$\mathbf{T}_1 \cdot \mathbf{T}_{\bar{1}} \sim$



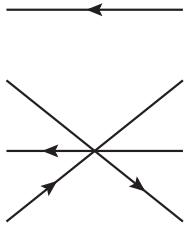
$\mathbf{T}_1 \cdot \mathbf{T}_{\bar{2}} \sim$



$\mathbf{T}_1 \cdot \mathbf{T}_2 \sim$



$\mathbf{T}_{\bar{1}} \cdot \mathbf{T}_{\bar{2}} \sim$



Colour 'reconnectors'
$\rightarrow$ genuinely $1/N$ supressed.

$\Sigma_{\tau\sigma} \neq 0$ *only* if $\sigma = \text{transposition}_{ij}(\tau)$

Soft anomalous dimensions.

Transparent organization of powers of N:

$$\boldsymbol{\Gamma} \equiv N\underline{\Gamma} + \underline{\Sigma} + \frac{1}{N}\rho\underline{1}$$

Leading-N $\rightarrow$ diagonal
Next-to-leading-N $\rightarrow$ colour reconnections, diagonal elements vanish
Next-to-next-to-leading-N $\rightarrow$ trivial

Can we learn something on exponentiation?
What is the 'leading-N' structure of the evolution matrix $e^{\boldsymbol{\Gamma}}$?

NB: (Integrated) subtraction terms at NLO exhibit the same structure as $\boldsymbol{\Gamma}$.

Large-N towers.

What is the 'leading-N' structure of the evolution matrix $e^{\boldsymbol{\Gamma}}$?

'Never use brute force to fight an exponential!'
Perturbation theory in $1/N$ won't work.
Need resummation to get at least close to the exact result.

$$\exp \boldsymbol{\Gamma} \equiv \exp\left(N\underline{\Gamma} + \underline{\Sigma} + \frac{1}{N}\rho\underline{1}\right)$$

Power counting: Need to resum if $\gamma N = \mathcal{O}(1)$.
Treat remainder as higher-order effects/perturbation, but on same footing.

$$\begin{aligned}
\text{at LC} \quad &: \quad 1 + \gamma N + \gamma^2 N^2 + \ldots \\
\text{at NLC} \quad &: \quad \left(\gamma + \frac{\gamma}{N}\right)(1 + \gamma N + \gamma^2 N^2 + \ldots) \\
\text{at NNLC} \quad &: \quad \left(\gamma + \frac{\gamma}{N}\right)^2 (1 + \gamma N + \gamma^2 N^2 + \ldots)
\end{aligned}$$

Will also consider a 'primed' resummation: Absorb ρ contribution into redifinition of Γ.

How to get there?

Work out the exponential ...

$$[\tau|e^{\Gamma}|\sigma] =$$

$$\sum_{l=0}^{\infty}\frac{(-1)^l}{N^l}\sum_{k=0}^{l}\frac{(-\rho)^k}{k!}\sum_{\sigma_0,\ldots,\sigma_{l-k}}\delta_{\tau\sigma_0}\delta_{\sigma_{l-k}\sigma}\left(\prod_{\alpha=0}^{l-k-1}\Sigma_{\sigma_\alpha\sigma_{\alpha+1}}\right)R(\{\sigma_0,\ldots,\sigma_{l-k}\},\{\Gamma_\sigma\})$$

$$R(\sigma,\Gamma) = \left[\prod_{\alpha=0}^{\#\mathrm{uniq}(\sigma)-1}\frac{1}{d_\alpha(\sigma)!}\frac{\partial^{d_\alpha(\sigma)}}{\partial^{d_\alpha(\sigma)}\Gamma_{\sigma_\alpha}}\right]\sum_{\alpha=0}^{\#\mathrm{uniq}(\sigma)-1}e^{-N\Gamma_{\sigma_\alpha}}\prod_{\beta=0,\beta\neq\alpha}^{\#\mathrm{uniq}(\sigma)-1}\frac{(\Gamma_{\sigma_\beta}/\Gamma_{\sigma_\alpha})^{d_\beta(\sigma)}}{\Gamma_{\sigma_\alpha}-\Gamma_{\sigma_\beta}}$$

(next-to-)^d-leading color is truncating the series at d.
Note that the leading colour contributions are exponentiated.

Some first observations.

Next-to-leading color:

$$[\tau|e^{\mathbf{\Gamma}}|\sigma] = \delta_{\tau\sigma} e^{-N\Gamma_\sigma}\left(1+\frac{\rho}{N}\right) - \Sigma_{\tau\sigma}\frac{1}{N}\frac{e^{-N\Gamma_\tau}-e^{-N\Gamma_\sigma}}{\Gamma_\tau-\Gamma_\sigma} + \text{NNLC}$$

At N^dLC need d matrix multiplications.
Or not even this: Σ is very sparse, with easily calculable non-vanishing elements.
$\rightarrow$ Think of a Monte Carlo over colour structures.

In a resummation context, to be matched to NLO:
NLC is in principle sufficient for the matching.

$$[\tau|e^{\mathbf{\Gamma}}|\sigma]\Big|_{\text{NLC}} = \delta_{\tau\sigma} + [\tau|\mathbf{\Gamma}|\sigma] + \mathcal{O}(\gamma^2)$$

For a large number of legs, a sufficiently high order is required to get at least a lowest order for transition elements involving many transpositions.
$\rightarrow$ Impact of these contributions to be checked at the level of amplitudes squared.

Higher orders & numerics.

E.g. N^3LC ingredients:

$$R(\{\sigma_0,\sigma_1,\sigma_2,\sigma_3\},\Gamma) = \frac{e^{-N\Gamma_{\sigma_0}}}{(\Gamma_{\sigma_0}-\Gamma_{\sigma_1})(\Gamma_{\sigma_0}-\Gamma_{\sigma_2})(\Gamma_{\sigma_0}-\Gamma_{\sigma_3})} + (0\leftrightarrow 1)+(0\leftrightarrow 2)+(0\leftrightarrow 3)$$

$$R(\{\sigma_0,\sigma_0,\sigma_0,\sigma_1\},\Gamma) = \frac{N^2}{2}\frac{e^{-N\Gamma_{\sigma_0}}}{\Gamma_{\sigma_0}-\Gamma_{\sigma_1}} + N\frac{e^{-N\Gamma_{\sigma_0}}}{(\Gamma_{\sigma_0}-\Gamma_{\sigma_1})^2} + \frac{e^{-N\Gamma_{\sigma_0}}-e^{-N\Gamma_{\sigma_1}}}{(\Gamma_{\sigma_0}-\Gamma_{\sigma_1})^3}$$

$$R(\{\sigma_0,\sigma_0,\sigma_1,\sigma_2\},\Gamma) = -N\frac{e^{-N\Gamma_{\sigma_0}}}{(\Gamma_{\sigma_0}-\Gamma_{\sigma_1})(\Gamma_{\sigma_0}-\Gamma_{\sigma_2})} + \frac{(\Gamma_{\sigma_1}+\Gamma_{\sigma_2}-2\Gamma_{\sigma_0})\,e^{-N\Gamma_{\sigma_0}}}{(\Gamma_{\sigma_0}-\Gamma_{\sigma_1})^2(\Gamma_{\sigma_0}-\Gamma_{\sigma_2})^2} + \left(\frac{e^{-N\Gamma_{\sigma_1}}}{(\Gamma_{\sigma_0}-\Gamma_{\sigma_1})^2(\Gamma_{\sigma_1}-\Gamma_{\sigma_2})} + (1\leftrightarrow 2)\right)$$

$$R(\{\sigma_0,\sigma_0,\sigma_1,\sigma_1\},\Gamma) = -N\frac{e^{-N\Gamma_{\sigma_0}}+e^{-N\Gamma_{\sigma_1}}}{(\Gamma_{\sigma_0}-\Gamma_{\sigma_1})^2} - 2\frac{e^{-N\Gamma_{\sigma_0}}-e^{-N\Gamma_{\sigma_1}}}{(\Gamma_{\sigma_0}-\Gamma_{\sigma_1})^3}$$

$$R(\{\sigma_0,\sigma_0,\sigma_0,\sigma_0\},\Gamma) = -\frac{N^3}{6}e^{-N\Gamma_{\sigma_0}}$$

`CVolver` C++ library implements all required ingredients at arbitrary N^dLC order.

- Transparent interface to implement anomalous dimensions and hard amplitudes.
- Prepared to perform Monte Carlo over colour flows.

Squaring evolved amplitudes.

Colour bases are typically not orthogonal:

$$|\mathcal{M}|^2 = \langle \mathcal{M} | \mathcal{M} \rangle = \mathcal{M}_\sigma^\dagger S_{\sigma\tau} \mathcal{M}_\tau$$

Colour flow basis for m colour flows:

$$S_{\sigma\tau} = \langle \sigma | \tau \rangle = N^{m - \#\text{transpositions}(\sigma,\tau)}$$

Approximate squared amplitudes on large-N approximation?
$\rightarrow$ No. Unrelated to approximation of soft-gluon evolution, will obscure convergence.

Again, amazingly simple structure:
Don't even need to do linear algebra, but Monte Carlo over colour flows.

Numerical results.

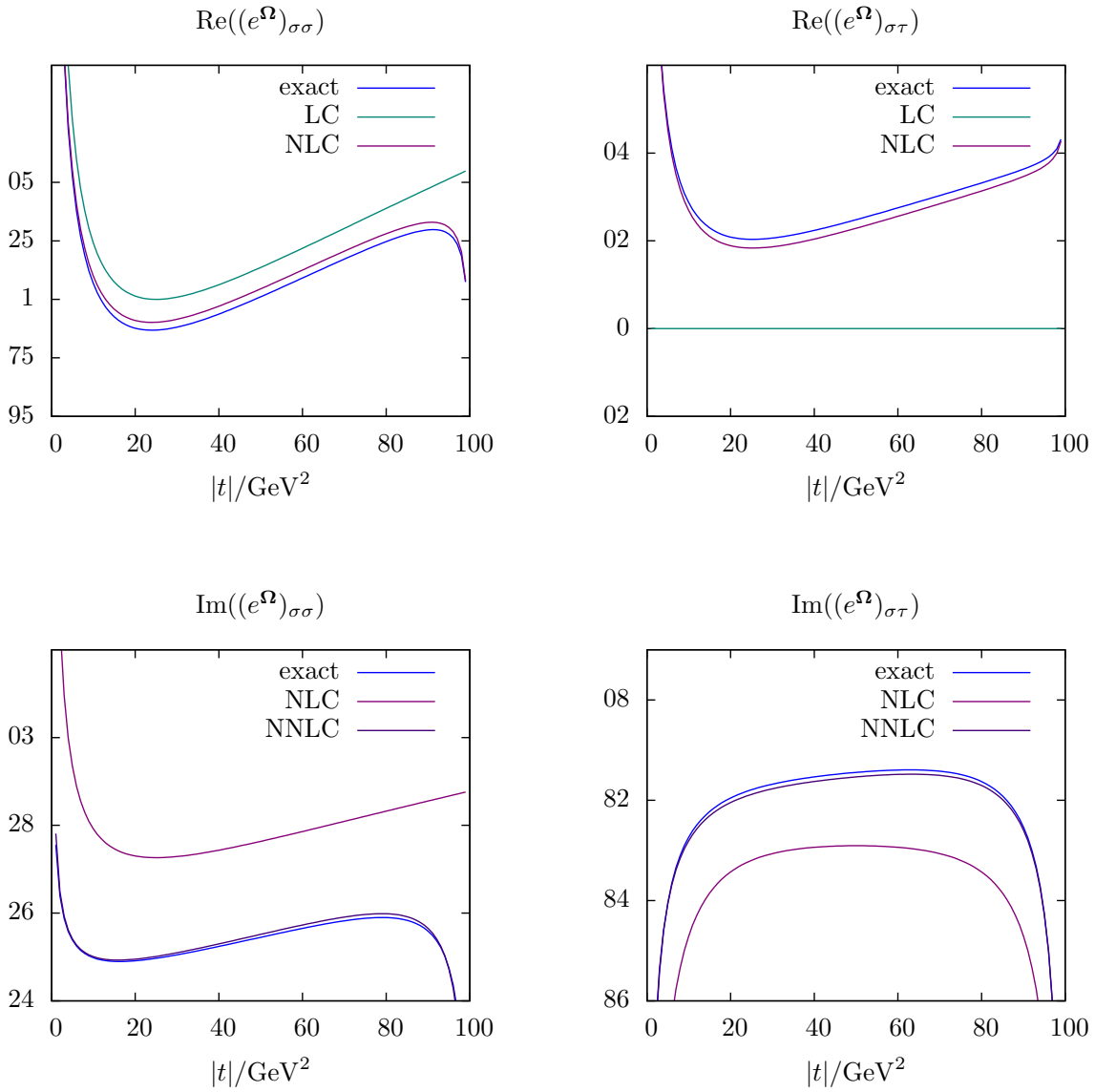
Testbed: (Dipole) shower emission veto in dijets $p_1, p_2 \to p_3, p_4$.
Each of the possible II, IF, FF dipoles can radiate with $\mu^2 < p_\perp^2 \lesssim 2p_i \cdot p_j$.

$$\begin{aligned}
\Gamma^{12} = \Gamma^{34} &= \frac{\alpha_s}{4\pi}\left(\frac{1}{2}\ln^2\frac{s}{\mu^2} - i\pi\ln\frac{s}{\mu^2}\right) \\
\Gamma^{13} = \Gamma^{24} &= \frac{\alpha_s}{8\pi}\ln^2\frac{|t|}{\mu^2} \\
\Gamma^{14} = \Gamma^{23} &= \frac{\alpha_s}{8\pi}\ln^2\frac{|u|}{\mu^2}
\end{aligned}$$

Exact solution for $q\bar{q}q\bar{q}$ amplitudes can be obtained straightforwardly.
Now check convergence of the N^{d}LC approximations.

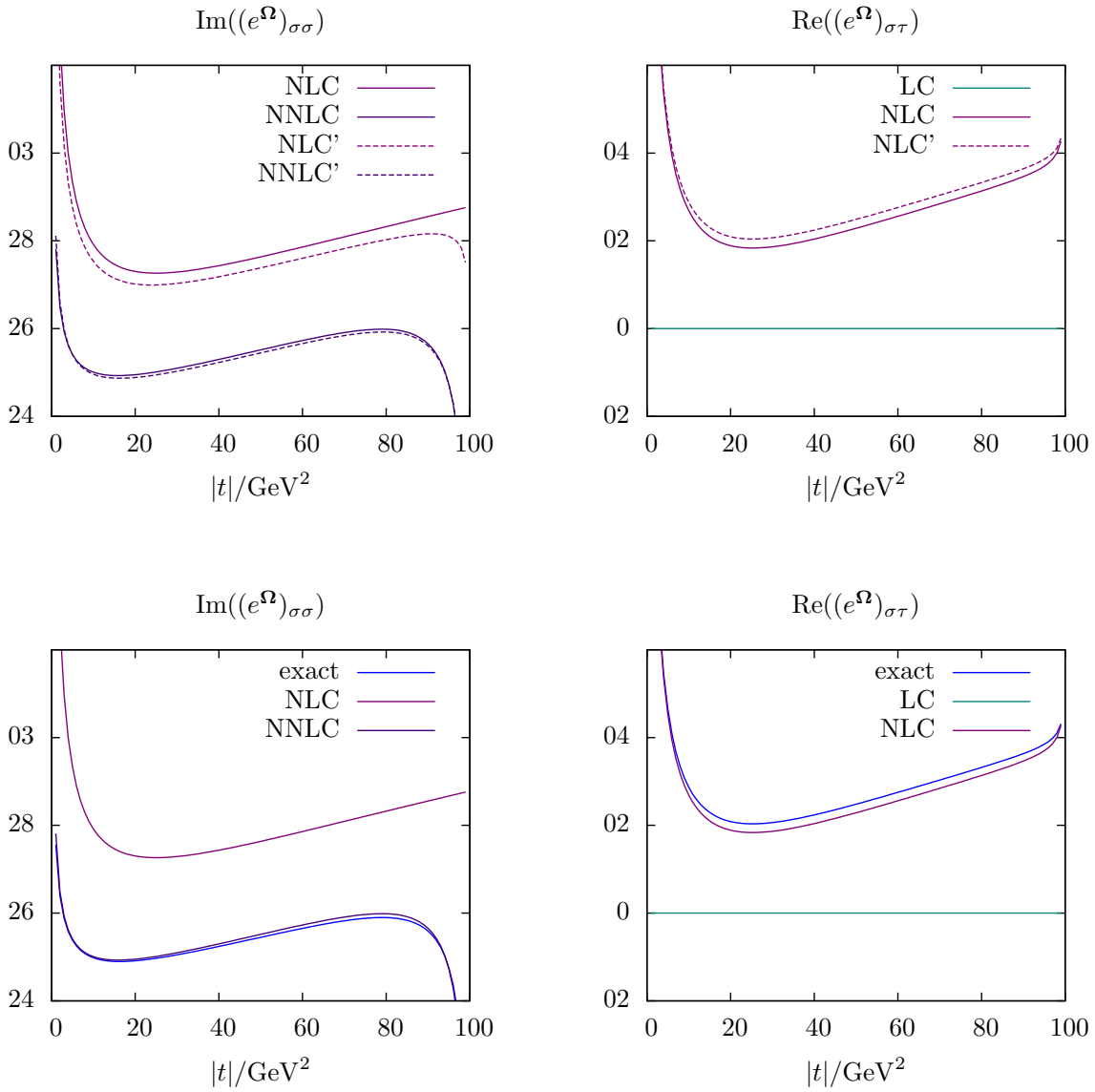
Numerical results.

C++ library CVolver will be public at some point



Numerical results: Primed resummation.

C++ library CVolver will be public at some point



Another observation.

Look at the exact solution for two colour flows:

$$e^{\Gamma} \equiv \frac{e^{\frac{1}{N}\rho} e^{-\frac{N}{2}(\Gamma_{12}+\Gamma_{21})}}{\kappa} \begin{pmatrix} -\Delta \sinh\frac{\kappa}{2} + \kappa\cosh\frac{\kappa}{2} & 2\Sigma_{1212}\sinh\frac{\kappa}{2} \\ 2\Sigma_{1221}\sinh\frac{\kappa}{2} & \Delta\sinh\frac{\kappa}{2} + \kappa\cosh\frac{\kappa}{2} \end{pmatrix}$$

$\Delta = N(\Gamma_{12} - \Gamma_{21})$, $\kappa = \sqrt{\Delta^2 + 4\Sigma_{1212}\Sigma_{1221}}$

Then find:

$$e^{\Gamma} \to e^{\Gamma}\Big|_{\text{NLC'}} \qquad \Delta^2 \gg 4\Sigma_{1212}\Sigma_{1221}$$

Delicate correlation between kinematic limits and large-N limit.
Hint why large-N for showers seems to work quite well for a lot of observables?
Note showers do primed evolution: Primed vs. non-primed is essentially C_F vs $C_A/2$.

Applications.

Soft gluon evolution, exact (few legs):

- Colour structure of anomalous dimension in flow basis.
- Use powerful colour flow based ME generators.

Soft gluon evolution, approximate (many legs):

- Check convergence of successive approximations.
- No matrix exponentiation, only (sparse) matrix multiplication needed.
- Ultimately: Do this within shower algorithms $\rightarrow$ NLL @ $\mathrm{N}^d\mathrm{LC}$

No need for linear algebra operations by performing a Monte Carlo over colour flows.

Not covered here: Insight into colour reconnection models.
So far only based on simple phenomenological reasoning.

Conclusions.

Soft gluon evolution in colour flow basis has a very transparent structure.

Exponentiation of the anomalous dimension can be performed approximately,
by resumming large-N enhanced terms.

Checking successive approximations for convergence,
evolution of a large number of legs is feasible.

Outlook:
Study more complicated colour correlations.
Identify phase space regions where subleading-N is (un)important.