Quantum geometry of refined topological strings

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Based on:

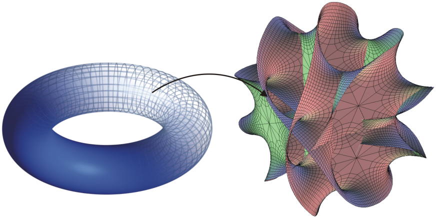
M.-x. Huang, A. Klemm, J. R. & M. Schiereck: arXiv:1401.4723 [hep-th]

Overview

1. What is refined topological string theory?
2. Refining special geometry
3. Conclusion & Outlook

The topological string

- topological subsector of physical string
- targetspace X is Calabi-Yau 3-fold
- two possibilities: A- and B-model



- A- and B-model connected by mirror symmetry
- A-model depends only on Kähler moduli, B-model on complex structure moduli
- applications: geometrical engineering, black hole partition functions, Gromov-Witten invariants

Refinement

- two parameter deformation of topological string
- genus expansion of topological string theory gets

$$F(\epsilon_1, \epsilon_2) = \sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} F^{(n,g)}$$

- for $\epsilon_1 = -\epsilon_2$ get the topological string partition function
- particular simple limit: Nekrasov-Shatashvili limit

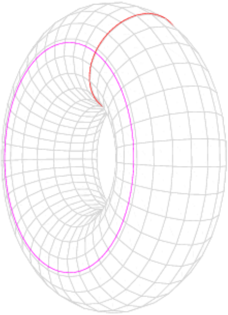
$$\epsilon_i = 0 \text{ for } i = 1 \text{ or } 2$$

- applications: Nekrasov partition function, quantum integrable system
Nekrasov, Shatashvili

Periods

- for local B-model geometries, periods are given by periods of meromorphic one form $\lambda = p\mathrm{d}x$ on the Riemann surface
- $2g$ cycles on a Riemann surface of genus g: $A^i,\ B_i$
- intersection numbers of cycles: $A^i \cap B_j = \delta^i_j$
- then periods are

$$a^i = \oint_{A^i} \lambda, \quad a_{D,i} = \oint_{B_i} \lambda$$



Special geometry

- prepotential can be calculated from the periods by use of special geometry

$$a_D = \frac{\partial F_0}{\partial a}$$

- how can we use this for refined free energies?

Refining the differential

- the β ensemble is a proposition for a matrix model for refined topological strings
  Dijkgraaf, Vafa

$$Z = \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^{-2\epsilon_1/\epsilon_2} e^{-\frac{1}{\epsilon_2}\sum_i V(\lambda_i)}$$

- inserting brane operators into the path integral leads to an equation for the brane wave-functions
  Aganagic et al.

$$\left(-\epsilon_\alpha^2 \frac{\partial^2}{\partial x^2} + W'(x)^2 + f(x) + g_s^2 \sum_{n=0}^{g} x^n \partial_{(n)}\right) \Psi_\alpha(x) = 0$$

- simplifies in Nekrasov-Shatashvili limit
  $(\epsilon_2 \to 0 \Rightarrow g_s = \epsilon_1\epsilon_2 \to 0,\ \epsilon_1 = \hbar)$
  $\to$ just quantum version of defining equation for Riemann surface

$$H(x, \mathrm{i}\hbar\partial_{\mathrm{x}})\Psi(\mathrm{x}) = 0$$

Refined periods

- can be solved for example by a WKB approximation in $\hbar$
  $\Rightarrow$ Ansatz: $\Psi(x,\hbar) = \exp\left(\frac{1}{\hbar} S(x,\hbar)\right)$
- plugging WKB ansatz into Schrödinger equation find for zeroth order

$$S'(x,\hbar)|_{\hbar^0} = -p(x)$$

$\Rightarrow$ propose to use $S'(x)$ as quantum deformation **Aganagic et al.**

- integration of $\lambda = S'(x,\hbar)\mathrm{d}x$ around cycles of geometry gives quantum periods
- for A-periods integral reduces to a residue $\Rightarrow$ easy to calculate
- But: in general the cycles get also quantum deformed
  $\Rightarrow$ not clear or very complicted to calculate B-periods

Operators for the refined B-periods

- can find operators which send zeroth order of quantum differential to the higher orders plus exact terms **Huang; Aganagic et al.; Mironov, Morozov**
- operators have the general form

$$\mathcal{D}_{2n} = a(u, \vec{m})\Theta_u + b(u, \vec{m})\Theta_u^2$$

  u are renormalizable moduli, m are non-renormalizable moduli
- renormalizable moduli: correspond to cycles of Riemann surface
- non-renormalizable moduli: just residues of the meromorphic differential
- operators can be exchanged with integration
  $\Rightarrow$ exact for periods

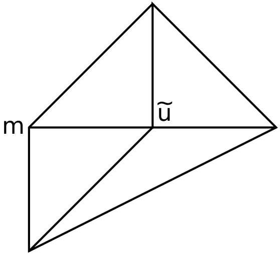
1. use Picard-Fuchs equation to find zeroth order periods
2. apply operators to find quantum corrections

Example: local $\mathbb{F}_1$

Huang, Klemm, Poretschkin

- $\mathcal{O}(-K_{\mathbb{F}_1}) \to \mathbb{F}_1$ is a toric geometry
- quantum corrected mirror curve is

$$H(x,p) = -1 + e^x + mu^2\, e^{-x} + e^p + e^{-\hbar/2}\,\frac{u}{m}\, e^x e^{-p}$$



- the first operator for the quantum correction is

$$\mathcal{D}_2 = \frac{mu^2\,(4m-9u)}{6(-8m+9u)}\Theta_u + \frac{4m-3u-16m^2u^2+36u^3m}{24(-8m+9u)}\Theta_u^2$$

- integration of special geometry relation leads to the instanton numbers

Example: Instanton numbers

$\hbar^0$	d_1	0	1	2	3
d_2					
0			1		
1		-2	3		
2			5	-6	
3			7	-32	27

$\hbar^2$	d_1	0	1	2	3
d_2					
0					
1		-1	4		
2			20	-35	
3			56	-368	396

Conclusion & Outlook

Conclusion

- could check calculations for refined free energies with different approaches and extend them to the orbifold and conifold point in the moduli space
- all operators were second order differential operators in the renormalizable moduli u

Outlook

- are operators in general of second order?
- can one generalize the Picard-Fuchs equations to quantum Picard-Fuchs equations?
- extend this procedure to higher genus curves
- Not yet clear if the β ensemble is the right deformation for topological string matrix models. If not, why gives this procedure the right results?