Three-loop Heavy Flavor Corrections to Deep Inelastic Scattering.

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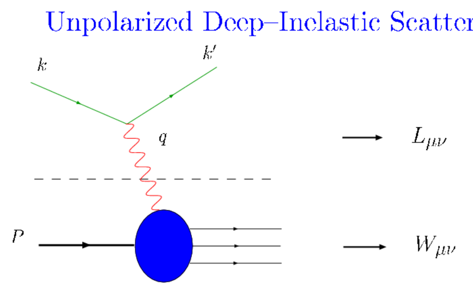
Matter & Universe - Fundamental Particles and Forces

Unpolarized Deep–Inelastic Scattering (DIS):



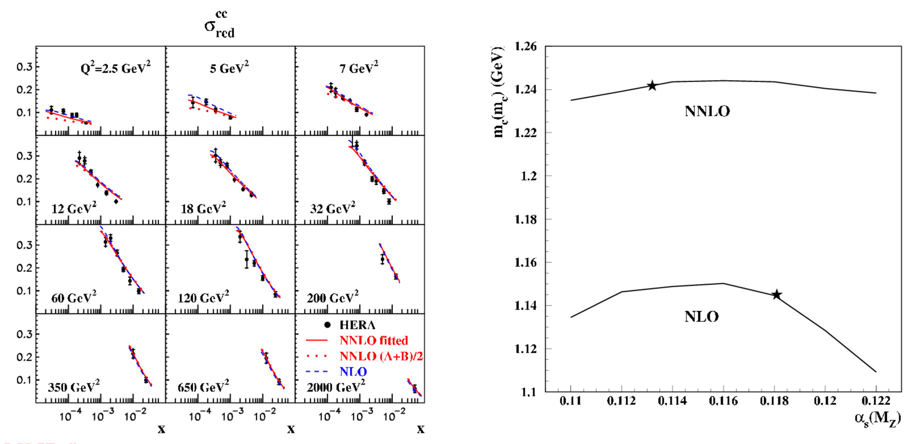
$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken } x$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu}L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2}g_{\mu\nu}\right)F_2(x,Q^2)\ .$$

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions.



NNLO:
S. Alekhin, J. Blümlein, K. Daum, K. Lipka, Phys.Lett. B720 (2013) 172 [1212.2355]

$$m_c(m_c) = 1.24 \pm 0.03(exp)\ {}^{+0.03}_{-0.02}\ (scale)\ {}^{+0.00}_{-0.07}\ (thy),$$

$\alpha_s(M_Z^2) = 0.1132 \pm 0.011$

Yet approximate NNLO treatment [Kawamura et al. [1205.5227].

$\alpha_s(M_Z^2)$ from NNLO DIS(+) analyses [from ABM13]

	$\alpha_s(M_Z^2)$	
BBG	$0.1134\ {}^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	0.1135 ± 0.0014	HQ: FFNS $N_f = 3$
JR	0.1128 ± 0.0010	dynamical approach
JR	0.1140 ± 0.0006	including NLO-jets
MSTW	0.1171 ± 0.0014	
MSTW	$0.1155 - 0.1175$	(2013)
ABM11$_J$	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	0.1133 ± 0.0011	
ABM13	0.1132 ± 0.0011	(without jets)
CTEQ	0.1159..0.1162	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrmann et al.	$0.1131\ {}^{+0.0028}_{-0.0022}$	e^+e^- thrust
Abbate et al.	0.1140 ± 0.0015	e^+e^- thrust
BBG	$0.1141\ {}^{+0.0020}_{-0.0022}$	valence analysis, N^{3}LO

$$\Delta_{\rm TH}\alpha_s = \alpha_s({\rm N^3LO}) - \alpha_s({\rm NNLO}) + \Delta_{\rm HQ} = +0.0009 \pm 0.0006_{\rm HQ}$$

NNLO accuracy is needed to analyze the world data. $\Longrightarrow$ NNLO HQ corrections needed.

Goals

- Complete the NNLO heavy flavor Wilson coefficients for twist-2 in the dynamical safe region $Q^2 > 20GeV^2$ (no higher twist) for $F_2(x,Q^2)$
- Measure m_c and α_s as precisely as possible
- Provide precise CC heavy flavor corrections
- Consequences for LHC:
  - NNLO VFNS will be provided
  - better constraint on sea quarks and the gluon
  - precise m_c and α_s on input

Factorization of the structure functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\ \delta(x - yz) f(y) g(z)\ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx\ x^{N-1} f(x)\ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2}, N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2}, N\right) = \langle j \mid O_i \mid j\rangle\ .$$

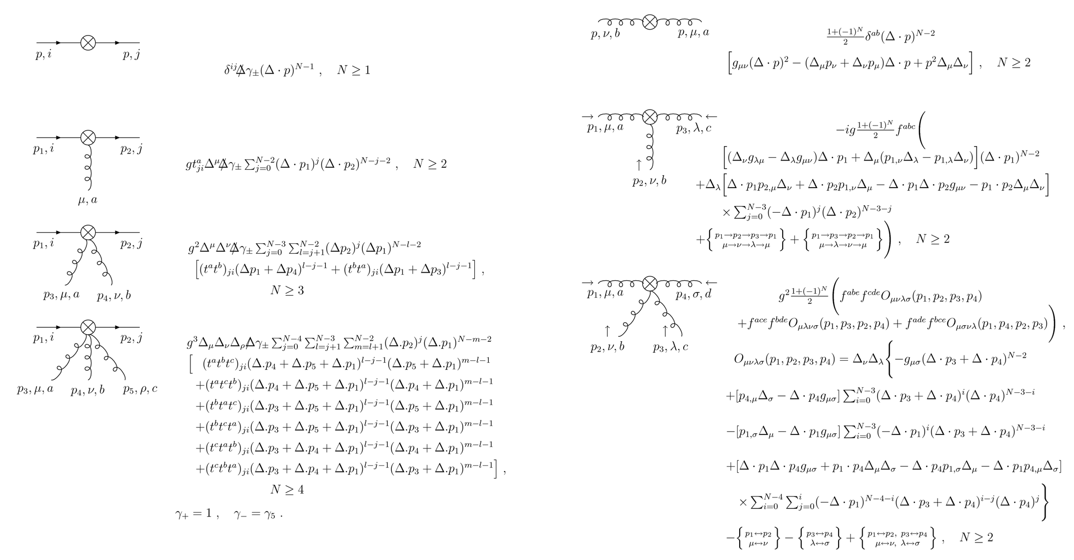
$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

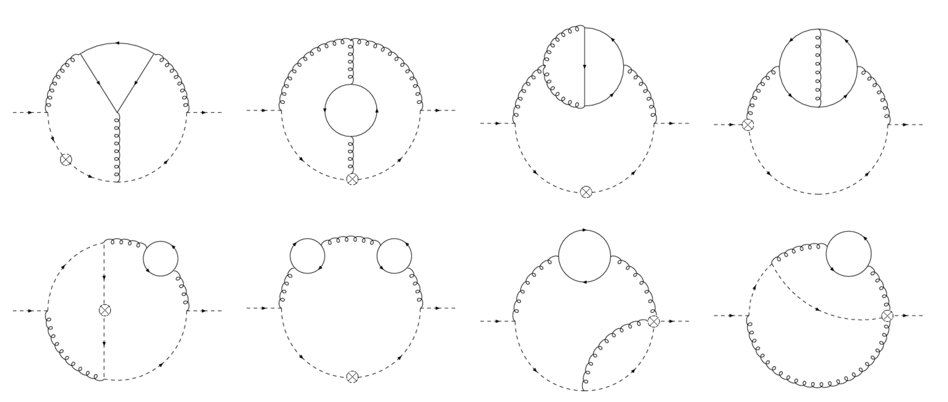
For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Calculation of the diagrams

The OMEs are calculated using the standard QCD Feynman rules together with the following operator insertion Feynman rules:



We generate the diagrams using QGRAF.



A Form program was written in order to replace the propagators, vertices and operator insertions appearing in the output of QGRAF by the corresponding Feynman rules, and also to introduce the corresponding projectors and perform the gamma matrix algebra in the numerator of the diagrams. The diagrams end up being expressed as linear combinations of scalar integrals.

Feynman integrals

Integration by parts

We use Reduze [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a C++ program based on Laporta's algorithm. It is somewhat difficult to adapt this algorithm to the case where we have operator insertions, due to the dependance on the arbitrary parameter N. For this reason we apply the following trick:

$$(\Delta\cdot k)^N \rightarrow \sum_{N=0}^{\infty} x^N (\Delta\cdot k)^N = \frac{1}{1 - x\Delta\cdot k}$$

This can be then treated as an additional propagator, and Laporta's algorithm can be applied without further modification.

If we denote the master integrals by M_i, then the reduction algorithm will allow us to express any given integral I as

$$I = \sum_i c_i(x) M_i(x)$$

In fact, any given diagram D will be written this way: $D = \sum_i c_i(x) M_i(x)$
In order to obtain each diagram $D(N)$ as a function of N. We proceed as follows:

1. Calculate the master integrals $M_i(N)$ as functions of N.
2. Evaluate $M_i(x) = \sum_{N=0}^{\infty} x^N M_i(N)$.
3. Insert the results in $D(x) = \sum_i c_i(x) M_i(x)$.
4. Obtain $D(N)$ by extracting the Nth term in the Taylor expansion of $D(x)$.

Calculation of the master integrals

For the calculation of the master integrals we use a wide variety of tools:

- Hypergeometric functions.
- Summation methods based on Zeilberger's algorithm, implemented in the Mathematica program Sigma [C. Schneider, 2005].
  - Reduction of the sums to a small number of key sums.
  - Expansion the summands in ε.
  - Harmonic sums are algebraically reduced using the package HarmonicSums (Ablinger) [Ablinger, Blümlein, Schneider 2011].
- Mellin-Barnes representations.
- Differential (difference) equations.
- In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].

Generalized sums have emerged in due course of these calculations, for example,

$$\begin{aligned}
&\sum_{i=1}^{N}\binom{2i}{i}(-2)^i\sum_{j=1}^{i}\frac{1}{j\binom{2j}{j}}S_{1,2}\left(\tfrac{1}{2},-1;j\right)\\
&= \int_0^1 dx\frac{x^N-1}{x-1}\sqrt{\frac{x}{8+x}}\left[{\rm H}^*_{w_{17},-1,0}(x) - 2{\rm H}^*_{w_{18},-1,0}(x)\right]\\
&+\frac{\zeta_2}{2}\int_0^1 dx\frac{(-x)^N-1}{x+1}\sqrt{\frac{x}{8+x}}\left[{\rm H}^*_{12}(x) - 2{\rm H}^*_{13}(x)\right]\\
&+c_3\int_0^1 dx\frac{(-8x)^N-1}{x+\frac{1}{8}}\sqrt{\frac{x}{1-x}}\ ,
\end{aligned}$$

$$w_{13} = \frac{1}{(2-x)\sqrt{x(8-x)}},\quad w_{12} = \frac{1}{\sqrt{x(8-x)}},\quad w_{17} = \frac{1}{\sqrt{x(8+x)}},\quad w_{18} = \frac{1}{(2+x)\sqrt{x(8+x)}}\ .$$

RESULTS

Anomalous dimensions

$$\begin{aligned}
\gamma_{\rm NS,TR}^{(0),qq}(N) &= 2C_F\left[4S_1 - 3\right]\\
\gamma_{\rm NS,TR}^{(k),\pm}(N) &= \gamma_{\rm NS,TR}^{(k),qq}(N) \pm \gamma_{\rm NS,TR}^{(k),q\bar{q}}(N),\quad k = 1,2.\\
\gamma_{\rm NS,TR}^{(1),qq}(N) &= \frac{1}{2}C_F\left(C_F - \frac{C_A}{2}\right)\left[128S_{-2,1} + \frac{4(17N^2+17N-12)}{3N(N-1)} - 128S_{-2}S_1\right.\\
&\left.-\frac{2144}{9}S_1 + \frac{352}{3}S_2 - 64S_3 - 64S_{-3}\right]\\
&+\frac{1}{2}C_F^2\left[S_1\left(\frac{2144}{9} - 64S_2\right) - \frac{208}{3}S_2 - \frac{86}{3}\right] + C_FT_FN_F\left[-\frac{160}{9}S_1 + \frac{32}{3}S_2 + \frac{4}{3}\right]\\
\gamma_{\rm NS,TR}^{(1),q\bar{q}}(N) &= C_F\left(C_F - \frac{C_A}{2}\right)\frac{8}{N(N+1)}\\
\hat{\gamma}_{\rm NS,TR}^{(2),qq}(N) &= C_F^2T_F\left\{-\frac{256}{3}S_{3,1} + \left[-\frac{8(1331N^2+1331N-36)}{27N(N-1)} - 128\zeta_3 + \frac{1280}{9}S_2 - \frac{128}{3}S_3\right]S_1\right.\\
&\left.-\frac{4(153N^2+153N-176)}{9N(N+1)} - \frac{128}{3}S_2^2 + \frac{9968}{27}S_2 - \frac{832}{9}S_3 + \frac{128}{3}S_4 + 96\zeta_3\right\}\\
&+C_FT_F\left(C_F - \frac{C_A}{2}\right)\left\{\left[-\frac{512}{3}S_{-2,1} + \frac{32(209N^2+209N-9)}{27N(N+1)} - 128S_3 + 256\zeta_3\right]S_1\right.\\
&+\frac{512}{3}S_{3,1} - \frac{2560}{9}S_{-2,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} + \frac{32(15N^3+30N^2+12N-5)}{3N(N+1)^2}\\
&+\left(\frac{1280}{9} - \frac{256}{3}S_1\right)S_{-3} + \left(\frac{2560}{9}S_1 - \frac{256}{3}S_2\right)S_{-2} - \frac{10688}{27}S_2 + \frac{896}{3}S_3 - \frac{640}{3}S_4\\
&\left.-\frac{256}{3}S_{-4} - 192\zeta_3\right\}\\
&+C_FT_F^2(2N_F+1)\left[+\frac{8(17N^2+17N-8)}{9N(N+1)} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3\right]\\
\hat{\gamma}_{\rm NS,TR}^{(2),q\bar{q}}(N) &= C_F\left(C_F - \frac{C_A}{C_F}\right)\left[\frac{64}{3N(N+1)}S_1 - \frac{32(13N-7)}{9N(N+1)^2}\right]
\end{aligned}$$

- Independent confirmation of full two-loop results.
- 1st ab initio calculation of the contribution $\propto T_F$ at 3 loops.
- Note a typo in the 15th moment in 1203.1022.
- Independent calculation of the anmalous dimensions ($\propto T_F$) $\gamma_{qq}^{\rm NS\pm}$ and γ_{gq} at 3 loops.

Wilson Coefficients

$L_{q,(2,L)}^{(3),\rm NS}$, $L_{q,(2,L)}^{(3),\rm PS}$ $L_{g,(2,L)}^{(3),\rm S}$ and $H_{q,(2,L)}^{(3),\rm PS}$ are by now available. Here we present $L_{q,2}^{(3),\rm PS}$:

$$\begin{aligned}
L_{q,2}^{(3),PS}(N) = &\ C_FN_FT_F^2\times\\
&\left\{-\frac{32(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)}\ln^3\left(\frac{m^2}{Q^2}\right) - \frac{32P_3}{9(N-1)N^3(N+1)^3(N+2)^2}\ln^2\left(\frac{m^2}{Q^2}\right)\right.\\
&+\left[-\frac{32P_5}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{64P_1}{3(N-1)N^3(N+1)^3(N+2)^2}S_1\right.\\
&\left.+\frac{32(N^2+N+2)^2}{3(N-1)N^2(N+1)^2(N+2)}\left[S_1^2 - S_2\right]\right]\ln\left(\frac{m^2}{Q^2}\right)\\
&-\frac{32P_7}{243(N-1)N^5(N+1)^5(N+2)^4} - \frac{16P_2}{27(N-1)N^3(N+1)^3(N+2)^2}S_1^2\\
&-\frac{16P_4}{27(N-1)N^3(N+1)^3(N+2)^2}S_2 + \left[\frac{32P_6}{81(N-1)N^4(N+1)^4(N+2)^3}\right.\\
&\left.+\frac{32(N^2+N+2)^2S_2}{9(N-1)N^2(N+1)^2(N+2)}\right]S_1 - \frac{64(N^2+N+2)^2}{27(N-1)N^2(N+1)^2(N+2)}S_1^3\\
&\left.+\frac{160(N^2+N+2)^2S_3}{27(N-1)N^2(N+1)^2(N+2)} + \frac{256(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)}\zeta_3\right\} + N_f\hat{\tilde{C}}_{q,2}^{\rm PS(3)}(N,N_f)
\end{aligned}$$

Operator Matrix Elements

We have now the full results for $A_{qq,Q}^{(3),\rm NS}$, $A_{qq,Q}^{(3),\rm TR}$, $A_{gq,Q}^{(3)}$ and $A_{Qq}^{(3),\rm PS}$. Here we present a piece of the constant part of the pure singlet OME:

$$\begin{aligned}
a_{Qq}^{(3),\rm PS}(N) = &\ C_F^2T_F\left\{\frac{64(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}S_{2,2}(2,\tfrac{1}{2}) - \frac{64(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}S_{3,1}(2,\tfrac{1}{2})\right.\\
&+2^N\left[-\frac{32P_3S_{2,1}(1,\frac{1}{2},N)}{(N-1)^2N^3(N+1)^2(N+2)} - \frac{32P_3S_{1,1,1}(\frac{1}{2},1,1,N)}{(N-1)^2N^3(N+1)^2(N+2)} + \frac{32P_4S_{1,1}(1, \ldots}{(N-1)^3N^4(N+ \ldots}\right.\\
&\left.+2^{-N}\left[-\frac{64(N^2+N+2)^2S_{1,2,1}(2,\frac{1}{2},1,N)}{(N-1)N^2(N+1)^2(N+2)} + \frac{64(N^2+N+2)^2S_{1,2,1}(2,1,\frac{1}{2},N)}{(N-1)N^2(N+1)^2(N+2)} + \cdots\right] + \right.\\
&+C_FT_F^2N_F\left\{-\frac{16(N^2+N+2)^2S_1(N)^3}{27(N-1)N^2(N+1)^2(N+2)} + \frac{16P_9S_1(N)^2}{27(N-1)N^3(N+1)^3(N+2)^2}\right.\\
&+\left[-\frac{208(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)}S_2 - \frac{32P_{23}}{81(N-1)N^4(N+1)^4(N+2)^3}\right]S_1\\
&\left.+\frac{32P_{31}}{243(N-1)N^5(N+1)^5(N+2)^4} + \frac{224(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)}\zeta_3 + \cdots\right\}\\
&+C_FC_AT_F\left\{\frac{2(N^2+N+2)^2S_1(N)^4}{9(N-1)N^2(N+1)^2(N+2)} + \frac{4(N^2+N+2)P_6S_1(N)^3}{27(N-1)^2N^3(N+1)^3(N+2)^2}\right.\\
&+2^{-N}\left[\frac{16P_2S_3(2,N)}{(N-1)N^3(N+1)^2} - \frac{16P_2S_{1,2}(2,1,N)}{(N-1)N^3(N+1)^2} + \frac{16P_2S_{2,1}(2,1,N)}{(N-1)N^3(N+1)^2} - \frac{16P_2S_{1,1,1}(2. \ldots}{(N-1)N^3(\ldots}\right.\\
&-\frac{32(N^2+N+2)^2S_{1,1,2}(2,\frac{1}{2},1,N)}{(N-1)N^2(N+1)^2(N+2)} + \frac{32(N^2+N+2)^2S_{1,1,2}(2,1,\frac{1}{2},N)}{(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{32(N^2+N+2)^2S_{1,2,1}(2,\frac{1}{2},1,N)}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2S_{1,2,1}(2,1,\frac{1}{2},N)}{(N-1)N^2(N+1)^2(N+2)}\\
&\left.-\frac{32(N^2+N+2)^2S_{1,1,1,1}(2,\frac{1}{2},1,1,N)}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2S_{1,1,1,1}(2,1,\frac{1}{2},1,N)}{(N-1)N^2(N+1)^2(N+2)} + \cdots\right\} +
\end{aligned}$$

For $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ we have partial results for some color factors.