

# Hyperkähler Metrics from Monopole Walls

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- Introduction
  - A) What's Hyperkähler
  - B) Moduli Space of BPS Monopoles
  - C) Today's Talk
- Asymptotic Metrics of Monopole walls
  - A) What's Monopole Walls
  - B) Setup
  - C) The Metric of SU(2) Monopole Walls
- Summary

# What's Hyperkähler

## Definition of the hyperkähler metrics

*In mathematics:*

A hyperkähler manifold is a Riemannian manifold of dimension  $4k$  and holonomy group contained in  $\text{Sp}(k)$ . Hyperkähler manifolds are special classes of Kähler manifolds. They can be thought of as quaternionic analogues of Kähler manifolds...



Ricci-flat

*In physics:*

*Gravitational instantons.* (Vacuum solutions of the self-dual Einstein eqs. in 4-dim. Euclidean space.)

ex. Gibbons-Hawking metric

$$ds^2 = U dx \cdot dx + U^{-1} (d\theta + \mathbf{W} \cdot dx)^2,$$

where  $\text{grad } U = \text{rot } \mathbf{W}$ ,

$$U = \lambda + 2M \sum_{j=1}^k \frac{1}{|x - x_j|}.$$

$\lambda = 0, k = 2$  : Eguchi-Hanson (ALE)

-> *Asymptotically Locally Euclidean*

$\lambda = 1$  : Taub-NUT (ALF)

-> *Asymptotically Locally Flat*

# Moduli Space of BPS Monopoles

*Super Yang-Mills theory with eight supercharges -> hyperkähler*

The moduli space of  $k$  BPS monopoles

$$\mathcal{M}_k = \mathbb{R}^3 \times (S^1 \times \widetilde{\mathcal{M}}_k^0) / \mathbb{Z}_k, \dim \mathcal{M}_k = 4k$$

(Position moduli of 3-dim. and *phase moduli of dyons*)

ex. Atiyah-Hitchin metric  $\widetilde{\mathcal{M}}_2^0$  (2-monopole)

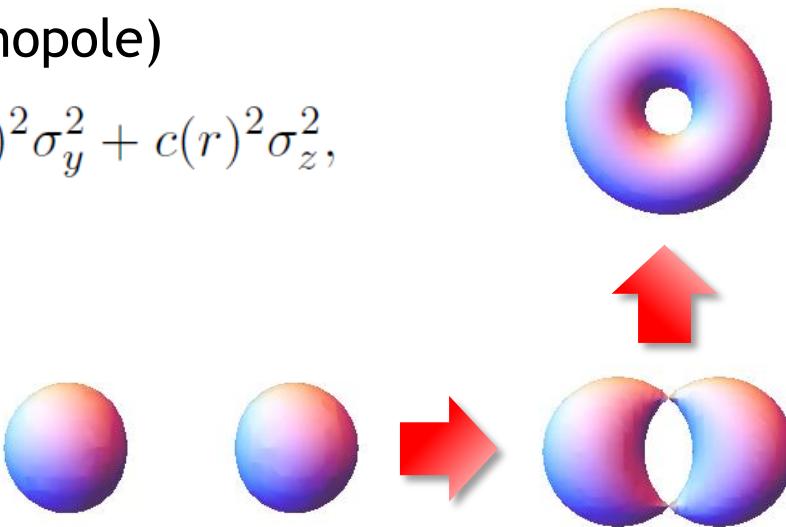
$$ds^2 = f(r)^2 dr^2 + a(r)^2 \sigma_x^2 + b(r)^2 \sigma_y^2 + c(r)^2 \sigma_z^2,$$

where

$$\sigma_x = \sin \psi d\theta - \cos \psi \sin \theta d\phi,$$

$$\sigma_y = \cos \psi d\theta + \sin \psi \sin \theta d\phi,$$

$$\sigma_z = d\psi + \cos \theta d\phi.$$



The metric of the  $k = 2$  BPS monopole (Atiyah-Hitchin metric)

M. F. Atiyah and N. J. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton University Press (1988)

# Today's Talk

Asymptotic metrics of the moduli space of BPS monopoles  
 -> *Gibbons-Manton metric*

The correspondence of periodicity, super YM and asymptotic behavior

Periodicity of monopole	Super Yang-Mills theory	Asymptotic behavior (4d topology)
$\mathbb{R}^3$ (non-periodic)	$\mathcal{N} = 4$ SYM on $\mathbb{R}^3$	ALF : $S^1$ fibration on $\mathbb{R}^3$
$S^1 \times \mathbb{R}^2$ (periodic)	$\mathcal{N} = 2$ SYM on $\mathbb{R}^3 \times S^1$	ALG : $T^2$ fibration on $\mathbb{C}$
$T^2 \times \mathbb{R}$ (doubly-periodic)	$\mathcal{N} = 1$ SYM on $\mathbb{R}^3 \times T^2$	ALH : $T^3$ fibration on $\mathbb{R}$

Asymptotic metric of non-periodic  $k$  BPS monopoles (ALF)

G.W. Gibbons and N.S. Manton, arXiv:hep-th/9506052

Asymptotic metric of periodic  $k$  BPS monopoles (ALG)

S.A. Cherkis and A. Kapustin, arXiv:hep-th/0109141

Asymptotic metric of doubly-periodic  $k$  BPS monopoles (ALH)

M. Hamanaka, H. Kanno and D. Muranaka, arXiv:1311.7143 [hep-th]

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# What's Monopole Walls

Moduli of monopole walls and amoebas

S.A. Cherkis and R.S. Ward, arXiv:1202.1294 [hep-th]

Space:  $T^2 \times \mathbb{R}$ , Coordinates:  $x^\alpha := (x, y, z)$  ( $\alpha = 1, 2, 3$ ),  
 Periodicities:  $x \sim x + 1, y \sim y + 1$ .

## Bogomolny equation

$$*D_A\phi = -F,$$

where  $D_A\phi := d\phi + [A, \phi]$  and  $F := dA + A \wedge A$ .

## Boundary condition

$$\text{EigVal } \phi = 2\pi i Q_\pm z + O(1) \quad \text{as } z \rightarrow \pm\infty,$$

## Chern number

$$Q_\pm = \int_{T_z} c_1(E_\pm) = \frac{i}{2\pi} \int_{T_z} \text{tr } F_\pm,$$

# What's Monopole Walls

ex. A U(1) solution with Dirac-type singularities

$$\phi = v - \frac{1}{r} - \sum_{m,n \in \mathbb{Z}} \left[ \frac{1}{\sqrt{(x-m)^2 + (y-n)^2 + z^2}} - \frac{1}{\sqrt{m^2 + n^2}} \right]$$

Note: The double-sum will not converge toward the z-direction.

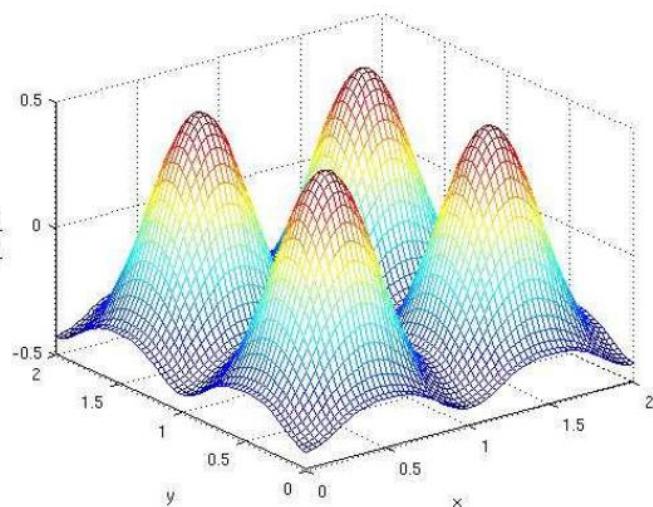
ex. The SU(2) constant-energy solution

$$\phi = 2\pi iz\sigma_3, \quad A = \pi i(ydx - xdy)\sigma_3.$$

Perturbative approach



Note: Doubly-periodic up to gauge.  
The four-moduli is proved.



An SU(2) monopole wall with charge  $(Q_-, Q_+) = (1, 1)$

R.S. Ward, arXiv:hep-th/0505254

An SU(2) monopole wall with charge  $(Q_-, Q_+) = (0, 1)$

R.S. Ward, arXiv:hep-th/0612047

# Setup

## Gibbons-Hawking ansatz

$$ds^2 = U \mathbf{d}\mathbf{x} \cdot \mathbf{d}\mathbf{x} + U^{-1} (\mathbf{d}\theta + \mathbf{W} \cdot \mathbf{d}\mathbf{x})^2,$$

where  $\text{grad } U = \text{rot } \mathbf{W}$ ,  $U = \lambda + 2M \sum_{j=1}^k \frac{1}{|\mathbf{x} - \mathbf{x}_j|}$ .

Corresponding Lagrangian:

$$\mathcal{L} = \frac{1}{2} g_{ij} \mathbf{V}_i \cdot \mathbf{V}_j + \frac{1}{2} h_{ij} (\dot{\theta}_i + \mathbf{W}_{ik} \cdot \mathbf{V}_k) (\dot{\theta}_j + \mathbf{W}_{jl} \cdot \mathbf{V}_l).$$

Electric charges of slowly moving dyons:  $q_i = \kappa h_{ij} (\dot{\theta}_j + \mathbf{V}_{jl} \cdot \mathbf{V}_l)$ .

Sigma model Lagrangian for well-separated BPS dyons

$$L_\ell = \underbrace{-(g^2 + q_\ell^2)^{1/2} \phi (1 - \mathbf{V}_\ell^2)^{1/2}}_{\text{Scalar interaction}} + \underbrace{q_\ell \mathbf{V}_\ell \cdot \mathbf{A} - q_\ell A_0}_{\text{Electric interaction}} + \underbrace{g \mathbf{V}_\ell \cdot \tilde{\mathbf{A}} - g \tilde{A}_0}_{\text{Magnetic interaction}}$$

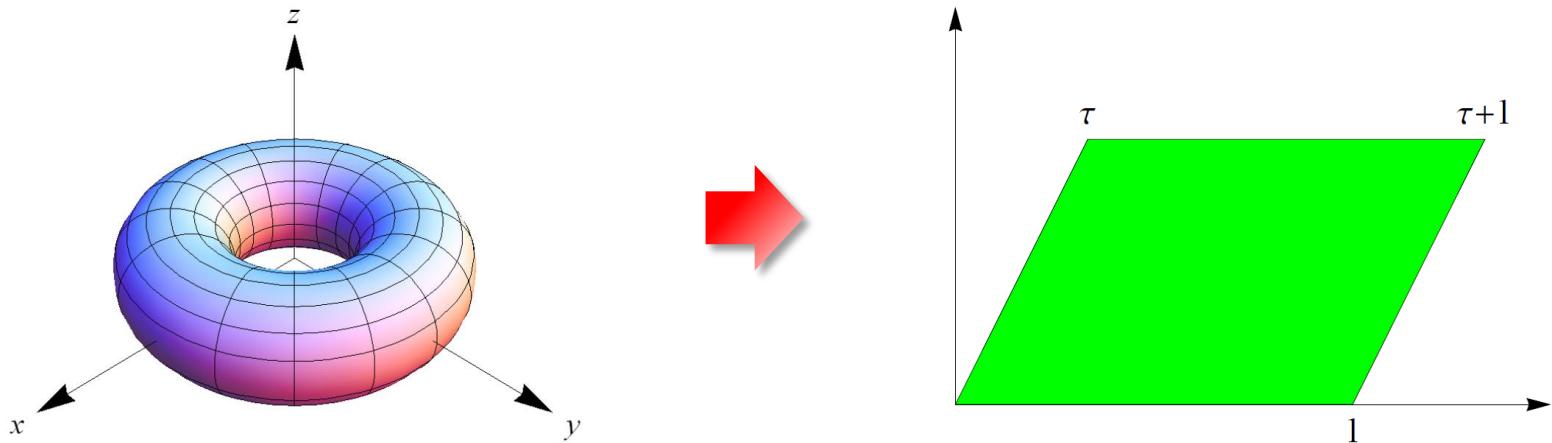
where  $\tilde{\mathbf{A}}, \tilde{A}_0$  are dual potentials.

# Setup

## Complex torus coordinates

$$\xi := x + \tau y,$$

where  $\tau := \tau_1 + i\tau_2$  ( $\tau_1, \tau_2 \in \mathbb{R}$ ),  $\xi \sim \xi + m + n\tau$  ( $m, n \in \mathbb{Z}$ ).



## The metric of complex torus

$$dx \cdot dx := g_{ij} dx^i dx^j := \frac{\nu}{\tau_2} (dx^2 + 2\tau_1 dx dy + |\tau|^2 dy^2) + dz^2.$$

where  $\nu := \sqrt{\det g}$ ,  $g := (g_{ij})$ .

# The Metric of SU(2) Monopole Walls

## The ansatz of the background fields

$$\phi(\mathbf{x}) = v + \sum_{j=1}^k \phi^j(\mathbf{x} - \mathbf{a}_j), \quad A_\xi(\mathbf{x}) = b + \sum_{j=1}^k A_\xi^j(\mathbf{x} - \mathbf{a}_j), \quad A_z(\mathbf{x}) = 0$$

$$\phi^j(\mathbf{x}) = \frac{1}{4\pi} \sum_{m=-M}^M \sum_{n=-N}^N \frac{-g}{\sqrt{|\xi - m - n\tau|^2 + z^2}} = \frac{g}{2}|z| - gC_{M,N} \quad \text{as } |z| \rightarrow \infty$$

→  $\phi(\mathbf{x}) = v_{\text{ren}} + \frac{g}{2} \sum_{j=1}^k |z - z_j|.$        $v_{\text{ren}} := v - kgC_{M,N}$

Converge      Diverge

↓ Bogomolny eq.

## The background gauge fields

$$A_\xi^j(\mathbf{x}) = \frac{i\nu g}{8\tau_2} \text{sign}(z) \bar{\xi}, \quad A_z^j(\mathbf{x}) = 0,$$

*Doubly-periodic  
up to gauge*

# The Metric of SU(2) Monopole Walls

## Periodic boundary conditions for phases

$$\begin{aligned}\theta &\mapsto \theta + \frac{\nu g}{4} \operatorname{sign}(z) y & \text{when } \xi &\mapsto \xi + 1, \\ \theta &\mapsto \theta - \frac{\nu g}{4} \operatorname{sign}(z) x & \text{when } \xi &\mapsto \xi + \tau.\end{aligned}$$

## The harmonic function and the Dirac potential

$$u(z) = \frac{1}{2}|z| - C_{M,N}, \quad w(\mathbf{x}) = \frac{i\nu}{8\tau_2} \operatorname{sign}(z) \bar{\xi},$$



Dyonize and Lorentz boost

## The background fields of moving dyonic monopole walls

$$\begin{aligned}\phi^j(\mathbf{x}) &\simeq (g^2 + q_j^2)^{1/2} u(z) (1 - V_j^2)^{1/2}, \\ A_\xi^j(\mathbf{x}) &\simeq -q_j u(z) V_{j\xi} + g w(\mathbf{x}), & \tilde{A}_\xi^j(\mathbf{x}) &\simeq -g u(z) V_{j\xi} - q_j w(\mathbf{x}), \\ A_z^j(\mathbf{x}) &\simeq -q_j u(z) V_{jz}, & \tilde{A}_z^j(\mathbf{x}) &\simeq -g u(z) V_{jz}, \\ A_0^j(\mathbf{x}) &\simeq -q_j u(z) + g(w V_j^\xi + \bar{w} V_j^{\bar{\xi}}), & \tilde{A}_0^j(\mathbf{x}) &\simeq -g u(z) - q_j(w V_j^\xi + \bar{w} V_j^{\bar{\xi}}),\end{aligned}$$

# The Metric of SU(2) Monopole Walls



Substitute into the Lagrangian

## Sigma model Lagrangian for $k = 2$

$L_{21} = L_{\text{CM}} + L_{\text{rel}}$ , where

$$L_{\text{CM}} = \frac{vg}{4}(\mathbf{V}_2 + \mathbf{V}_1)^2 - \frac{v}{4g}(q_2 + q_1)^2 + \frac{b}{2}(q_2 + q_1)(V_2^\xi + V_1^\xi) + \frac{\bar{b}}{2}(q_2 + q_1)(V_2^{\bar{\xi}} + V_1^{\bar{\xi}}),$$

$$\begin{aligned} L_{\text{rel}} = & \frac{g^2}{2} \left( \frac{v_{\text{ren}}}{2g} + \frac{1}{2}|z_2 - z_1| \right) (\mathbf{V}_2 - \mathbf{V}_1)^2 - \frac{1}{2} \left( \frac{v_{\text{ren}}}{2g} + \frac{1}{2}|z_2 - z_1| \right) (q_2 - q_1)^2 \\ & + \left\{ \frac{b}{2} + \frac{i\nu g}{8\tau_2} \text{sign}(z_2 - z_1) (\bar{\xi}_2 - \bar{\xi}_1) \right\} (q_2 - q_1)(V_2^\xi - V_1^\xi) \\ & + \left\{ \frac{\bar{b}}{2} - \frac{i\nu g}{8\tau_2} \text{sign}(z_2 - z_1) (\xi_2 - \xi_1) \right\} (q_2 - q_1)(V_2^{\bar{\xi}} - V_1^{\bar{\xi}}). \end{aligned}$$

$L_{\text{CM}}$  diverge, while

$L_{\text{rel}}$  **converge** -> change the electric charge to the phases  
in terms of the Legendre transformation. Then...

# The Metric of SU(2) Monopole Walls

## The asymptotic metric from 2-monopole walls

$$\frac{1}{g} ds^2 = U \mathbf{d}\mathbf{x} \cdot \mathbf{d}\mathbf{x} + \frac{1}{U} (\mathbf{d}\theta + \mathbf{W} \cdot \mathbf{d}\mathbf{x})^2.$$

where

$$U = \frac{v_{\text{ren}}}{2} + \frac{g}{2}|z|, \quad W_\xi = \frac{b}{2} + \frac{i\nu g}{8\tau_2} \text{sign}(z) \bar{\xi}, \quad W_{\bar{\xi}} = \bar{W}_\xi, \quad W_z = 0.$$

and relative coordinates:

$$\xi := \xi_2 - \xi_1, z := z_2 - z_1, \mathbf{V} := \mathbf{V}_2 - \mathbf{V}_1, q := q_2 - q_1$$

- *The metric of complex torus* is included in dot products.
- *Doubly-periodic* (cf. the boundary condition for the phase).
- *Modular invariant*: for any  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ ,

$$\xi \mapsto \frac{\xi}{c\tau + d}, \quad \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \tau_2 \mapsto \frac{\tau_2}{|c\tau + d|^2}$$

# The Metric of SU(2) Monopole Walls

## The asymptotic metric from multi-monopole walls

$$\frac{1}{g} ds^2 = U_{IJ} d\mathbf{X}_I \cdot d\mathbf{X}_J + U_{IJ}^{-1} (d\Theta_I + \mathbf{W}_{IK} \cdot d\mathbf{X}_K) (d\Theta_J + \mathbf{W}_{JL} \cdot d\mathbf{X}_L)$$

where

$$U_{JJ} = (k-1) \frac{v_{\text{ren}}}{k} + \frac{g}{2} \sum_{I \neq J} |Z_J - Z_I|,$$

$$U_{IJ} = -\frac{v_{\text{ren}}}{k} - \frac{g}{2} |Z_J - Z_I|, \quad (I \neq J)$$

$$(W_\xi)_{JJ} = (k-1) \frac{b}{k} + \frac{i\nu g}{8\tau_2} \sum_{I \neq J} \text{sign}(Z_J - Z_I) (\bar{\Xi}_J - \bar{\Xi}_I),$$

$$(W_\xi)_{IJ} = -\frac{b}{k} - \frac{i\nu g}{8\tau_2} \text{sign}(Z_J - Z_I) (\bar{\Xi}_J - \bar{\Xi}_I), \quad (I \neq J)$$

$$(W_{\bar{\xi}})_{IJ} = (\bar{W}_\xi)_{IJ}, \quad (W_z)_{IJ} = 0.$$

and relative coordinates:  $(I, J, K, L = 1, \dots, k-1)$

$$\mathbf{X}_J := (\Xi_J, Z_J) \quad \Xi_J := \xi_J - \xi_k, \quad Z_J := z_J - z_k, \quad \Theta_J := \theta_J - \theta_k$$

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- Summary

- We have obtained new hyperkähler metrics whose asymptotic behavior is the ALH type from the moduli space of monopole walls. The metrics
  - contain the metric of complex torus,
  - are doubly-periodic,
  - are modular invariant.
- More results: (cf. [arXiv:1311.7143\[hep-th\]](#))  
The asymptotic metrics of the moduli space of monopole walls with Dirac-type singularities can also be obtained, in which the maximum number of singularities are  $2k$  in  $SU(2)$  case.
- Future works:
  - Global metrics on the moduli space of monopole walls.
  - Monopole wall scattering.

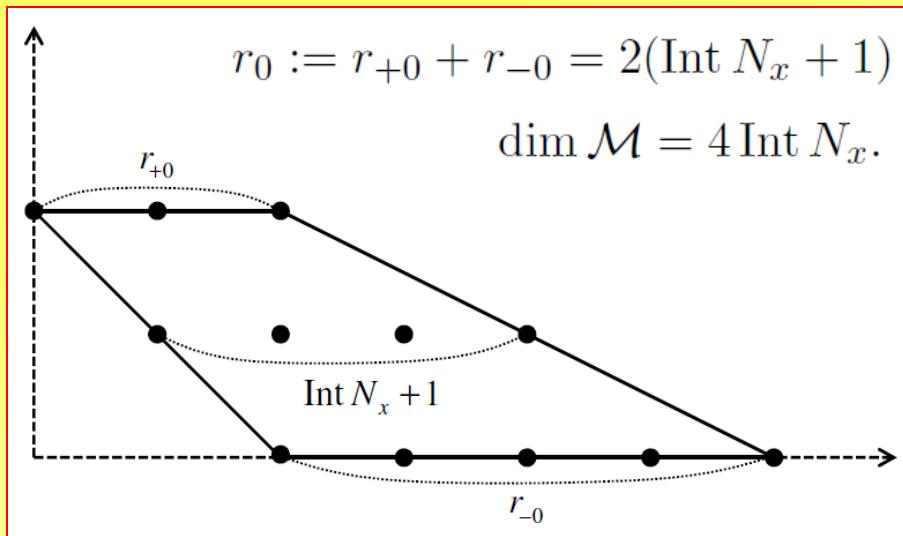
*Thank you !*

# The Metric with Dirac Singularities

The maximum number of Dirac singularities of  $SU(2)$  walls

$$r_0 \leq \frac{1}{2} \dim \mathcal{M} + 2.$$

Proof: The maximum Newton polygon of  $U(2)$  walls  $\rightarrow$  trapezoid



Note: The dimension of the relative moduli space is  $4(k - 1) \rightarrow r_0 = 2k$

- = The maximal number of the matter hypermultiplets in the fundamental rep. of  $SU(k)$  super YM theory with eight supercharges.

# The Metric with Dirac Singularities

Replacing by  $v + \sum_{\ell=1}^{r_0} g_\ell u(r_{\ell z} - z)$ ,  $b + \sum_{\ell=1}^{r_0} g_\ell w(r_\ell - x)$ , then

The asymptotic metric of 2-wall with Dirac singularities

$$U = \frac{v'_{\text{ren}}}{2} + \frac{g}{2}|z| + \frac{1}{4} \sum_{\ell=1}^{r_0} g_\ell \left| r_{\ell z} - \frac{z}{2} \right| + \frac{1}{4} \sum_{\ell=1}^{r_0} g_\ell \left| r_{\ell z} + \frac{z}{2} \right|,$$

$$\begin{aligned} W_\xi &= \frac{b}{2} + \frac{i\nu g}{8\tau_2} \text{sign}(z) \bar{\xi} + \frac{i\nu}{16\tau_2} \sum_{\ell=1}^{r_0} g_\ell \text{sign}\left(r_{\ell z} - \frac{z}{2}\right) \left(\bar{r}_{\ell\xi} - \frac{\bar{\xi}}{2}\right) \\ &\quad + \frac{i\nu}{16\tau_2} \sum_{\ell=1}^{r_0} g_\ell \text{sign}\left(r_{\ell z} + \frac{z}{2}\right) \left(\bar{r}_{\ell\xi} + \frac{\bar{\xi}}{2}\right), \end{aligned}$$

$$W_{\bar{\xi}} = \bar{W}_\xi, \quad W_z = 0.$$

where  $v'_{\text{ren}} := v - \left(2 + \sum_{\ell=1}^{r_0} \frac{g_\ell}{g}\right) g C_{M,N}$

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