Factorization and parton distribution functions

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What is a parton distribution function?

- $f_{i/h}(\xi, \mu^2)\, d\xi$ is supposed to represent the probability to find
  - a parton of flavor $i \in \{\mathrm{g}, \mathrm{u}, \bar{\mathrm{u}}, \mathrm{d}, \dots\}$ in a hadron of flavor h
  - carrying a fraction ξ of the hadron's momentum
  - when measured at a resolution scale μ^2.

- It is defined as the expectation value in a hadron state of a certain operator.

The definition (quarks)

- For hadron h moving in the $+z$ direction with momentum p.
- $v^{\pm} = (v^0 \pm v^3)/\sqrt{2}$.
- So p^+ is big and $p^- = m_h^2/(2p^+)$ is small.

$$f_{i/h}(\xi, \mu^2) = \frac{1}{2} \int \frac{dy^-}{2\pi}\; e^{i\xi p^+ y^-} \langle p \big| \bar{\psi}_i(0, y^-, \mathbf{0}) \gamma^+ F \psi_i(0) \big| p \rangle$$

$$F = \mathcal{P} \exp\left(-ig \int_0^{y^-} dz^- A_a^+(0, z^-, \mathbf{0}) t_a \right)$$

- Renormalize with the $\overline{\mathrm{MS}}$ prescription with scale μ^2.

Why?

$$f_{i/h}(\xi,\mu^2) = \frac{1}{2}\int \frac{dy^-}{2\pi}\; e^{i\xi p^+ y^-} \langle p \big| \bar{\psi}_i(0,y^-,\mathbf{0})\gamma^+ F \psi_i(0) \big| p\rangle$$

$$F = \mathcal{P}\exp\left(-ig\int_0^{y^-} dz^-\, A_a^+(0,z^-,\mathbf{0})t_a\right)$$

- Consider $A^+(x) = 0$ gauge, so $F = 0$.
- Use null-plane based field theory based on "time" $= x^+$.

- Fourier expansion of $\psi(x)$ on the $x^+ = 0$ plane is

$$
\begin{aligned}
&\psi_{\mathrm{good}}(x) \\
&= (2\pi)^{-3} \int_0^\infty \frac{k^+}{k^+} \int d\boldsymbol{k}_\perp \sum_s \Big\{ e^{-i(k^+ x^- - \boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp)} U_{\mathrm{good}}(k,s) b(k,s) \\
&\qquad + e^{+i(k^+ x^- - \boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp)} V_{\mathrm{good}}(k,s) d^\dagger(k,s) \Big\}
\end{aligned}
$$

- This applies to $\psi_{\mathrm{good}}(x) = \frac{1}{2}\gamma^-\gamma^+\psi(x)$.

- $b(\xi p^+, \boldsymbol{k}_\perp, s)$ is a quark destruction operator.
- $b^\dagger(\xi p^+, \boldsymbol{k}_\perp, s)$ is a quark creation operator.

- They obey canonical anticommutation relations.

- The operator

$$\rho(\xi p^+, \boldsymbol{k}_\perp, s) = \frac{1}{(2\pi)^3 \xi}\; b^\dagger(\xi p^+, \boldsymbol{k}_\perp, s) b(\xi p^+, \boldsymbol{k}_\perp, s)$$

measures the number density of quarks:

$$\rho(\xi p^+, \boldsymbol{k}_\perp, s) d\xi \, d\boldsymbol{k}$$

measures the number of quarks with momentum fraction in the interval $d\xi$ and transverse momentum in the range $d\boldsymbol{k}$.

- With a little manipulation of

$$f_{i/h}(\xi,\mu^2) = \frac{1}{2}\int \frac{dy^-}{2\pi}\; e^{i\xi p^+ y^-} \langle p|\bar{\psi}_i(0,y^-,\mathbf{0})\gamma^+\psi_i(0)|p\rangle$$

and

$$\rho(\xi p^+,\boldsymbol{k}_\perp,s) = \frac{1}{(2\pi)^3\xi}\; b^\dagger(\xi p^+,\boldsymbol{k}_\perp,s)b(\xi p^+,\boldsymbol{k}_\perp,s)$$

we obtain

$$f_{i/h}(\xi,\mu^2)\langle p'|p\rangle = \int d\boldsymbol{k}_\perp \sum_s \langle p'|\rho(\xi p^+,\boldsymbol{k}_\perp,s)|p\rangle$$

- We used a special gauge, $A^+(z) = 0$.
- But if we change the definition to

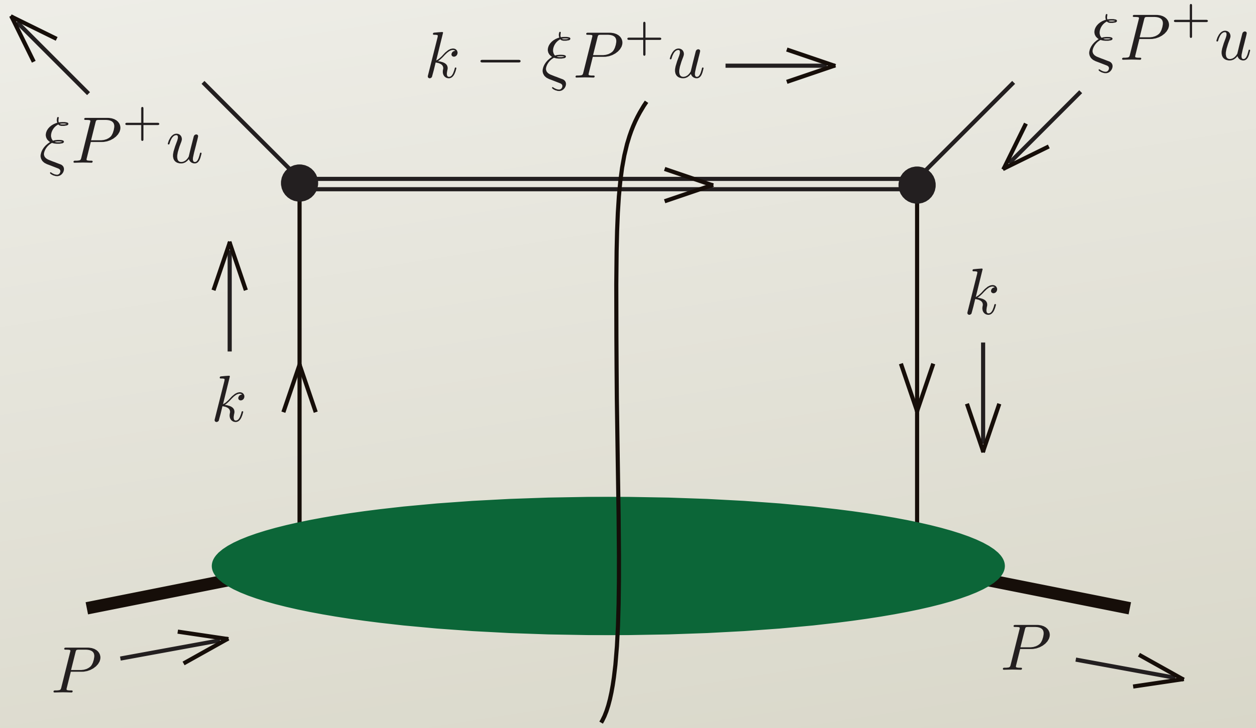
$$f_{i/h}(\xi,\mu^2) = \frac{1}{2}\int \frac{dy^-}{2\pi}\; e^{i\xi p^+ y^-} \langle p \big| \bar{\psi}_i(0,y^-,\mathbf{0})\gamma^+ F \psi_i(0) \big| p\rangle$$

$$F = \mathcal{P}\exp\left(-ig\int_0^{y^-} dz^- A_a^+(0,z^-,\mathbf{0})t_a\right)$$

- the definition is gauge invariant.

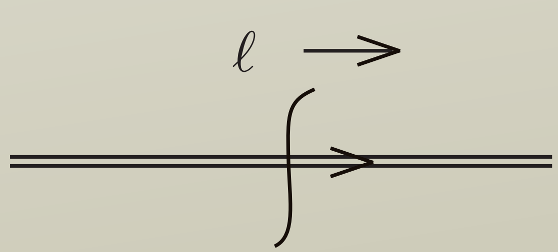
Feynman diagrams

- We can use Feynman diagrams for $f_{i/h}(\xi, \mu^2)$.



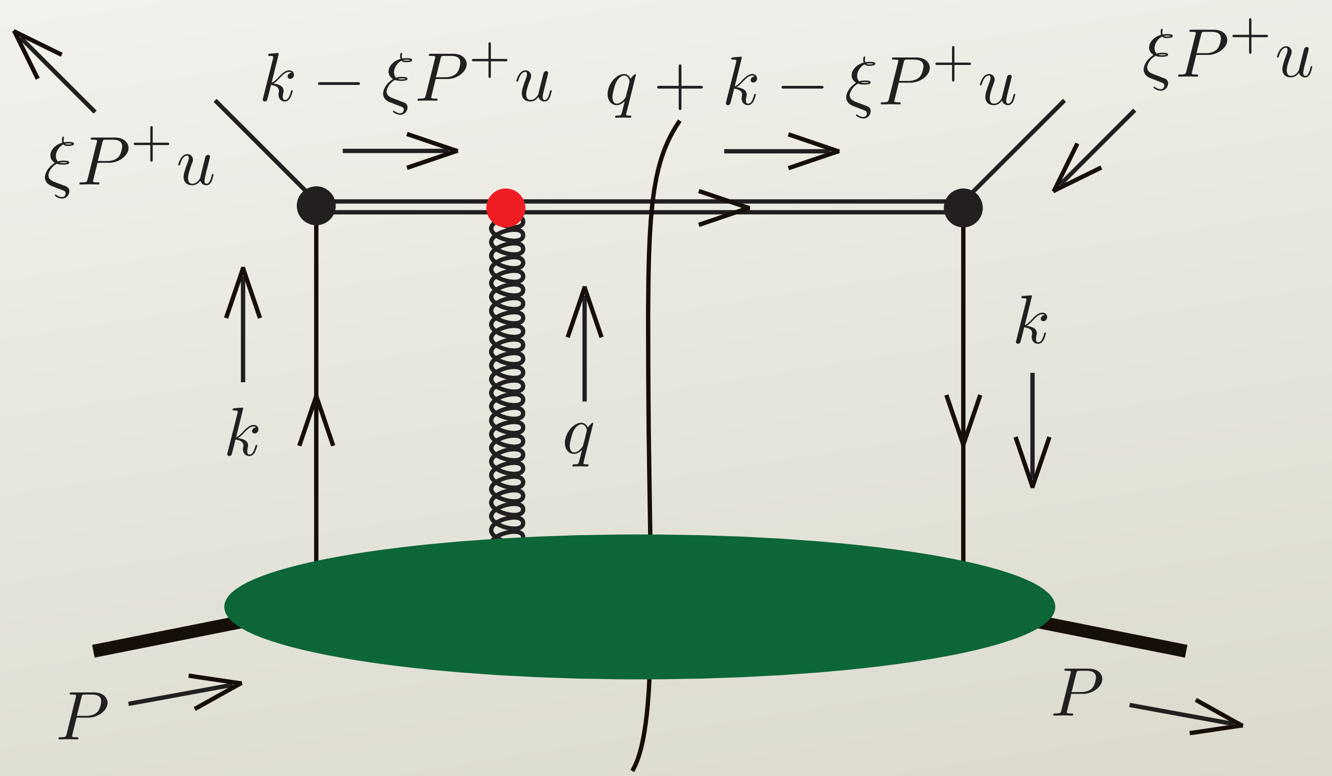
$$u^+ = 1$$
$$u^- = 0$$
$$u_\perp = 0$$

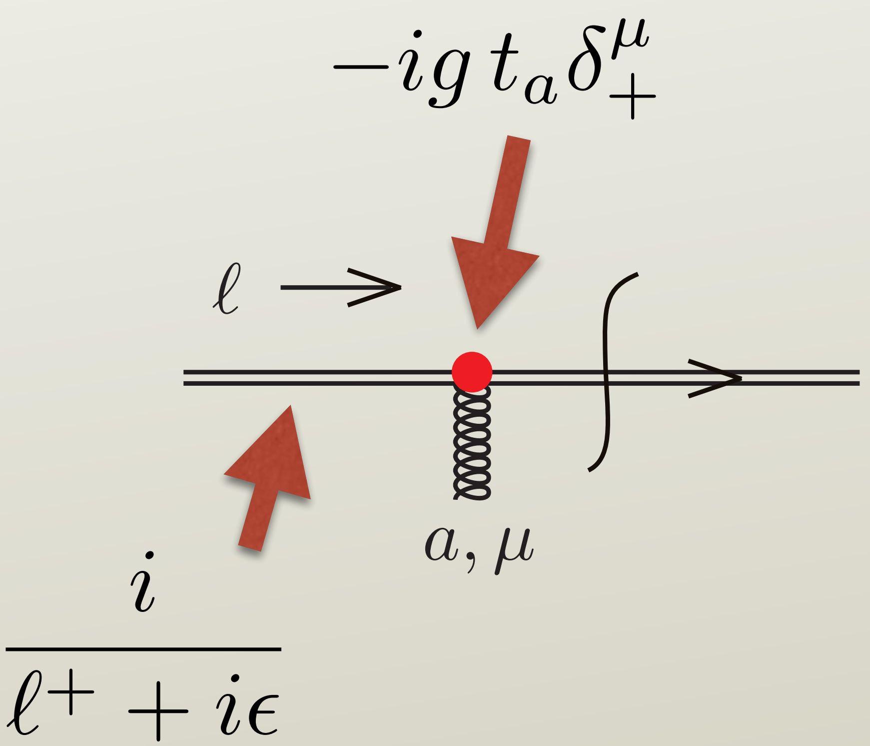
- There is a loop integration $\displaystyle\int \frac{d^4k}{(2\pi)^4}$
- The cut "eikonal line" is



$$= (2\pi)\delta(\ell^+)\ 1_{\rm color}\gamma^+$$

The eikonal line absorbs gluons

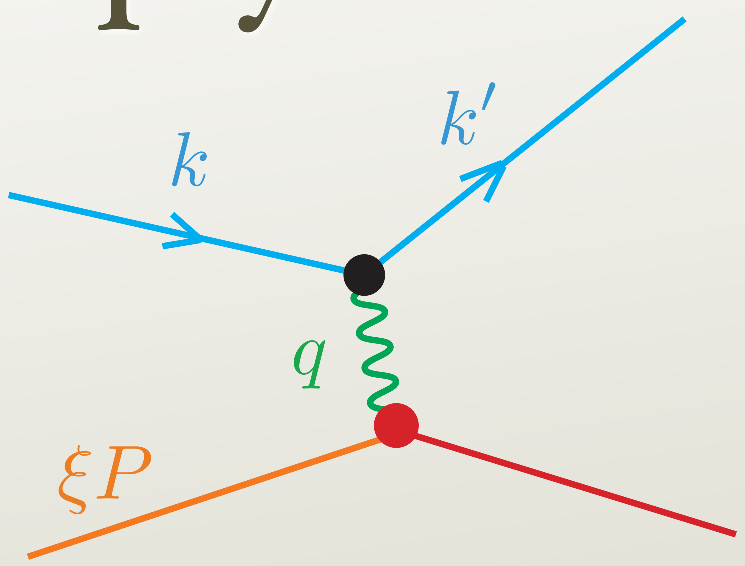




$$F = \mathcal{P} \exp\left(-ig \int_0^\infty dz^- A_a^+(0, z^-, \mathbf{0}) t_a \right)$$

$$\times\, \overline{\mathcal{P}} \exp\left(-ig \int_\infty^{y^-} dz^- A_a^+(0, z^-, \mathbf{0}) t_a \right)$$

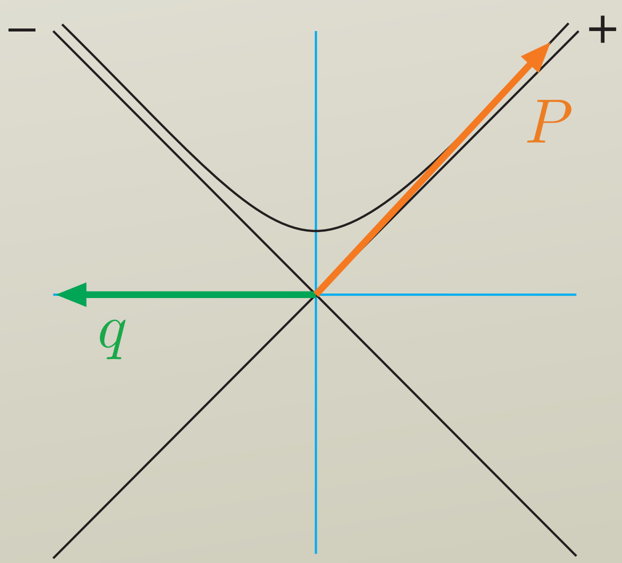
Look at deeply inelastic scattering



- A convenient reference frame is

$$(q^+, q^-, \boldsymbol{q}) = \frac{1}{\sqrt{2}}\,(-Q, Q, \mathbf{0})$$

$$(P^+, P^-, \boldsymbol{P}) \approx \frac{1}{\sqrt{2}}\,\left(\frac{Q}{x}, \frac{x m_h^2}{Q}, \mathbf{0}\right)$$



- Hadron momentum is big; momentum transfer is big.

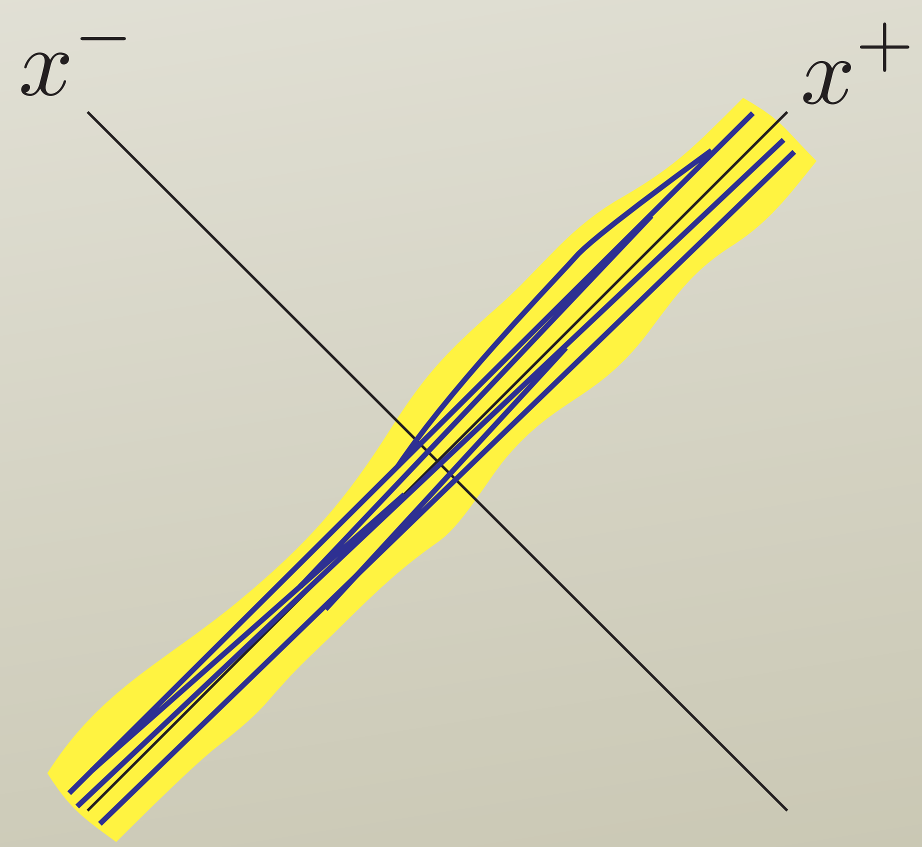
- Picture for a fast moving hadron is Lorentz transformation from rest frame picture.

$$e^{\omega} = \frac{P^+}{P^+_{\rm rest}} = \frac{Q}{mx}$$

- Separations Δx^μ between interactions:

$$\Delta x^+ \sim \frac{1}{m} \times \frac{Q}{mx} = \frac{Q}{m^2 x}$$

$$\Delta x^- \sim \frac{1}{m} \times \frac{mx}{Q} = \frac{x}{Q}$$



- The hadron meets the virtual photon.

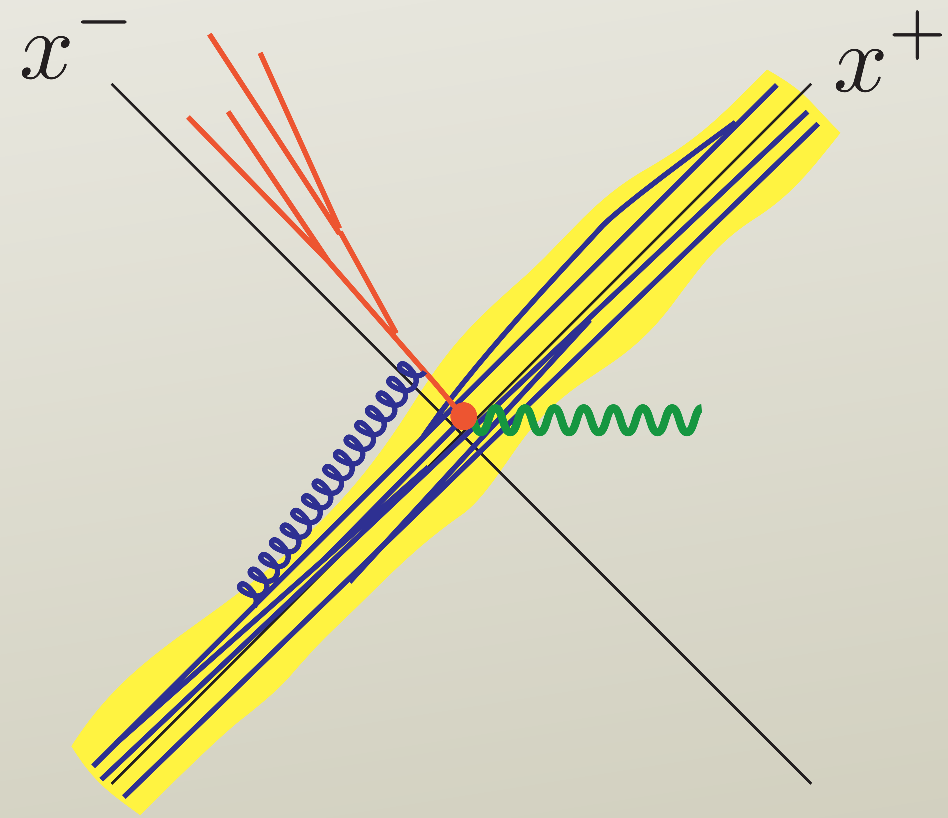
- Virtual photon has $q^- \sim Q$ so its interaction takes place over

$$\Delta x^+ \sim 1/Q$$

- But interactions in the proton happen at a scale

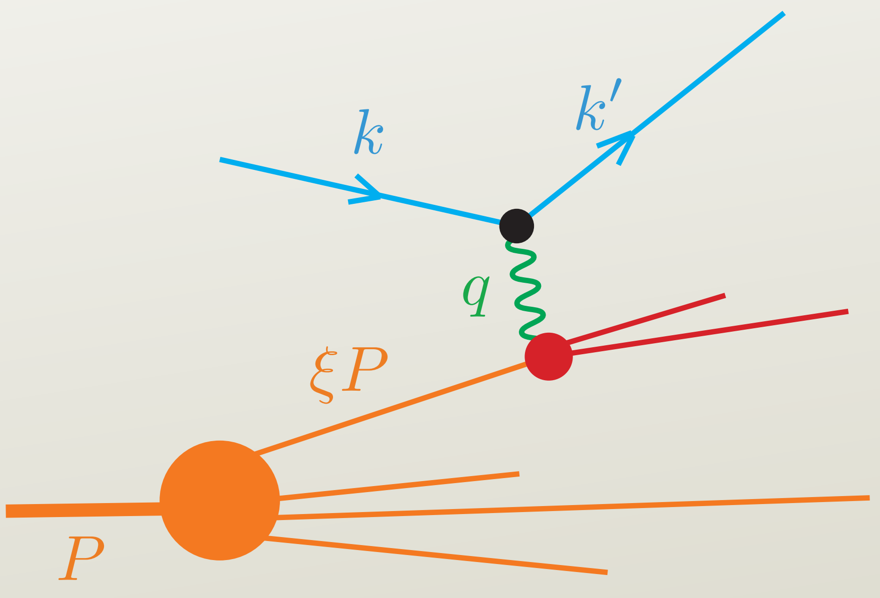
$$\Delta x^+ \sim Q/(m^2 x)$$

- so the "partons" in the hadron are effectively free as seen by the virtual photon.



Factored cross section

- This picture gives

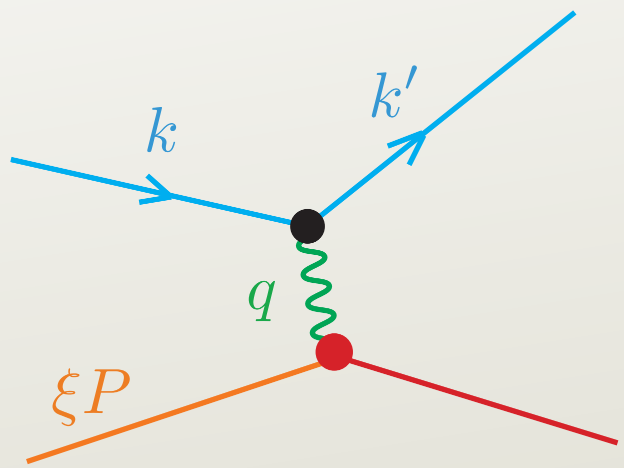


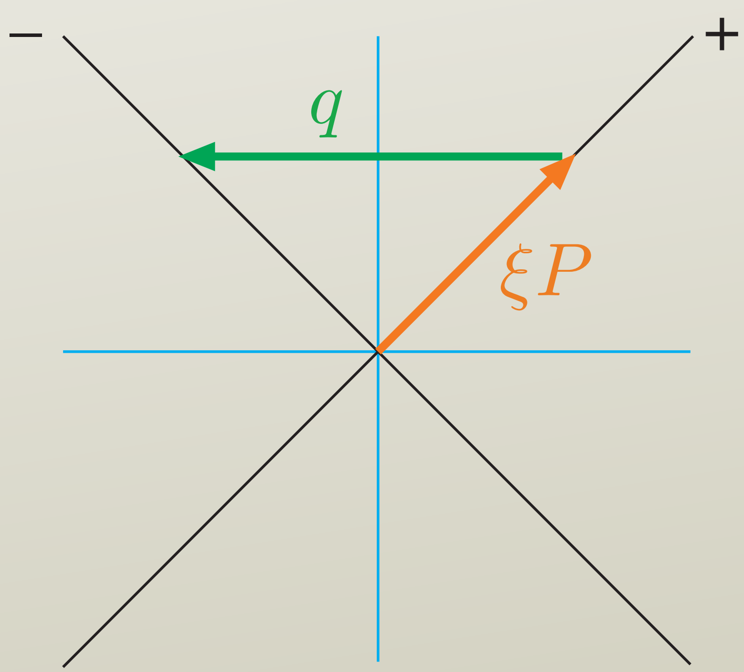
$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi,\mu)\; \frac{d\hat{\sigma}_a(\mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q)$$

$f_{a/h}(\xi,\mu)\; d\xi =$ probability to find a parton
with flavor $a = g, u, \bar{u}, d, \ldots,$
in hadron h,
carrying momentum fraction $\xi = p_i^+/p^+$.

$d\hat{\sigma}_a/dE'\,d\omega' =$ cross section for scattering that parton.

Kinematics of the leading order diagram





$$\xi P^+ + q^+ = 0$$

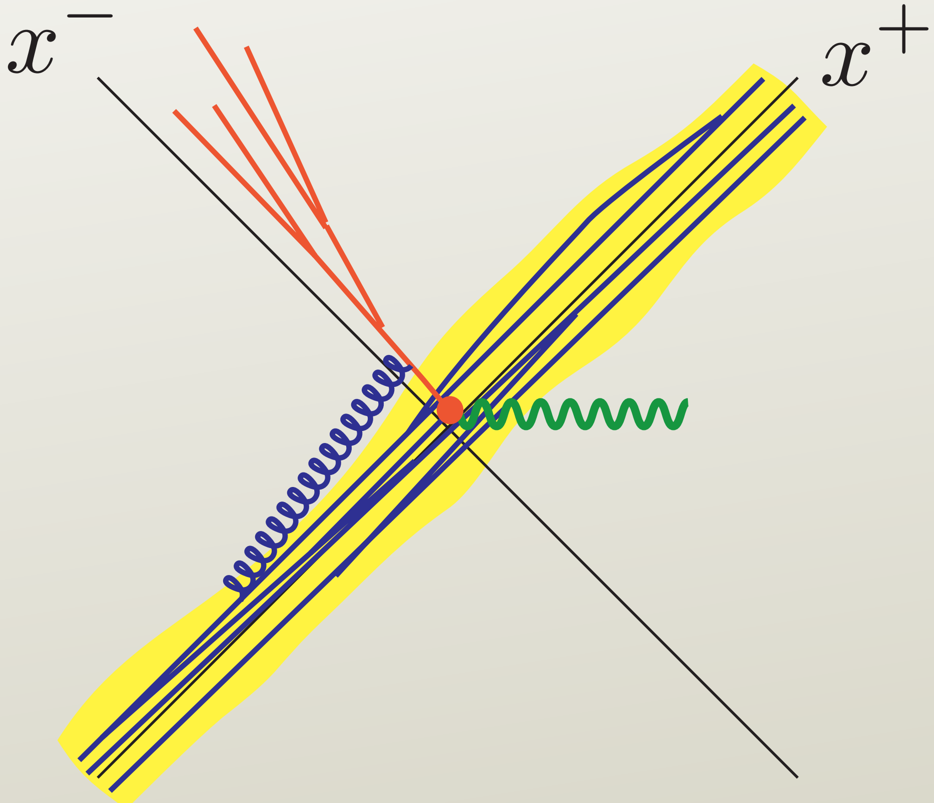
$$P^+ = \frac{Q}{x\sqrt{2}}$$

$$q^+ = -\frac{Q}{\sqrt{2}}$$

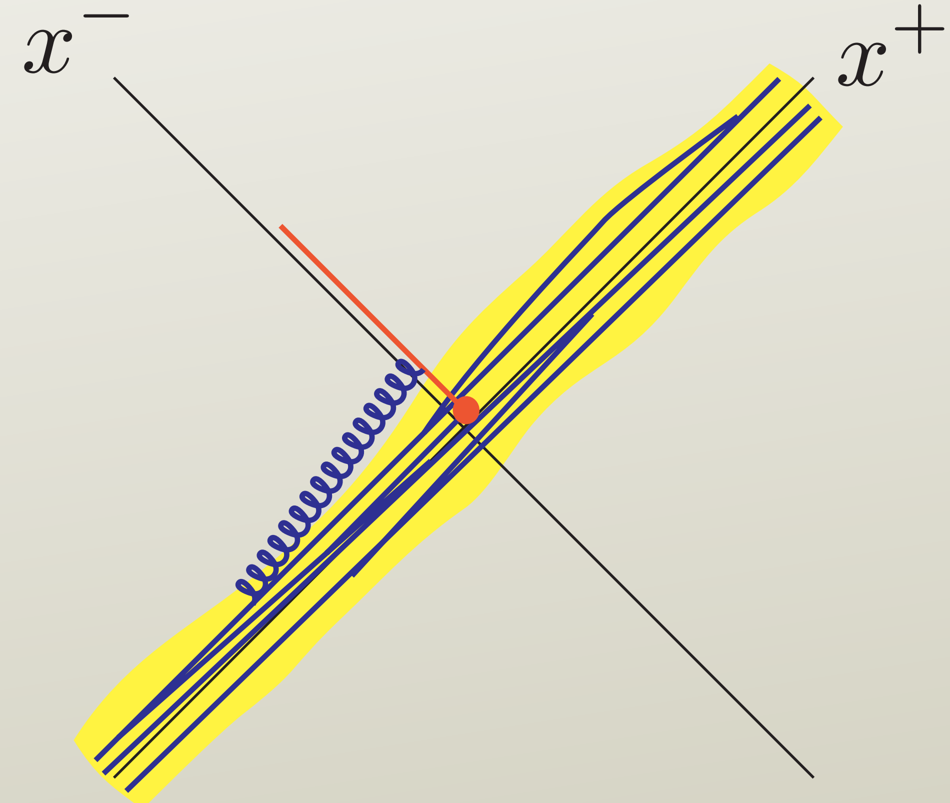
So

$$\boxed{\xi = x}$$

Physical role of the eikonal line

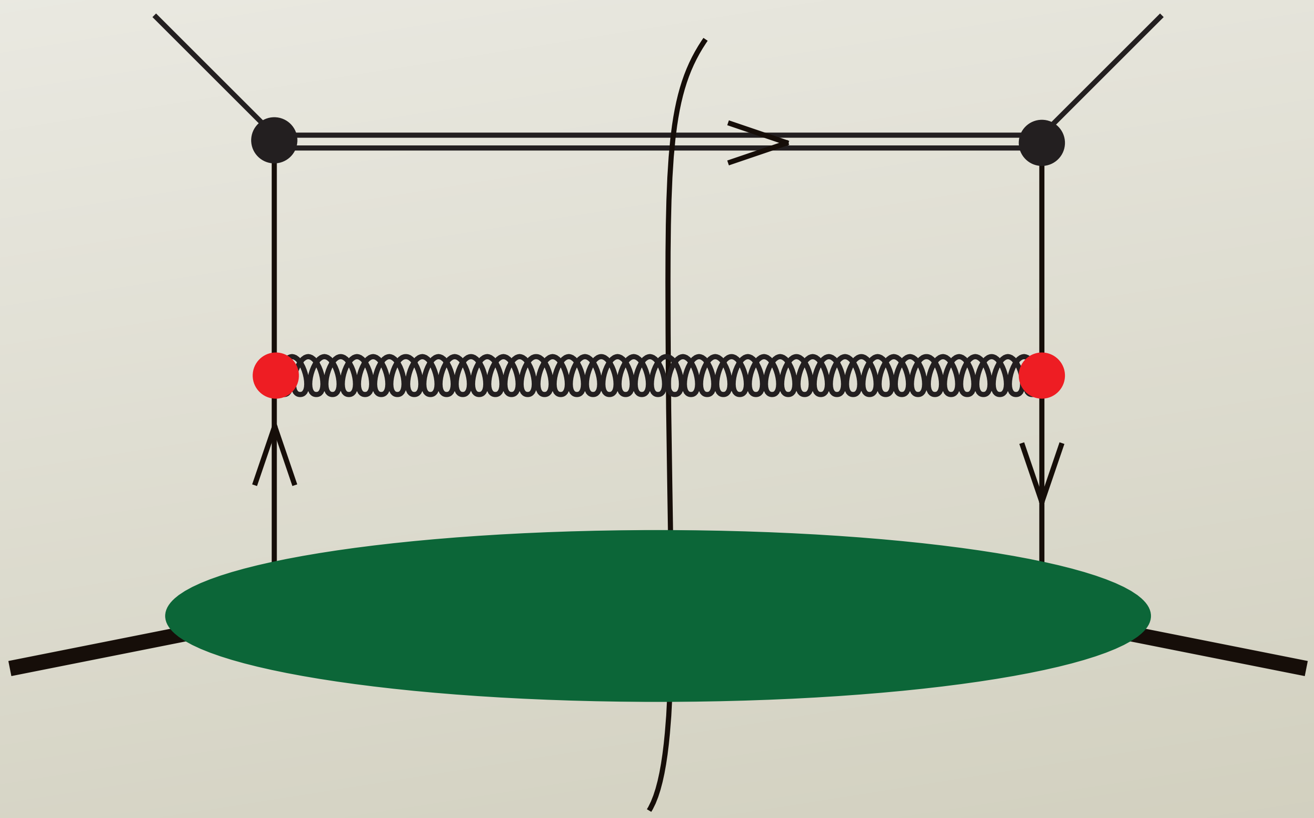


DIS



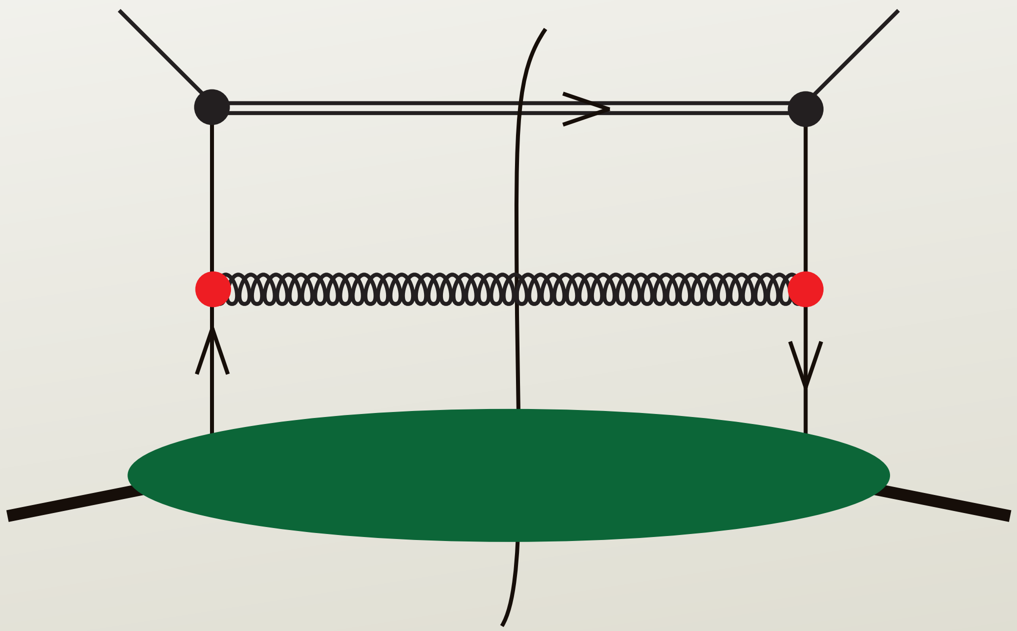
quark distribution function

The parton distribution function is ultraviolet divergent



- A one loop diagram with a divergent loop integration.

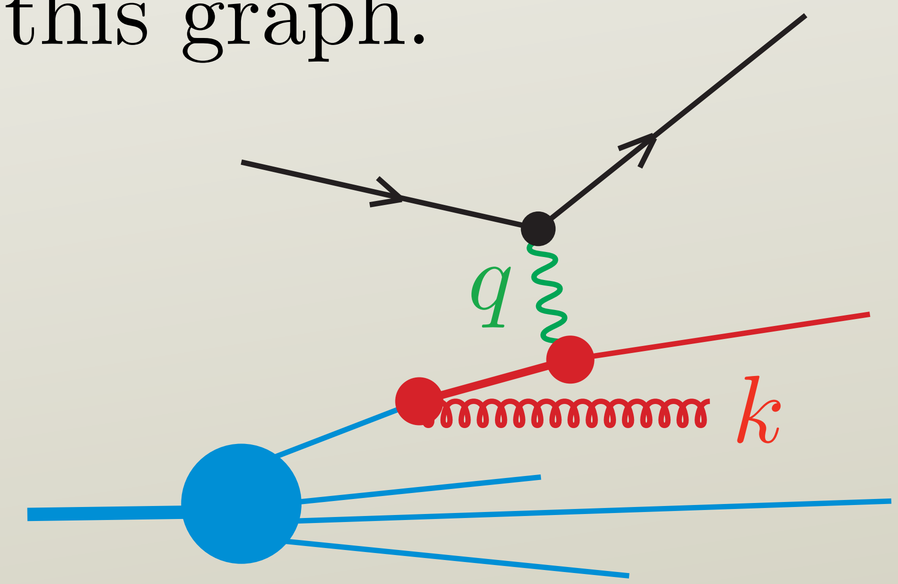
Renormalize it



- Use the $\overline{\text{MS}}$ prescription.
- This gives "$\overline{\text{MS}}$" parton distributions.
- Calculate in $4 - 2\epsilon$ dimensions.
- Subtract poles $1/\epsilon^n$.
- $f_{i/h}(\xi, \mu^2)$ now depends on a scale μ^2, often called $\mu_{\rm F}^2$

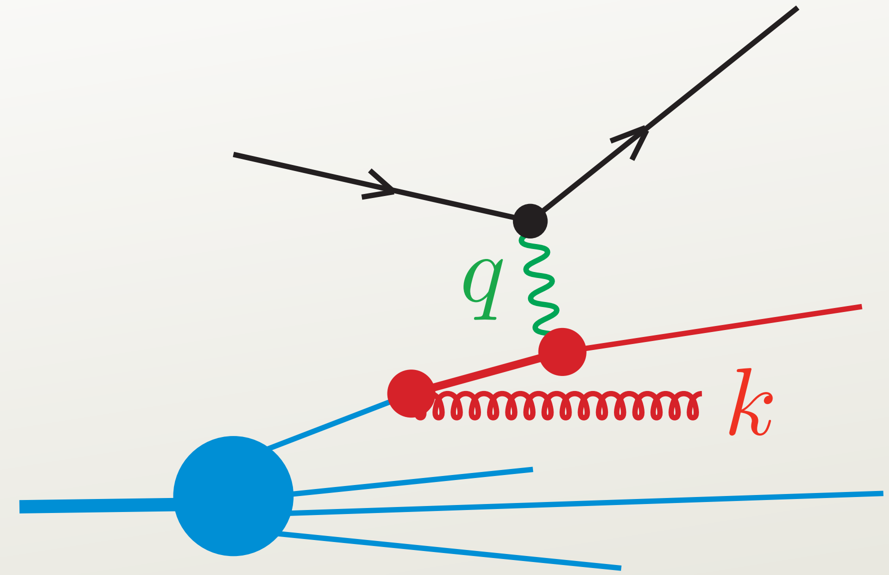
The factorization scale in DIS

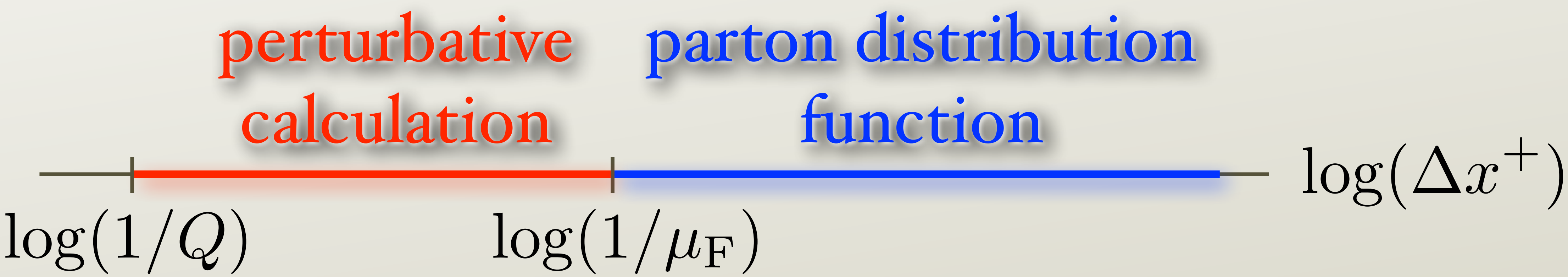
- We argued that $\Delta x^+ \sim Q/(xm^2) \gg 1/Q$.
- Thus we regarded the partons as "frozen."
- But look at this graph.



- Integration over k maps to integration over Δx^+
- $1/Q \lesssim \Delta x^+ \lesssim Q/(xm^2)$.
- So we were wrong.

- Solution: divide up the integration region.



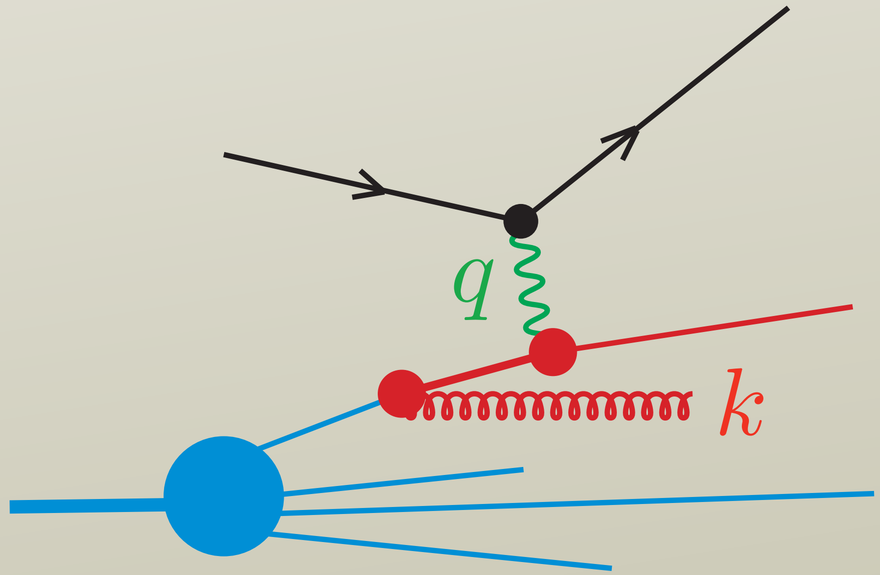


- We call $\mu_{\rm F}$ the factorization scale.
- $\overline{\rm MS}$ subtractions do not really give a hard cut.
- But the effect is similar.
- Note that an upper bound on $k_\perp^2$ included in the PDF corresponds to a lower bound on $k_\perp^2$ in the hard matrix element.

- Both $f_{a/h}(\xi,\mu_{\mathrm{F}})$ and $d\hat{\sigma}_a(\mu_{\mathrm{F}},\mu)$ depend on μ_{F}.

$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi,\mu_{\mathrm{F}}) \; \frac{d\hat{\sigma}_a(\mu_{\mathrm{F}},\mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q)$$

- This formula tells how to calculate $d\hat{\sigma}$.



Renormalization group equation

- The $\overline{\mathrm{MS}}$ renormalization of $f_{i/h}(x,\mu^2)$ introduces the scale μ^2.
- This determines how $f_{i/h}(x,\mu^2)$ depends on μ^2:

$$\mu^2 \frac{d}{d\mu^2} f_{i/h}(x,\mu^2) = \sum_j \int_x^1 \frac{d\xi}{\xi}\, P_{ij}(x/\xi, \alpha_s(\mu^2))\, f_{j/h}(\xi,\mu^2)$$

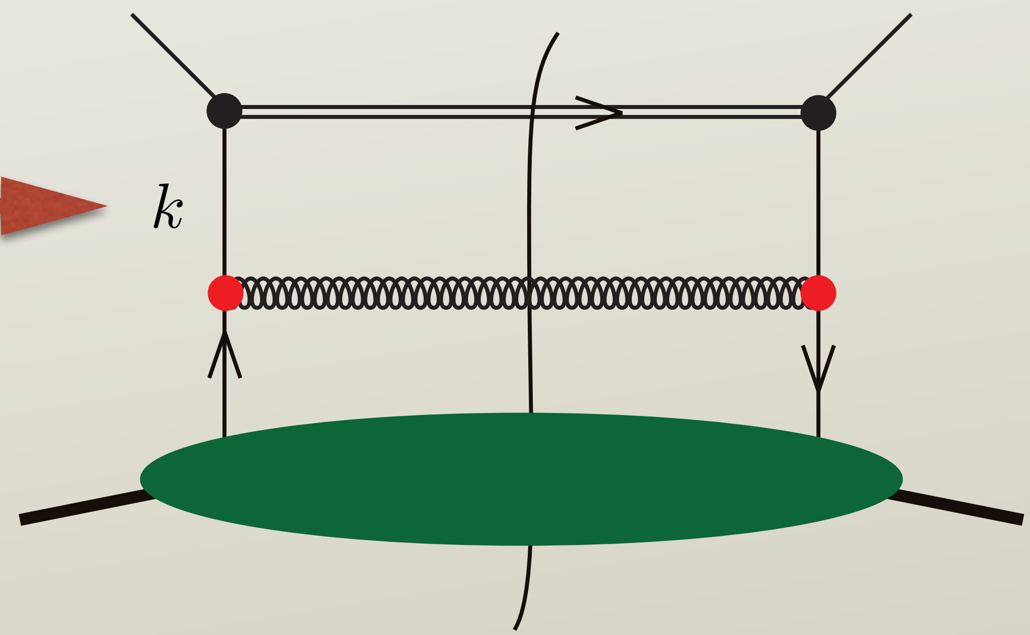
- Here $P_{ij}(x/\xi, \alpha_s(\mu^2))$ is the DGLAP evolution kernel.

$$P_{ij}(x/\xi, \alpha_s(\mu^2)) = \frac{\alpha_s(\mu^2)}{2\pi} P_{ij}^{(1)}(x/\xi) + \left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2 P_{ij}^{(2)}(x/\xi) + \cdots$$

Quark masses

- The Feynman diagrams for $f_{i/h}(x,\mu^2)$ include quark masses.

$$\frac{\not{k}+m}{k^2-m^2+i\epsilon}$$



- But for very large k, m^2 is negligible.
- So the renormalization counter terms do not depend on m.
- So $P_{ij}(x/\xi,\alpha_s(\mu^2))$ does not depend on m.

- You can use $f_{i/h}(x,\mu^2)$ including $i = \mathrm{b}$ or not including $i = \mathrm{b}$.

- You should use $f_{i/h}(x,\mu^2)$ including $i = \mathrm{b}$ when $\mu^2 \gg m_{\mathrm{b}}^2$.

- You should not use $f_{i/h}(x,\mu^2)$ including $i = \mathrm{b}$ when $\mu^2 \approx m_{\mathrm{b}}^2$ or $\mu^2 < m_{\mathrm{b}}^2$.

- There are matching conditions between the two sets of $f_{i/h}(x,\mu^2)$.

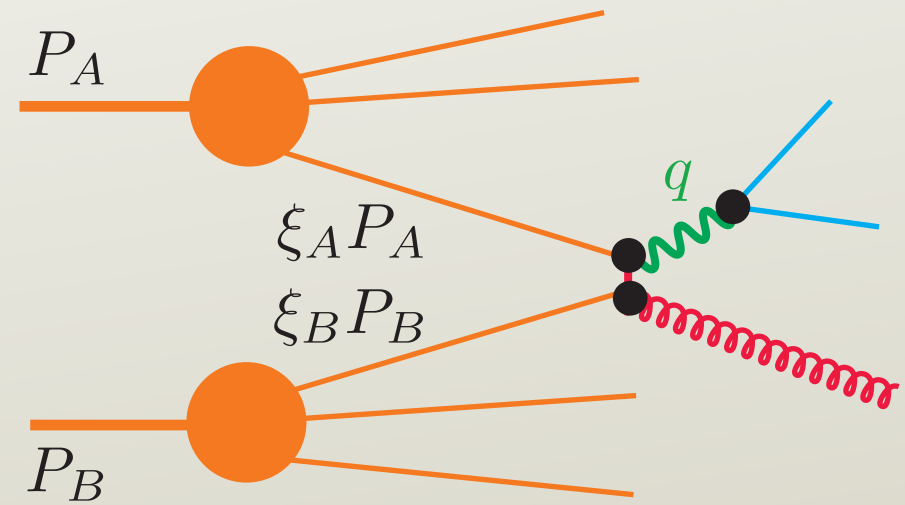
- When $i = \mathrm{b}$ is included, $f_{\mathrm{b}/h}(x,\mu^2) \to 0$ for $\mu^2 \to m_{\mathrm{b}}^2$.

- There are ways to make cross sections interpolate smoothly.

Factorization in hadron-hadron collisions

- Consider $d\sigma/dy$ for

$$A + B \to Z + X$$



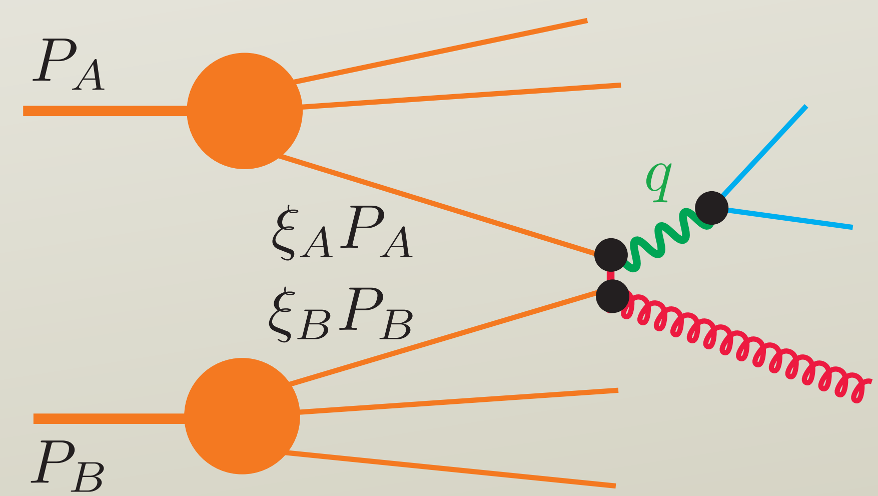
- Factored form of cross section

$$\frac{d\sigma}{dy} = \sum_{a,b} \int_{x_{\mathrm{A}}}^{1} d\xi_{\mathrm{A}} \int_{x_{\mathrm{B}}}^{1} d\xi_{\mathrm{B}}\; f_{a/\mathrm{A}}(\xi_{\mathrm{A}}, \mu^2) f_{b/\mathrm{B}}(\xi_{\mathrm{B}}, \mu^2)\; \frac{d\hat{\sigma}_{ab}(\mu^2)}{dy}$$
$$+ \mathcal{O}(m/M)$$

$$x_A = e^{y}\sqrt{M^2/s} \qquad x_B = e^{-y}\sqrt{M^2/s}$$

$$\frac{d\sigma}{dy} = \sum_{a,b} \int_{x_{\rm A}}^{1} d\xi_{\rm A} \int_{x_{\rm B}}^{1} d\xi_{\rm B}\; f_{a/{\rm A}}(\xi_{\rm A},\mu^2) f_{b/{\rm B}}(\xi_{\rm B},\mu^2)\, \frac{d\hat\sigma_{ab}(\mu^2)}{dy}$$
$$+\, \mathcal{O}(m/M)$$

- The factored formula has power suppressed corrections.
- When $d\hat\sigma_{ab}/dy$ is evaluated at order α_s^n, there are also corrections of order α_s^{n+1}.



- We integrate over $\boldsymbol{q}_T$. The Z boson has mostly $\boldsymbol{q}_T^2 \lesssim M^2$.

Scale dependence

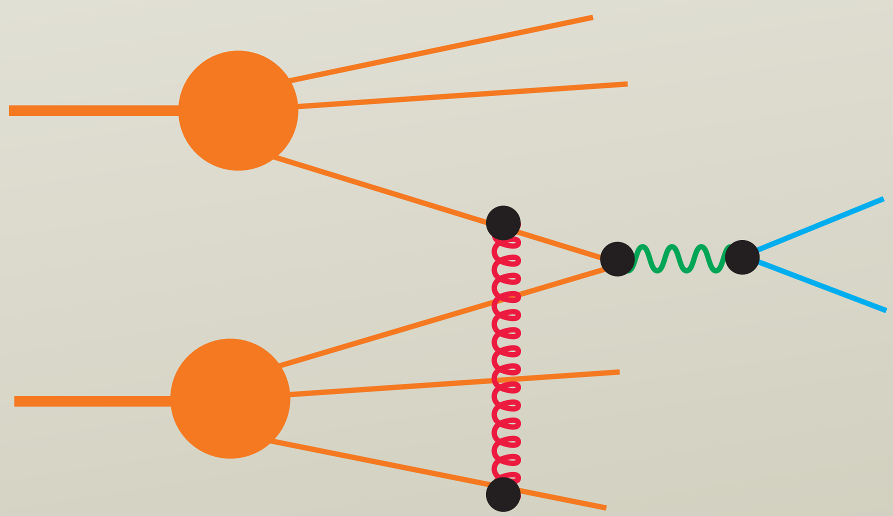
$$\frac{d\sigma}{dy} = \sum_{a,b} \int_{x_\mathrm{A}}^{1} d\xi_\mathrm{A} \int_{x_\mathrm{B}}^{1} d\xi_\mathrm{B}\ f_{a/\mathrm{A}}(\xi_\mathrm{A},\mu^2) f_{b/\mathrm{B}}(\xi_\mathrm{B},\mu^2)\ \frac{d\hat{\sigma}_{ab}(\mu^2)}{dy}$$
$$+ \mathcal{O}(m/M)$$

- If evaluated exactly, $d\sigma/dy$ does not depend on μ^2.
- But if $d\hat{\sigma}_{ab}(\mu^2)/dy$ is evaluated at order n, then there are corrections of order α_s^{n+1}.
- So the derivative with respect to $\log(\mu^2)$ of $d\sigma/dy$ is of order α_s^{n+1}.
- It is usually best to choose μ^2 of the order of the hard scale in the process.

Discussion of factorization

$$\frac{d\sigma}{dy} = \sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B \; f_{a/A}(\xi_A,\mu)\; f_{b/B}(\xi_B,\mu)\; \frac{d\hat{\sigma}_{ab}(\mu)}{dy}$$
$$+\mathcal{O}(m/M)$$

- This is not obvious.

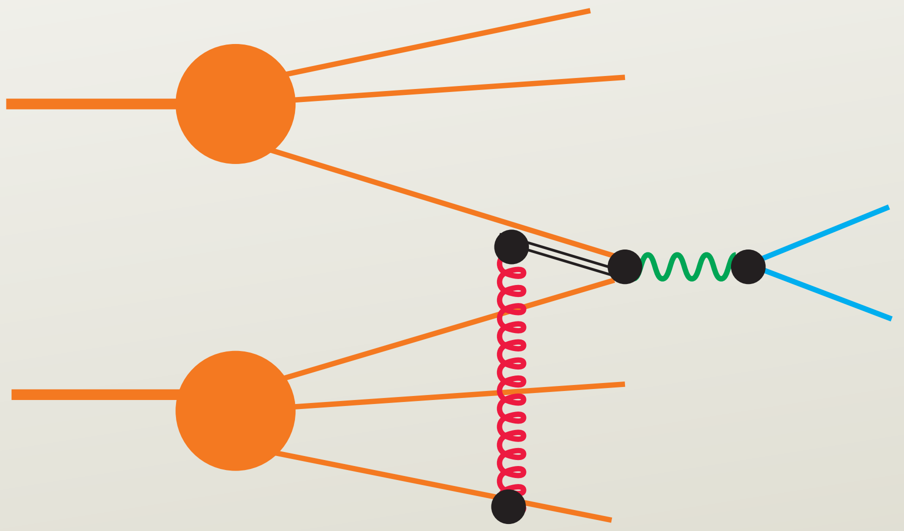


- Need unitarity, causality, gauge invariance.

Understanding this example

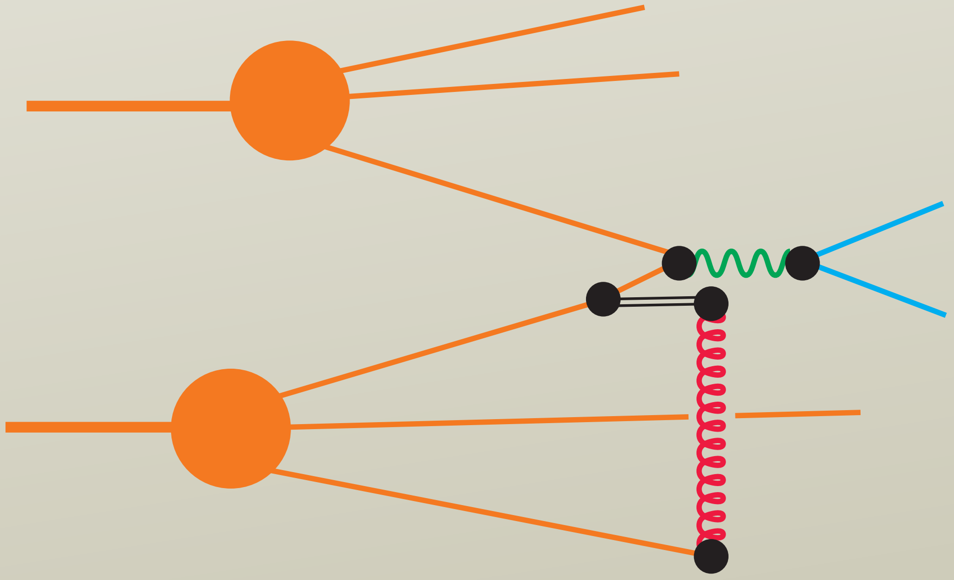
- The soft gluon attachment to the incoming quark is equivalent to an attachment to an eikonal line.

- It is an incoming eikonal line.



- But that can be replaced by an attachment to an outgoing eikonal line.

- The outgoing eikonal line is part of the definition of the parton distribution function.



Review

- A parton distribution function is defined as a matrix element in a hadron state of a certain operator.
- The operator needs renormalization, which leads to the DGLAP evolution equation.
- Cross sections factor into parton distribution functions convoluted with hard parton scattering cross sections.
- The factorization is based on unitarity, causality, and gauge invariance and on how the parton distribution functions are defined.