Instantons on Special Geometries

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Aim of my work:
Better understanding of instantons on special geometries

Instantons in Higher Dimensions

- **Recall:**
  instantons in four dimensions satisfy the anti-self-duality equation

$$*\mathcal{F} = -\mathcal{F}$$

$\mathcal{F}$: curvature of connection $\mathcal{A}$ on a principal G-bundle over M^4

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  Consider a **coset space** $X^n = G/H$
  G: *compact, semisimple Lie group*
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- **Generalization to higher dimensions:**
  Consider a **coset space** $X^n = G/H$
  G: *compact, semisimple Lie group*
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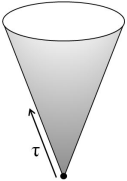
- We work with the **cone over this coset space**:

$$\mathcal{C}(G/H) = (\mathbb{R}_+ \times G/H, g_{cone})$$
$$g_{cone} = e^{2\tau}(d\tau^2 + g_{G/H})$$

τ: parameter in $\mathbb{R}$-direction



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- **Remark:** Instanton equation implies Yang-Mills equation with torsion

$$\mathcal{D} * \mathcal{F} + \underbrace{*\mathcal{H} \wedge \mathcal{F}}_{\text{Torsion}} = 0$$

where $*\mathcal{H} = d*Q$

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- Explicit higher-dimensional instanton solutions exist on Euclidean spaces and on certain conical manifolds
- Instantons on **special holonomy manifolds** can be lifted to heterotic supergravity solutions
- Special holonomy manifolds can be constructed as cones over
  - 5-dimensional Sasaki manifolds
  - 6-dimensional nearly Kähler manifolds
  - 7-dimensional 3-Sasakian manifolds
  - 7-dimensional nearly parallel G_2-manifolds

Many examples of these manifolds are **coset spaces**

Lechtenfeld et.al [arxiv 1108.3951, 1202.5046]
Bär [Commun. Math. Phys., 154(3):509-521, 1993]

Reduction to Matrix Equations

- **Instanton equation** $*\mathcal{F} = -(*Q) \wedge \mathcal{F}$ in components:

$$\mathcal{F}_{0a} = -\frac{1}{2} Q_{0acd} \mathcal{F}_{cd} \qquad \mathcal{F}_{ab} = \frac{1}{2} \left(Q_{0eab} Q_{0ecd} - Q_{abcd} \right) \mathcal{F}_{cd}$$

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- Ansatz for the **gauge potential**:

$$\mathcal{A} = e^i I_i + e^a X_a(\tau)$$

$X_a(\tau) = X_a^b(\tau) I_b$: matrix-valued function that satisfies the **invariance condition** $[I_i, X_a] = f_{ia}^b X_b$

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- **Curvature:**

$$\mathcal{F} = \dot{X}_a e^0 \wedge e^a - \frac{1}{2}\left(f_{ab}^i I_i + f_{ab}^e X_e - [X_a, X_b]\right) e^a \wedge e^b$$

Notation:

I_a	generators of G	$\{0, a\}$	indices on $\mathbb{R} \times G/H$
f_{ab}^c	structure constants of G	$\{i\}$	indices on H
e^a	coframe on $\mathbb{R} \times G/H$		

Reduction to Matrix Equations

- Assume $g_{G/H} \propto \delta_{ab} e^a e^b$ to define a 3-form:

$$f = \frac{1}{3!} f_{abc} e^a \wedge e^b \wedge e^c$$

- Use this to construct the 4-form Q:

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- With $\mathcal{F}$ and Q, the instanton equations reduce to matrix equations:

$$\dot{X}_a = \frac{\tilde{\beta}_1}{2} f_{acd} (f^a_{cd} X_a - [X_c, X_d]) \tag{1}$$

$$f^i_{ab} I_i + f^e_{ab} X_e - [X_a, X_b] = \left(\frac{\tilde{\beta}_1^2}{2} f^e_{ab} f^e_{cd} + 3\beta_2 f^e_{[ab} f^e_{cd]} \right) (f^i_{cd} I_i + f^a_{cd} X_a - [X_c, X_d]) \tag{2}$$

- Interpretation in terms of **quiver gauge theory:** instanton conditions match the relations of a certain quiver

Example (simplest possible case)

- Suppose $\mathcal{A}$ is parametrized by only one function $\phi(\tau)$:

$$X_a = \phi(\tau) I_a \qquad \mathcal{A} = e^i I_i + \phi e^a I_a$$

$$\mathcal{F} = \dot{\phi} e^0 \wedge e^a - \frac{1}{2} \left((1-\phi^2) f^i_{ab} I_i + (\phi - \phi^2) f^e_{ab} I_e \right) e^a \wedge e^b$$

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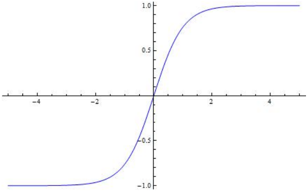
- Condition (2) is identically satisfied.
  Condition (1) turns into
  the Kink equation:

$$\dot{\phi} = \frac{1}{2}(\phi^2 - \phi)$$

Interpretation:
Instanton solution $\mathcal{A}(\tau)$ interpolates between vacuum configurations $\mathcal{A}(-\infty)$ and $\mathcal{A}(+\infty)$



Lechtenfeld, Rahn et.al [arxiv 0904.0654]

Equivariant Dimensional Reduction

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- This is the case when isometries of the extra spacetime dimensions can be compensated by gauge tansformations (G-**equivariance**)
- Yang-Mills theory on G/H reduces to a quiver gauge theory on M^4

Structure of the Equivariant Bundle
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- $SU(3)$-equivariance:
  $SU(3)$ must act on the fibers in a $U(n)$ representation
- We choose the fundamental representation **3** of $SU(3)$
  $\rightarrow$ structure group of ε becomes $U(3)$

Structure of the Equivariant Bundle

- G-equivariance determines the vector bundle structure:

$$\begin{array}{c} \varepsilon = (\mathbb{R} \otimes \mathcal{L}_1) \oplus (\mathbb{R} \otimes \mathcal{L}_2) \oplus (\mathbb{R} \otimes \mathcal{L}_3) \\ \big\downarrow \\ \mathbb{R} \times G/H \end{array}$$

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- **Idea:**
  - The representation **3** decomposes under restriction to H as

$$\mathbf{3}|_{U(1)\times U(1)} = \bigoplus_{\alpha} \rho_\alpha = \rho_1 \oplus \rho_2 \oplus \rho_3$$

  - H-representation induces G-equivariant vector bundle over G/H
  - $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$: complex line bundles over G/H

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  - H-representation induces G-equivariant vector bundle over G/H
  - $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$: complex line bundles over G/H
- The single summands are related by G-action
- Gauge group $U(3)$ of ε is broken to $\prod_{\alpha=1}^{3} U(1)$

Quiver Gauge Theory

- Connection on $\varepsilon = (\mathbb{R} \otimes \mathcal{L}_1) \oplus (\mathbb{R} \otimes \mathcal{L}_2) \oplus (\mathbb{R} \otimes \mathcal{L}_3)$:

$$\mathcal{A} = \begin{pmatrix} a_1 & -\bar{\phi}_{12} \otimes \beta^{12} & -\phi_{13} \otimes \bar{\beta}^{13} \\ \phi_{12} \otimes \bar{\beta}^{12} & a_2 & -\bar{\phi}_{23} \otimes \beta^{23} \\ \bar{\phi}_{13} \otimes \beta^{13} & \phi_{23} \otimes \bar{\beta}^{23} & a_3 \end{pmatrix}$$

$\phi_{\alpha\beta} \in \mathrm{Hom}(\mathbb{C}, \mathbb{C})$	Higgs fields
$\beta^{\alpha\beta} \in \Omega^1(G/H)$	one-forms
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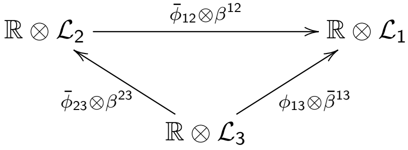
- This gives rise to a **quiver gauge theory:**



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- Requiring $\mathcal{A}$ to satisfy the instanton equation implies the **Quiver relations:**

$$\phi_{13} = \bar{\phi}_{12}\bar{\phi}_{23} \qquad \bar{\phi}_{12} = \phi_{13}\phi_{23} \qquad \bar{\phi}_{23} = \phi_{12}\phi_{13} \tag{3}$$

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- **Direct computation:**
  quiver relations (3) match the matrix instanton equations:

$$f_{ab}^{i} I_i + f_{ab}^{e} X_e - [X_a, X_b] = \left(\frac{\tilde{\beta}_1^2}{2} f_{ab}^{e} f_{cd}^{e} + 3\beta_2 f_{[ab}^{e} f_{cd]}^{e} \right) \left(f_{cd}^{i} I_i + f_{cd}^{a} X_a - [X_c, X_d] \right)$$

Summary

- Higher-dimensional instantons satisfy $*\mathcal{F} = -(*Q) \wedge \mathcal{F}$
- Interesting objects in string theory,
  especially on cones over special holonomy manifolds
- We have constructed general instanton conditions on $\mathcal{C}(X^n)$
  that can be solved under special assumptions
- Equivariant dimensional reduction: quiver gauge theory
- Quiver relations match the instanton matrix equations
- **Outlook:**
  Instanton conditions with deformed metric;
  Lifting of the building blocks to heterotic supergravity solutions;
  Relation to heterotic string model building