Partition Function of Chiral Boson on 2-Torus from Floreanini-Jackiw Lagrangian

based on arXiv: 1307.2172 [hep-th]

KHOO Fech Scen

Jacobs University

Nordic String Meeting 2014
AEI, Golm

February 3, 2014

Motivation

- **To establish a Lagrangian formulation for the quantum theory of a chiral boson, accommodating a wider class of chiral boson theories.**

Chiral Boson

- Self-dual gauge theory/chiral boson theory lives in $4k + 2$ spacetime dimensions, for integer k.
- There exists a $2k$-form field/chiral boson, ϕ.
- Field strength of $(2k + 1)$-form is self-dual, i.e. $d\phi = *d\phi$.

In 2 dimensions:

- Chiral scalar field, ϕ
- Self-duality: $\dot{\phi} = \phi'$

Outline

- Henningson-Nilsson-Salomonson - Holomorphic decomposition
- Witten - Holomorphic line bundle
- **Our proposal** - Path integral for FJ model

Introduction

In the following discussions, we will be operating with **three** different actions S on a 2-torus Σ, defined by

$$z \sim z + m + n\tau \qquad (m, n \in \mathbb{Z})$$

and a circle target space of the scalar field ϕ obeying the periodic boundary condition

$$\phi(z + m + n\tau, \bar{z} + m + n\bar{\tau}) = \phi(z, \bar{z}) + 2\pi(m\omega_1 + n\omega_2), \qquad (1)$$

for arbitrary winding numbers $\omega_1, \omega_2 \in \mathbb{Z}$.

Introduction

The goal is to compute the partition function Z $(= \int D\phi \, e^{-S})$ for a chiral boson.

The partition function is a product of zero-mode- and non-zero-mode-,

$$Z[A] = Z_0[A^{(0)}]\tilde{Z}[\tilde{A}].$$

Formula:

General Jacobi ϑ function:

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (y;\tau) = \sum_{n\in\mathbb{Z}} \exp\left[\pi i(n+\alpha)^2\tau\right] \exp\left[2\pi i(n+\alpha)(y+\beta)\right]$$

with real characteristics/spin structures α, β.

Holomorphic decomposition

(Henningson et al., hep-th/9908107)

Idea:

$$Z_{\text{non-chiral}}[A_{\bar{z}}, A_z] = Z_{\text{hol}}[A_{\bar{z}}] Z_{\text{anti-hol}}[A_z]$$

Holomorphic decomposition

(Henningson et al., hep-th/9908107)

Idea:

$$Z_{\text{non-chiral}}[A_{\bar{z}}, A_z] = Z_{\text{hol}}[A_{\bar{z}}] Z_{\text{anti-hol}}[A_z]$$

However for nontrivial topology,

$$Z_{\text{non-chiral}}[A_{\bar{z}}, A_z] \sim \sum_s Z^{(s)}_{\text{hol}}[A_{\bar{z}}] Z^{(s)}_{\text{anti-hol}}[A_z]$$

where s is spin structure.

Holomorphic decomposition

(Henningson et al., hep-th/9908107)

Idea:

$$Z_{\text{non-chiral}}[A_{\bar z}, A_z] = Z_{\text{hol}}[A_{\bar z}] Z_{\text{anti-hol}}[A_z]$$

However for nontrivial topology,

$$Z_{\text{non-chiral}}[A_{\bar z}, A_z] \sim \sum_s Z^{(s)}_{\text{hol}}[A_{\bar z}] Z^{(s)}_{\text{anti-hol}}[A_z]$$

where s is spin structure.

Action:

$$S[\phi, A] = \frac{1}{\pi g^2} \int dz d\bar z \; \partial_z \phi \partial_{\bar z} \phi + \frac{1}{\pi g} \int dz d\bar z \; (A_{\bar z} \partial_z \phi + A_z \partial_{\bar z} \phi)$$

Calculations[1]

The non-chiral zero-mode partition function obtained is

$$Z_0[y,\bar{y}] = \frac{k_2^{-1}\mathcal{W}[y,\bar{y}] \sum_{n_1=0}^{k_1-1} \sum_{n_2=0}^{k_2-1} \vartheta \begin{bmatrix} \frac{n_1}{k_1} \\ \frac{n_2}{k_2} \end{bmatrix} \left(\sqrt{\frac{k_1}{k_2}} y; \frac{k_1}{k_2}\tau \right)}{\vartheta \begin{bmatrix} \frac{n_1}{k_1} \\ \frac{n_2}{k_2} \end{bmatrix} \left(\pm\sqrt{\frac{k_1}{k_2}} y; \frac{k_1}{k_2}\tau \right)}, \qquad (2)$$

where

$$\mathcal{W}[y,\bar{y}] = \sqrt{\frac{g^2\tau_2}{2}} \exp\left[\frac{\pi}{2\tau_2}(y+\bar{y})^2 \right],$$

and $y = \frac{i\tau_2}{\pi} A_{\bar{z}}^{(0)}$, $\qquad \bar{y} = -\frac{i\tau_2}{\pi} A_z^{(0)}$.

$+ : g = \sqrt{k_1 k_2}$, $\qquad - : g = \frac{2}{\sqrt{k_1 k_2}}$.

Calculations^2

- There is a **T-duality** relating the two coupling constants $\left(g \text{ and } \quad g' = \frac{2}{g}\right)$.
- **Self-duality** is realized at $g^2 = 2$ with the only possibility from $k_1 = 1, k_2 = 2$.

Calculations²

- There is a **T-duality** relating the two coupling constants $\left(g \text{ and } \quad g' = \frac{2}{g}\right)$.
- **Self-duality** is realized at $g^2 = 2$ with the only possibility from $k_1 = 1, k_2 = 2$.

The **zero-mode partition function of a chiral boson** should be identified with the holomorphic factor,

$$Z_0^{\text{chiral}}[y] = \vartheta \begin{bmatrix} \frac{n_1}{k_1} \\ \frac{n_2}{k_2} \end{bmatrix} \left(\sqrt{\frac{k_1}{k_2}} y; \frac{k_1}{k_2} \tau \right). \tag{3}$$

Discussions

Disadvantages in this approach:

- A Lagrangian of a non-chiral boson is assumed.
- An anomalous by-product $\mathcal{W}$.

Results:

- Modular parameter defining the base space z and base space $A^{(0)}$ can differ by a fraction k_1/k_2.
- Obtained more general rational spin structures. (Could be lifted to arbitrary real spin structures.)

Holomorphic line bundle

(Witten, hep-th/9610234)

Idea:

Anti-chiral component decoupled in a Lorentz invariant action.

Action:

$$S[\phi, A] = \frac{1}{\pi g^2}\int_\Sigma dz d\bar{z}\, [(\partial_z\phi + gA_z/2)(\partial_{\bar{z}}\phi + gA_{\bar{z}}/2) - g\phi F_{z\bar{z}}/2]$$
$$= \frac{1}{\pi g^2}\int_\Sigma dz d\bar{z}\, \left(\partial_z\phi\partial_{\bar{z}}\phi + gA_{\bar{z}}\partial_z\phi + g^2 A_z A_{\bar{z}}/4\right).$$

Holomorphic line bundle

Z satisfies the two differential equations

$$\frac{D}{DA_z} Z[A] = 0, \tag{4}$$

$$\left[\partial_z \frac{D}{DA_z} + \partial_{\bar{z}} \frac{D}{DA_{\bar{z}}} - \frac{F_{z\bar{z}}}{2\pi}\right] Z[A] = 0, \tag{5}$$

on which the line bundle is defined with the covariant derivatives

$$\frac{D}{DA_z} = \frac{\delta}{\delta A_z} + \frac{A_{\bar{z}}}{4\pi}, \qquad \frac{D}{DA_{\bar{z}}} = \frac{\delta}{\delta A_{\bar{z}}} - \frac{A_z}{4\pi}.$$

By these transformations,

$$\delta A_{\bar{z}} = -\partial_{\bar{z}} \epsilon_z, \qquad \delta A_z = \lambda,$$

we can get to configurations

$$A_{\bar{z}} = A_{\bar{z}}^{(0)} \quad \text{and} \quad A_z = A_z^{(0)}.$$

The space of $A_{\bar{z}}^{(0)}$ forms a torus, parametrized by

$$A_{\bar{z}}^{(0)} \to A_{\bar{z}}^{(0)} + \frac{2\pi i}{g\tau_2}(m\tau - n) \qquad (m, n \in \mathbb{Z}). \tag{6}$$

- Z will not be a function of $A_{\bar{z}}^{(0)}$.

The space of $A_{\bar{z}}^{(0)}$ forms a torus, parametrized by

$$A_{\bar{z}}^{(0)} \to A_{\bar{z}}^{(0)} + \frac{2\pi i}{g\tau_2}(m\tau - n) \qquad (m, n \in \mathbb{Z}). \tag{6}$$

- Z will not be a function of $A_{\bar{z}}^{(0)}$.

Calculations:

Let $y = i\tau_2 A_{\bar{z}}^{(0)}/\pi$ and $\bar{y} = -i\tau_2 A_{z}^{(0)}/\pi$, the covariant derivatives are

$$D_{\bar{y}} = \frac{\partial}{\partial \bar{y}} - \frac{F}{2}y, \qquad D_{y} = \frac{\partial}{\partial y} + \frac{F}{2}\bar{y}, \tag{7}$$

where field strength $F = -\pi/\tau_2$.

Calculations[2]

From

$$U_{(m,n)} D_i(y;\bar{y}) U^{-1}_{(m,n)} = D_i\left(y + \frac{2}{g}(m+n\tau); \bar{y} + \frac{2}{g}(m+n\bar{\tau})\right), \tag{8}$$

we find the transition function $U_{(m,n)}$ is

$$U_{(m,n)} = e^{-\frac{F}{g}(m+n\bar{\tau})y + \frac{F}{g}(m+n\tau)\bar{y} + f(m,n)}.$$

A section Y of the bundle should satisfy

$$U_{(m,n)} Y(y,\bar{y}) = Y\left(y + \frac{2}{g}(m+n\tau),\ \bar{y} + \frac{2}{g}(m+n\bar{\tau})\right). \tag{9}$$

By further satisfying the self-duality condition: $D_{\bar{y}} Y = 0$, we have

$$Y(y,\bar{y}) = e^{\frac{F}{2}y\bar{y} - \frac{F}{2}y^2} Y_0(y).$$

Calculations[3]

Given a choice of $f(m, n)$, we can have

$$Y_0(y) = \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (ay; b\tau).$$

This is realized from a property of the ϑ function,

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (y + n\tau + m; \tau) = e^{-i2\pi ny - i\pi n^2\tau + i2\pi(m\alpha - n\beta)} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (y; \tau).$$

We found that a, b could be parametrized by integers k_1, k_2,

$$a = \sqrt{\frac{k_1}{k_2}}, \qquad b = \frac{k_1}{k_2},$$

where $g = \frac{2}{\sqrt{k_1 k_2}}$.
So the **zero-mode partition function for chiral boson** is

$$Z_0^{(s)}[y, \bar{y}] = \mathcal{N}^{(s)} e^{\frac{F}{2} y\bar{y} - \frac{F}{2} y^2} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\sqrt{\frac{k_1}{k_2}} y; \frac{k_1}{k_2} \tau \right), \qquad (10)$$

where spin structure $s = (\alpha, \beta)$.

Discussions

Shortcoming in this approach:

- The action

$$S[\phi, A] = \frac{1}{\pi g^2} \int_\Sigma dz d\bar{z} \left[(\partial_z \phi + g A_z/2)(\partial_{\bar{z}} \phi + g A_{\bar{z}}/2) - g \phi F_{z\bar{z}}/2 \right]$$

was tailor-made.

Result:

- Obtained arbitrary real spin structures.

Path integral for FJ model

(Chen et al., hep-th/1307.2172)

Idea: Direct computation by a modified chiral boson Lagrangian.

Path integral for FJ model

(Chen et al., hep-th/1307.2172)

Idea: Direct computation by a modified chiral boson Lagrangian.

Action we propose:

$$\begin{aligned} S_\phi &= S_0 + S_A[A + A_0] + S_B + S_\Psi \\ &= \frac{1}{4\pi g^2}\int dz d\bar{z}(\partial_z + \partial_{\bar{z}})\phi\partial_{\bar{z}}\phi \\ &\quad + \frac{1}{2\pi g}\int dz d\bar{z}\,(\partial_z + \partial_{\bar{z}})\phi(A + A_0) \\ &\quad + i\int d\sigma_{\mathcal{B}} \wedge d\sigma_{\mathcal{A}}\, B(\sigma_{\mathcal{B}})\partial_{\mathcal{B}}\phi(z,\bar{z}) \\ &\quad + \frac{i}{2\pi k_1}\int (d\phi + d\Gamma) \wedge d\Psi, \end{aligned} \tag{11}$$

to compute the zero-mode partition function.

Notations: $\sigma_{\mathcal{A}} = \frac{-\bar{\tau}z + \tau\bar{z}}{\tau - \bar{\tau}}$, $\quad \sigma_{\mathcal{B}} = \frac{z - \bar{z}}{\tau - \bar{\tau}}$

Local physics[1]

The original **FJ action**

$$S_0 = \frac{1}{4\pi g^2} \int dz d\bar{z} \, (\partial_{\mathcal{A}} \phi)(\partial_{\bar{z}} \phi) \tag{12}$$

has the equation of motion

$$\partial_{\mathcal{A}} \partial_{\bar{z}} \phi = 0.$$

where $\partial_{\mathcal{A}} = \partial_z + \partial_{\bar{z}}, \partial_{\mathcal{B}} = \tau \partial_z + \bar{\tau} \partial_{\bar{z}}$.
Through gauge transformation

$$\phi \to \phi' = \phi + F(\sigma_{\mathcal{B}}), \tag{13}$$

it is equivalent to the self-duality condition

$$\partial_{\bar{z}} \phi = 0. \tag{14}$$

Local physics[2]

The source term of action

$$S_A[A] = \frac{1}{2\pi g} \int dz d\bar{z} \, (\partial_{\mathcal{A}} \phi) A. \tag{15}$$

When expressing in a new coordinate $y = \frac{i\tau_2}{\pi} A$, we notice the partition function is periodic,

$$Z_0[y + g] = Z_0[y]. \tag{16}$$

Gauge fixing

- The winding number ω_2 has become gauge-invariant, as a consequence from the symmetry $\phi \to \phi' = \phi + F(\sigma_{\mathcal{B}})$.
- Consider the gauge-fixing condition

$$\oint_{C_{\mathcal{A}}(\sigma_{\mathcal{B}})} d\sigma_{\mathcal{A}} \partial_{\mathcal{B}} \phi(z, \bar{z}) = 2\pi\omega_2 = \text{constant} \qquad \forall \sigma_{\mathcal{B}}.$$

- Impose another constraint by introducing a constant Lagrange multiplier $a \in \mathbb{R}$.
- We **set** ω_2 **to zero**.

This procedure contributes a term in action as

$$S_B = i \int d\sigma_{\mathcal{B}} \wedge d\sigma_{\mathcal{A}} \; B(\sigma_{\mathcal{B}}) \partial_{\mathcal{B}} \phi(z, \bar{z}), \tag{17}$$

where $\sigma_{\mathcal{A}} = \frac{-\bar{\tau} z + \tau \bar{z}}{\tau - \bar{\tau}}$, $\sigma_{\mathcal{B}} = \frac{z - \bar{z}}{\tau - \bar{\tau}}$.

Modifications[1]

We introduce **topological terms** into the action,

$$S_\Psi = \frac{i}{2\pi k_1} \int (d\phi + d\Gamma) \wedge d\Psi. \qquad (18)$$

Ψ is a scalar **auxiliary field**, with twisted periodicity

$$\Psi \sim \Psi + 2\pi(N + \beta) \qquad \forall\, N \in \mathbb{Z}, \beta \in \mathbb{R}.$$

The respective winding modes are

$$\begin{aligned} (d\phi)^{(0)} &= 2\pi\omega_1\, d\sigma_{\mathcal{A}} + 2\pi\omega_2\, d\sigma_{\mathcal{B}} \qquad (\omega_1, \omega_2 \in \mathbb{Z}), \ [\omega_2 \text{ is set to zero}] \\ (d\Psi)^{(0)} &= 2\pi m\, d\sigma_{\mathcal{A}} + 2\pi n\, d\sigma_{\mathcal{B}} \qquad (m, n \in \mathbb{Z}). \end{aligned}$$

The constant one-form $d\Gamma$ is defined by

$$d\Gamma = \gamma\, d\sigma_{\mathcal{A}} + \zeta\, d\sigma_{\mathcal{B}},$$

where $\gamma = -2\pi n_1$. We may ignore the effect of ζ which contributes only an overall factor.

Modifications[2]

[Recall from earlier $Z_0[y+g] = Z_0[y]$.]

The implication of the auxiliary field is to realize the **twisted quotient periodic boundary condition**,

$$Z_0^{new}[y] = e^{-i2\pi n_1/k_1} Z_0^{new}[y + g/k_1]. \tag{19}$$

This is satisfied when we define

$$Z_0^{new}[y] = \sum_{n=0}^{k_1-1} e^{-i2\pi n n_1/k_1} Z_0[y + gn/k_1]. \tag{20}$$

Or,

$$Z_0^{new}[y] = \frac{1}{\left(\sum_{r\in\mathbb{Z}} 1\right)} \sum_{n\in\mathbb{Z}} e^{-i2\pi n n_1/k_1} Z_0[y + gn/k_1],$$

under gauge symmetry

$$n \to n + rk_1 \qquad \forall r \in \mathbb{Z}.$$

Modifications[3]

The twisting of β in the boundary condition of Ψ introduces a background A_0, where we define

$$A_0 = \pi g \beta / i \tau_2 k_1$$

in $E_{\mathcal{B}}-$direction. This appears from a nontrivial topological term $\int d\phi \wedge d\kappa$, where $d\kappa$ is a constant one-form. That is, we have a term in action as

$$S_A[A_0] = \frac{1}{2\pi g} \int dz d\bar{z}\ (\partial_{\mathcal{A}} \phi) A_0. \tag{21}$$

This gives a shift in the source term A.

Calculations

The **zero-mode partition function** is computed to be

$$Z_0[y] = \mathcal{N} \sum_{\omega_1, n} e^{-S_\phi^{(0)}} = \vartheta \begin{bmatrix} n_1/k_1 \\ \beta \end{bmatrix} \left(\frac{k_1}{g} y ; \frac{k_1^2}{g^2} \tau \right) . \qquad (22)$$

To derive this result we have decomposed the variables

$$\omega_1 = k_1 p + m, \qquad n = k_1 q + r,$$

where $p, q \in \mathbb{Z}$ and $m, r = 0, 1, 2, \cdots, (k_1 - 1)$.
To reproduce the former results, $g = \sqrt{k_1 k_2}$.

Discussions

Our proposal:

- For a generic interacting chiral boson Lagrangian, of Lorentz-invariance or -variance, we propose to introduce the correction terms $S_A[A_0] + S_\Psi$ to the "local" action $S_0 + S_A[A]$.
- The quotient condition is the guiding principle. And it involves introducing an auxiliary field.
- All possible topological terms are exhausted and carefully defined.

Discussions

Our proposal:

- For a generic interacting chiral boson Lagrangian, of Lorentz-invariance or -variance, we propose to introduce the correction terms $S_A[A_0] + S_\Psi$ to the "local" action $S_0 + S_A[A]$.
- The quotient condition is the guiding principle. And it involves introducing an auxiliary field.
- All possible topological terms are exhausted and carefully defined.

Results:

- Obtained a single ϑ function as a solution for the chiral boson partition function.
- Spin structures can be lifted to be arbitrary real by twisting the boundary condition of ϕ.

Summary

- Henningson-Nilsson-Salomonson:

  - contains an anomalous piece that makes the decomposition not strictly holomorphic.

  - requires a non-chiral boson action to start with.

- Witten:

  - it is a customized action.

- **Our approach**:

  - clear emergence of a single ϑ function.

  - applicable to other generic actions.

Future works

- Interacting chiral boson Lagrangian
- Higher genus
- Higher dimension
- Curved spacetime

THANK YOU for your attention

Formulas[1]

General Jacobi ϑ function:

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(y;\tau)=\sum_{n\in\mathbb{Z}}\exp\left[\pi i(n+\alpha)^2\tau\right]\exp\left[2\pi i(n+\alpha)(y+\beta)\right]$$

with real characteristics/spin structures α, β.

Poisson resummation formula:

$$\sum_{m=-\infty}^{\infty}\exp\left[-\alpha(m+\beta)^2+\gamma(m+\beta)\right]$$
$$=\sum_{n=-\infty}^{\infty}\sqrt{\frac{\pi}{\alpha}}\exp\left[i2\pi n\beta+\frac{1}{4\alpha}(\gamma-i2\pi n)^2\right]$$

Identity:

$$k_2^{-1}\sum_{n_2=0}^{k_2-1}e^{2\pi i(m_1-m_2)n_2/k_2}=\sum_{\omega_2=-\infty}^{\infty}\delta_{m_1-m_2}^{\omega_2 k_2}$$

Formulas[2]

Modular transformations of Jacobi ϑ function:

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{y}{\tau};-\frac{1}{\tau}\right)=\sqrt{-i\tau}e^{i\pi y^2/\tau+2\pi i\alpha\beta}\vartheta\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}(y;\tau)$$

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\Big(y;\tau+1\Big)=e^{-i\pi\alpha(\alpha+1)}\vartheta\begin{bmatrix}\alpha\\ \beta+\alpha+\frac{1}{2}\end{bmatrix}(y;\tau)$$

For $m,n\in\mathbb{Z}$, the Jacobi ϑ function has the periodicity

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(y+n\tau+m;\tau)=e^{-i2\pi ny-i\pi n^2\tau+i2\pi(m\alpha-n\beta)}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(y;\tau).$$