Lie algebroids in gravity and string theory

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Contents

Motivation

- T-duals of flux compactifications Hull, Dabholkar, Shelton, Taylor, Wecht, Graña, Minasian, Petrini, Waldram, Halmagyi, Andriot, Hohm, Larfors, Lüst, Patalong ...and many others

$$H_{abc} \to f_{ab}{}^{c} \to Q_{a}{}^{bc} \to R^{abc}$$

- Structure of fluxes:

$$\partial_{[\underline{a}} B_{\underline{bc}]} = H_{abc}\ , \qquad [e_a, e_b] = f_{ab}{}^{c} e_c$$
$$\partial_a \beta^{bc} = Q_a{}^{bc}\ , \qquad \beta^{[\underline{an}} \partial_n \beta^{\underline{bc}]} = R^{abc}$$

- Gravity theory with metric G_{ab} and flux H_{abc}

$$S = \frac{1}{2\kappa^2} \int d^n x \sqrt{-|G|}\, e^{-2\phi} \left(R - \tfrac{1}{12} H_{abc} H^{abc} + 4 G^{ab} \partial_a \phi \partial_b \phi \right)$$

$\to$ corresponding theory for $(\hat{g}^{ab}, \beta^{ab})$ without double field theory?

- Poisson geometry

$$\{\{f,g\},h\} + \text{cycl.} = \beta^{[\underline{an}} \partial_n \beta^{\underline{bc}]}\, \partial_a f\, \partial_b g\, \partial_c h$$

Definition and examples

Definition: LIE ALGEBROID

- $E \to M$ vector bundle,
  $[\cdot,\cdot]_E : E \times E \to E$, skew symm., Jacobi identity
- $\rho : E \to TM$, bundle homomorphism (anchor)
- Leibniz rule: For $s_1, s_2 \in \Gamma(E), f \in \mathcal{C}^\infty(M)$

$$[s_1, fs_2]_E = \rho(s_1)(f)\, s_2 + f\ [s_1, s_2]_E$$

Examples

- M manifold, $(TM, [\cdot,\cdot], \rho = \mathrm{id})$...trivial
- M Poisson manifold
  $\to \beta = \frac{1}{2}\beta^{ij}\,\partial_i \wedge \partial_j$ Poisson tensor (i.e. $[\beta,\beta]_{SN} = 0$)
  Lie algebroid $(T^*M, [\cdot,\cdot]_{KS}, \beta^\sharp)$
  Anchor: $\beta^\sharp : T^*M \to TM$, $\beta^\sharp(dx^i) = \beta^{in}\partial_n$
  Bracket: $[\alpha,\gamma]_{KS} := L_{\beta^\sharp(\alpha)}\gamma - L_{\beta^\sharp(\gamma)}\alpha - d\,(\beta(\alpha,\gamma))$

Differential geometry with Lie algebroids

Mackenzie, Vaisman, Weinstein, Roytenberg, Gualtieri, Boucetta

Objects: $X \in \Gamma(TM) \ \to s \in \Gamma(E)$

Lie derivative

$$\mathcal{L}_s(f) = \rho(s)(f)\ , \quad \mathcal{L}_{s_1} s_2 = [s_1, s_2]_E$$
$$\text{e.g. } dx^i(f) = \beta^{in}\partial_n f,\ \left[dx^i, dx^j\right]_{KS} = \partial_k \beta^{ij}\, dx^k$$

Covariant derivative

$$\nabla_{fs_1} s_2 = f \nabla_{s_1} s_2$$
$$\nabla_{s_1}(f\, s_2) = \rho(s_1)(f) s_2 + f\, \nabla_{s_1} s_2$$
$$\to \nabla_{dx^i} dx^j = \Gamma_k{}^{ij}\, dx^k$$

Torsion: $T(s_1, s_2) = \nabla_{s_1} s_2 - \nabla_{s_2} s_1 - [s_1, s_2]_E$
Curvature: $R(s_1, s_2) s_3 = \nabla_{s_1}\nabla_{s_2} s_3 - \nabla_{s_2}\nabla_{s_1} s_3 - \nabla_{[s_1, s_2]_E} s_3$

Application to string theory

Bosonic low energy effective action:

$$S = \frac{1}{2\kappa^2} \int d^n x \sqrt{-|G|}\, e^{-2\phi} \left(R - \tfrac{1}{12} H_{abc} H^{abc} + 4 G^{ab} \partial_a \phi \partial_b \phi \right)$$

...invariant under diffeomorphisms, B-field gauge transformations

Seiberg, Witten '99 :

$$\beta^{ab} = (B^{-1})^{ab} , \quad \hat{g}^{ab} = \beta^{am} G_{mn} \beta^{mb}$$

Gauge trafos of B: $\Delta_\xi B = d\xi$ can be transformed:

- Gauge trafos of β:

$$\Delta_\xi \beta = L_{\beta^\sharp(\xi)} \beta - \mathcal{L}_\xi \beta - \wedge^2 \beta^\sharp (d\xi)$$

- Gauge transformations of the new metric $\hat{g}$:

$$\Delta_\xi \hat{g} = L_{\beta^\sharp(\xi)} \hat{g} - \mathcal{L}_\xi \hat{g}$$

β-diffeomorphisms

Blumenhagen, Deser, Plauschinn, Rennecke, arXiv: 1210.1591, 1211.0030

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- $\mathcal{L}_\xi \hat{g}$: "β-diffeomorphism":

$$\mathcal{L}_\xi(f) = \xi_m D^m f \,, \quad D^m = \beta^{mn}\partial_n$$
$$\mathcal{L}_\xi \eta = (\xi_m D^m \eta_a - \eta_m D^m \xi_a - \xi_m \eta_n Q_a{}^{mn})\, dx^a$$
$$\mathcal{L}_\xi X = (\xi_m D^m X^a + X^m D^a \xi_m - X^m \xi_n Q_m{}^{na})\, \partial_a$$

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Question: Is there an action $\int d^n x\, \hat{\mathfrak{L}}(\hat{g},\beta,\phi)$ invariant under diffeomorphisms and β-diffeomorphisms?

β-diffeomorphisms

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$$\Delta_\xi \hat{g} = L_{\beta^\sharp(\xi)} \hat{g} - \mathcal{L}_\xi \hat{g}$$

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Question: Is there an action $\int d^n x\, \hat{\mathfrak{L}}(\hat{g}, \beta, \phi)$ invariant under diffeomorphisms and β-diffeomorphisms?
$\rightarrow$ Idea:

- Find the right Lie algebroid $(E, [\cdot,\cdot]_E, \rho)$, such that $[\cdot,\cdot]_E$ maps β-tensors to β-tensors
- Apply differential geometry of E to get Ricci scalar, flux corresponding to H, dilaton kinetic term

The right Lie algebroid

Blumenhagen, Deser, Plauschinn, Rennecke, arXiv: 1210.1591, 1211.0030

$$(T^*M, [\cdot,\cdot]^H_{KS}, \beta^\sharp)$$

- $[\cdot,\cdot]^H_{KS}$: H-twisted Koszul-Schouten bracket (Severa, Weinstein)

$$[\xi,\eta]^H_{KS} = [\xi,\eta]_{KS} + \iota_{\beta^\sharp(\xi)}\iota_{\beta^\sharp(\eta)} H$$
$$[dx^a, dx^b]^H_{KS} = \left(Q_k{}^{ab} - \beta^{am}\beta^{bn}H_{mnk}\right) dx^k =: \mathcal{Q}_k{}^{ab} dx^k$$

- $(T^*M, [\cdot,\cdot]^H_{KS}, \beta^\sharp)$ is a Lie algebroid if

$$\beta^{am}\beta^{bn}\beta^{ck}H_{mnk} = R^{abc}$$
$$R := [\beta,\beta]_{SN}\ , \quad R^{abc} = \beta^{[\underline{a}p}\partial_p\beta^{\underline{bc}]}$$

The right Lie algebroid

Blumenhagen, Deser, Plauschinn, Rennecke, arXiv: 1210.1591, 1211.0030

- Metric on $E = T^*M$: $\hat{g}^{ab} = \hat{g}(dx^a, dx^b)$
- Covariant derivative $\nabla_{dx^a} dx^b = \Gamma_k{}^{ab} dx^k$
- Vanishing torsion, metricity:

$$\beta^\sharp(dx^i)(\hat{g}(dx^j, dx^k)) = \hat{g}(\nabla_{dx^i} dx^j, dx^k) + \hat{g}(dx^j, \nabla_{dx^i} dx^k)$$

- Unique Levi-Civita connection on $(T^*M, [\cdot,\cdot]^H_{KS}, \beta^\sharp)$:

$$\Gamma_k{}^{ab} = \frac{1}{2}\hat{g}_{km}\left(D^a\hat{g}^{bm} + D^b\hat{g}^{am} - D^m\hat{g}^{ab}\right) - \hat{g}_{km}\hat{g}^{(\underline{a}p}\mathcal{Q}_p{}^{\underline{b})m} + \frac{1}{2}\mathcal{Q}_k{}^{ab}$$

- Get explicit Ricci-scalar for $\Gamma_k{}^{ab}$

Main result

Blumenhagen, Deser, Plauschinn, Rennecke, arXiv: 1210.1591, 1211.0030

Action for $(\hat{g}, \beta, \phi)$ invariant under both types of diffeomorphisms

$$S = \frac{1}{2\kappa^2} \int d^n x \sqrt{-|\hat{g}|} |\beta^{-1}| e^{-2\phi} \left(\hat{R} - \tfrac{1}{12} R^{abc} R_{abc} + 4 \hat{g}_{ab} D^a \phi D^b \phi \right)$$

- $|\beta^{-1}|$ is needed for diffeomorphism and β-diffeomorphism invariance
- The bivector β enters with non-trivial dynamics:
  - $R^{abc} = \beta^{am} \partial_m \beta^{bc} + \ldots$
  - Ricci scalar contains e.g. $D^a D^b \hat{g}_{ab}$, $D^a(\hat{g}_{ab} \mathcal{Q}_m{}^{mb})$ ...
- Relation to standard bosonic action by Seiberg/Witten's field redefinition:

$$\mathfrak{L}_{NS}(G, B, \phi) = \hat{\mathfrak{L}}(\hat{g}(B, G), \beta(B, G), \phi)$$

where $\hat{g}^{ab} = \beta^{am} G_{mn} \beta^{nb}$, and $\beta^{ab} = (B^{-1})^{ab}$.

Generalization: Filippov 3-algebroids

Filippov, Vallejo, Grabowski, Marmo

Definition:

- $F \to M$ vector bundle,
- $[\cdot,\cdot,\cdot] : F \times F \times F \to F$ tri-linear, skew-symmetric 3-bracket, fundamental identity:

$$[u_1, u_2, [v_1, v_2, v_3]] = [[u_1, u_2, v_1], v_2, v_3] + [v_1, [u_1, u_2, v_2], v_3] + [v_1, v_2, [u_1, u_2, v_3]]$$

- Anchor-morphism $a : \Gamma(\wedge^2 F) \to TM$, Leibniz rule

$$[v_1, v_2, fv_3] = f[v_1, v_2, v_3] + a(v_1 \wedge v_2)(f)\, v_3$$

- Homomorphism property:

$$[a(u_1 \wedge u_2), a(v_1 \wedge v_2)]_L = a([u_1, u_2, v_1] \wedge v_2) + a(v_1 \wedge [u_1, u_2, v_2])$$

Differential geometry with 3-algebroids

Mimic constructions of the Lie algebroid case, e.g.:

- 3-Lie derivative:

$$\mathcal{L}_{s_1 \wedge s_2}(f) := a(s_1 \wedge s_2)(f) \,, \quad \mathcal{L}_{s_1 \wedge s_2} s_3 := [s_1, s_2, s_3]$$

  Commutator has to be modified according to the gen. Leibniz-rule:

$$[\mathcal{L}_{s_1 \wedge s_2}, \mathcal{L}_{e_1 \wedge e_2}]s = \mathcal{L}_{[s_1, s_2, e_1] \wedge e_2} s + \mathcal{L}_{e_1 \wedge [s_1, s_2, e_2]} s \,.$$

Differential geometry with 3-algebroids

- <u>3-Covariant derivative</u>:

$$\nabla : \quad \Gamma(F \wedge F) \times \Gamma(F) \to \Gamma(F) \, ,$$

Properties:

- $\nabla_{s_1 \wedge s_2 + e_1 \wedge e_2} = \nabla_{s_1 \wedge s_2} + \nabla_{e_1 \wedge e_2}$.
- $\nabla_{s_1 \wedge s_2}(e_1 + e_2) = \nabla_{s_1 \wedge s_2} e_1 + \nabla_{s_1 \wedge s_2} e_2$.
- $\nabla_{f \, s_1 \wedge s_2} = f \, \nabla_{s_1 \wedge s_2}$.
- $\nabla_{s_1 \wedge s_2}(f \, s_3) = a(s_1 \wedge s_2)(f) \, s_3 + f \, \nabla_{s_1 \wedge s_2} s_3$.

Differential geometry with 3-algebroids

- 3-Torsion

$$T(s_1, s_2, s_3) := \nabla_{s_1\wedge s_2} s_3 + \nabla_{s_2\wedge s_3} s_1 + \nabla_{s_3\wedge s_1} s_2 - [s_1, s_2, s_3] \, .$$

- 3-Curvature

$$\begin{aligned} R(s_1, s_2, s_3, s_4) = {} & [\nabla_{s_1\wedge s_2}, \nabla_{s_3\wedge s_4}] - \nabla_{[s_1,s_2,s_3]\wedge s_4} - \nabla_{s_3\wedge[s_1,s_2,s_4]} \\ & - [\nabla_{s_1\wedge s_4}, \nabla_{s_2\wedge s_3}] + \nabla_{[s_1,s_4,s_2]\wedge s_3} + \nabla_{s_2\wedge[s_1,s_4,s_3]} \\ & + [\nabla_{s_2\wedge s_4}, \nabla_{s_3\wedge s_1}] - \nabla_{[s_2,s_4,s_3]\wedge s_1} - \nabla_{s_3\wedge[s_2,s_4,s_1]} \, . \end{aligned}$$

Conclusion and discussion

- Action with dynamical fields $(\hat{g}^{ab}, \beta^{ab}, \phi)$
  - Invariant under diffeomorphisms and β-diffeomorphisms
  - Used the differential geometry of Lie algebroids
  - Relation to standard bosonic string action by Seiberg-Witten field redefinition
- Extendible to type IIA superstring, e.g. gravitino

$$\mathfrak{L}_{RS} = \bar{\hat{\Psi}}^a \gamma_{abc} \left(\hat{\nabla}^b - \frac{i}{4} \hat{\omega}^b{}_{\alpha\beta} \gamma^{\alpha\beta} \right) \hat{\Psi}^c$$

- Poisson tensor implemented non-trivial into gravity $\rightarrow$ deformation quantization?
  E.g. Fedosov construction of covariant star product...
- Connection to T-duality and $O(d,d)$-transformations
- Possibility to generalize to 3-algebroids and n-algebroids

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Thank you!