# Do time-like and space-like reductions always commute?

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# Motivation

Motivated by general results on time-like reductions of four-dimensional $N = 2$ supergravity ($c$-map):
V. Cortés, P. Dempster, T.M., O. Vaughan, to appear.

Here we will give a self-contained analysis of the space-like/time-like (ST) and time-like/space-like (TS) reduction of pure five-dimensional supergravity based on:
V. Cortés, P. Dempster, T.M., arXiv:1401.5672

# Previous results

- Scalar geometry is locally $G_{2(2)}/(SL(2) \cdot SL(2))$ for both ST reduction and TS reduction
- ST and TS reduction are related by analytic continuation to SS reduction (and hence one another).
- Map between ST and TS reduction ($(t, \psi)$-flip) is related to the '4D-5D lift'.

M. Berkooz and B. Pioline, JHEP **0805** (2008) 045 [arXiv:0802.1659] G. Compere, S. de Buyl, E. Jamsin and A. Virmani, Class. Quant. Grav. **26** (2009) 125016 [arXiv:0903.1645], G. Compere, S. de Buyl, S. Stotyn and A. Virmani, JHEP **1011** (2010) 133 [arXiv:1006.5464].

# Reduction of 5d supergravity

Bosonic part of action of pure 5d supergravity.

$$S = \int d^5x \left[ \sqrt{\hat{g}} \left( \frac{\hat{R}}{2} - \frac{1}{4}\mathcal{F}_{\hat{\mu}\hat{\nu}}\mathcal{F}^{\hat{\mu}\hat{\nu}} \right) + \frac{1}{6\sqrt{6}} \varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}} \mathcal{F}_{\hat{\mu}\hat{\nu}}\mathcal{F}_{\hat{\rho}\hat{\sigma}}\mathcal{A}_{\hat{\lambda}} \right] .$$

Reduction:

$$ds^2_{(5)} = -\epsilon_1 e^{2\sigma} \left( dx^0 + \mathcal{A}^0 \right)^2 - \epsilon_2 e^{2\phi-\sigma} \left( dx^4 + B \right)^2 + e^{-2\phi-\sigma} ds^2_{(3)},$$

$\epsilon_1 = \epsilon_2 = -1$ space/space
$\epsilon_1 = -1$, $\epsilon_2 = 1$ space/time
$\epsilon_1 = 1$, $\epsilon_2 = -1$ time/space
$\epsilon := -\epsilon_1 \epsilon_2$

Three-dimensional Lagrangian:

$$
\begin{aligned}
\mathcal{L}_3 \quad = \quad & \frac{R}{2} + \frac{3}{4y^2}\epsilon_1(\partial x)^2 - \frac{3}{4y^2}(\partial y)^2 - \frac{1}{4\phi^2}(\partial\phi)^2 \\
& + \frac{1}{4\phi^2}\epsilon_1\left(\partial\tilde{\phi} + p^I \overleftrightarrow{\partial} s_I\right)^2 \\
& + \frac{y^3}{12\phi}\epsilon(\partial p^0)^2 + \frac{y}{4\phi}\epsilon_2\left(\partial p^1 - x\partial p^0\right)^2 \\
& + \frac{3}{y^3\phi}\epsilon_2\left(\partial s_0 + x\partial s_1 - \frac{1}{6}x^3\partial p^0 + \frac{1}{2}x^2\partial p^1\right)^2 \\
& + \frac{1}{y\phi}\epsilon\left(\partial s_1 - \frac{1}{2}x^2\partial p^0 + x\partial p^1\right)^2 .
\end{aligned}
$$

8 scalars: $(x, y, \phi, \tilde{\phi}, p^0, p^1, s_0, s_1)$.

## Co-frame

$$(\theta^a) = (\eta^2, \xi_2, \alpha, \beta, \eta^0, \eta^1, \xi_0, \xi_1)\ .$$

where

$$\begin{aligned}
\eta^2 &= \frac{1}{\phi}\left(d\tilde{\phi} + p^I ds_I - s_I dp^I\right), \qquad \xi_2 = \frac{d\phi}{\phi}, \\
\alpha &= \frac{\sqrt{3}}{y}dx, \qquad \beta = \frac{\sqrt{3}}{y}dy, \\
\eta^0 &= \sqrt{\frac{y^3}{3\phi}}dp^0, \qquad \eta^1 = \sqrt{\frac{y}{\phi}}\left(dp^1 - xdp^0\right), \\
\xi_0 &= 2\sqrt{\frac{3}{y^3\phi}}\left(ds_0 + xds_1 + \frac{1}{2}x^2dp^1 - \frac{1}{6}x^3dp^0\right), \\
\xi_1 &= \frac{2}{\sqrt{y\phi}}\left(ds_1 + xdp^1 - \frac{1}{2}x^2dp^0\right).
\end{aligned}$$

## Scalar metric:

$$\begin{aligned}
4g_{SS/ST/TS} &= -\epsilon_1\eta^2 \otimes \eta^2 + \xi_2 \otimes \xi_2 - \epsilon_1\alpha \otimes \alpha + \beta \otimes \beta \\
&\quad -\epsilon\eta^0 \otimes \eta^0 - \epsilon_2\eta^1 \otimes \eta^1 - \epsilon_2\xi_0 \otimes \xi_0 - \epsilon\xi_1 \otimes \xi_1 .
\end{aligned}$$

Lie algebra structure on the co-frame:

$$d\theta^A = -c^A{}_{BC}\theta^B \wedge \theta^C$$

$$\begin{aligned}
d\eta^2 &= -\xi_0 \wedge \eta^0 - \xi_1 \wedge \eta^1 - \xi_2 \wedge \eta^2, \\
d\xi_2 &= 0, \\
d\alpha &= \frac{1}{\sqrt{3}}\alpha \wedge \beta, \\
d\beta &= 0, \\
d\eta^0 &= \frac{\sqrt{3}}{2}\beta \wedge \eta^0 - \frac{1}{2}\xi_2 \wedge \eta^0, \\
d\eta^1 &= \frac{1}{2\sqrt{3}}\beta \wedge \eta^1 - \frac{1}{2}\xi_2 \wedge \eta^1 - \alpha \wedge \eta^0, \\
d\xi_0 &= -\frac{\sqrt{3}}{2}\beta \wedge \xi_0 - \frac{1}{2}\xi_2 \wedge \xi_0 + \alpha \wedge \xi_1, \\
d\xi_1 &= -\frac{1}{2\sqrt{3}}\beta \wedge \xi_1 - \frac{1}{2}\xi_2 \wedge \xi_1 + \frac{2}{\sqrt{3}}\alpha \wedge \eta^1.
\end{aligned}$$

$\Rightarrow$ scalar metrics are left-invariant metrics on a Lie group $L \simeq \mathbb{R}^8$ with Lie algebra $\mathfrak{l}$: $(M_{SS}, g_{SS}) \cong (L, g_{SS})$, $(M_{ST}, g_{ST}) \cong (L, g_{ST})$, $(M_{TS}, g_{TS}) \cong (L, g_{TS})$.

Generators $(T_a) = (V_2, U^2, A, B, V_0, V_1, U^0, U^1)$.

$$
\begin{aligned}
[B, A] &= \frac{1}{\sqrt{3}} A, \qquad \left[U^2, V_2\right] = V_2, \\
\left[V_0, U^0\right] &= -V_2, \qquad \left[V_1, U^1\right] = -V_2, \\
\left[U^2, V_I\right] &= \frac{1}{2} V_I \quad \text{for } I = 0, 1, \qquad \left[U^2, U^I\right] = \frac{1}{2} U^I \quad \text{for } I = 0, 1, \\
[B, V_0] &= -\frac{\sqrt{3}}{2} V_0, \quad [B, V_1] = -\frac{1}{2\sqrt{3}} V_1, \quad \left[B, U^0\right] = \frac{\sqrt{3}}{2} U^0, \\
& \left[B, U^1\right] = \frac{1}{2\sqrt{3}} U^1, \\
[A, V_0] &= V_1, \qquad \left[A, U^1\right] = -U^0, \qquad [A, V_1] = -\frac{2}{\sqrt{3}} U^1.
\end{aligned}
$$

$\Rightarrow \mathfrak{l}$ is a solvable Lie algebra.

In fact, $\mathfrak{l}$ is the Iwasawa subalgebra of the Lie algebra $\mathfrak{g}$ of the Lie group $G_{2(2)}$.

# Iwasawa decomposition

Iwasawa decomposition of a real simple non-compact Lie group $G$:

$$G = LK$$

$L =$ Iwasawa subgroup (maximal solvable subgroup), $K =$ maximal compact subgroup.
$L$ acts simply transitively on the symmetric space $G/K$:

$$L \simeq \frac{G}{K}$$

Example:

$$L \simeq \frac{G_{2(2)}}{SO(4)}$$

This Riemannian symmetric space is quaternionic-Kähler, i.e.

$$\mathrm{Hol}(G/K) \subset Sp(1)_c \cdot Sp(n)_c \cong SU(2) \cdot USp(2n) \subset SO(4n)$$

Here $n = 2$.

# Iwasawa subgroup

If $K$ is replaced by a non-compact real form $H$, then $G/H$ is still a symmetric space, but $G \neq LH$. The Iwasawa subgroup $L$ does not act simply transitively, but can still act with open orbit:

$$L \subset \frac{G}{H}$$

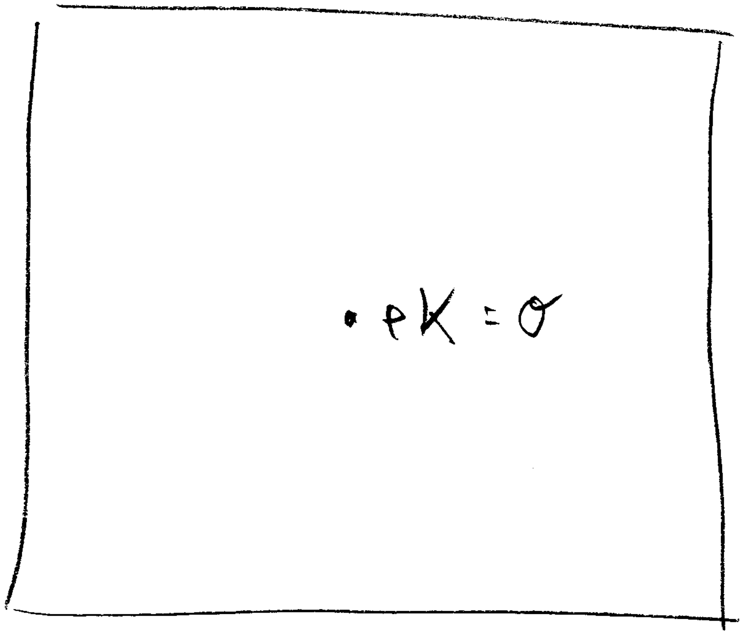
Example

$$L \subset \frac{G_{2(2)}}{SL(2) \cdot SL(2)}$$

This Riemannian symmetric space is para-quaternionic Kähler, i.e.

$$\mathrm{Hol}(G/H) \subset Sp(\mathbb{R}^2) \cdot Sp(\mathbb{R}^{2n}) \subset SO(2n, 2n)$$

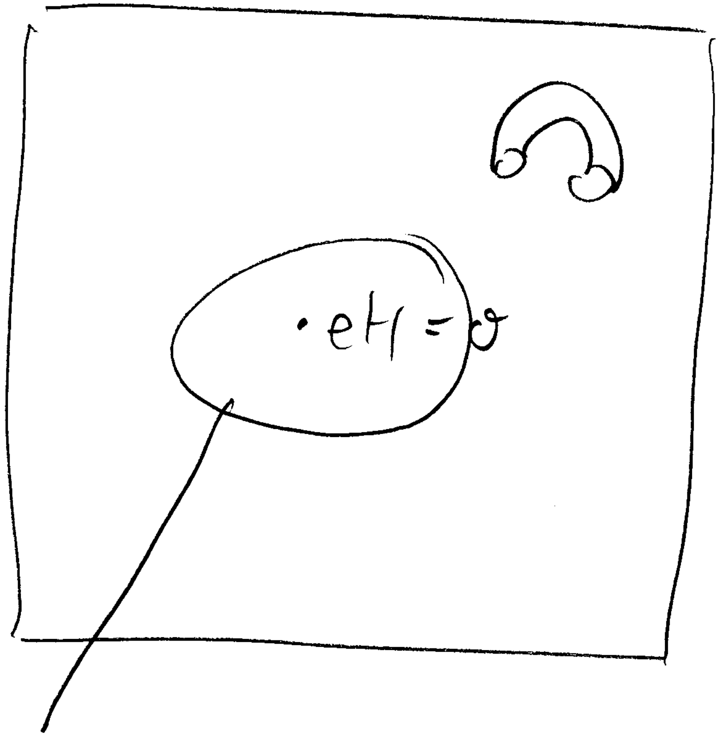
Here $n = 2$.




$$L \cong G/K \cong \mathbb{R}^m$$

$G = G_{2(2)}$, $K = SO(4)$

$m = 8$




$$L \subset G/H$$

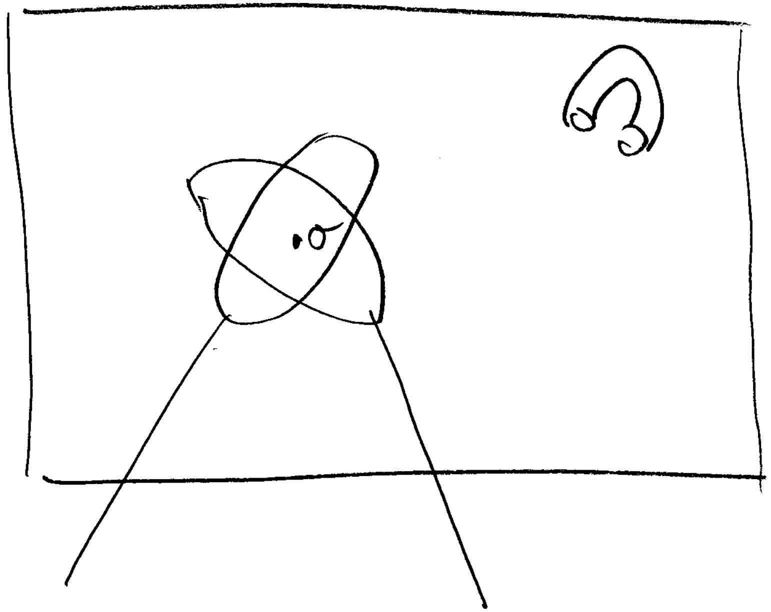
$H = SL(2)\cdot SL(2)$

# Open orbits

Goal: Identify the scalar manifolds $(M_{ST}, g_{ST})$ and $(M_{TS}, g_{TS})$ with open orbits of $L$ on $G_{2(2)}/(SL(2) \cdot SL(2))$.

Problem: The standard Iwasawa subgroup $L \subset G_{2(2)}$ does not act with open orbit on the standard base point
$eH \in G/H = G_{2(2)}/(SL(2) \cdot SL(2))$.

Equivalent options: (i) find conjugate Iwasawa subgroups $L_1$, $L_2$ which act with open orbit on $eH$. (ii) find points on $G/H$ on which $L$ acts with open orbit. We used (i).

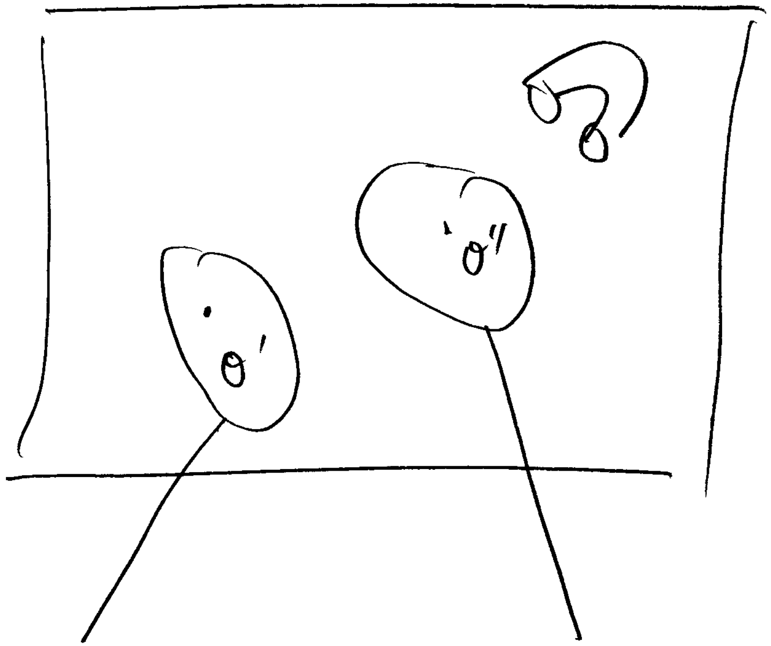



$L_1 \sigma$ $\quad$ $L_2 \sigma$

$L_i = a_i L a_i^{-1}$

$\sigma = eH$

$\Leftrightarrow$




$L\sigma'$ $\quad$ $L\sigma''$

$\sigma' = a_1^{-1}\sigma$ $\quad$ $\sigma'' = a_2^{-1}\sigma$

**Result:** we have found $a_i \in G_{2(2)}$, $i = 1, 2$ such that $L_i = a_i L a_i^{-1}$ act with open orbit, with induced metrics

$$\begin{aligned} g_1 &\propto \text{diag}(-1, 1, -1, 1, -1, 1, 1, -1) \propto g_{TS} \\ g_2 &\propto \text{diag}(1, 1, 1, 1, -1, -1, -1, -1) \propto g_{ST} \end{aligned}$$

**Result:** The automorphism group of $L$ is

$$\mathrm{Aut}(L) \cong \mathrm{Aut}(\mathfrak{l}) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathrm{Inn}(\mathfrak{l}), \quad \text{where} \quad \mathrm{Inn}(\mathfrak{l}) \cong L$$

**Consquence:** $(L, g_1) \cong (M_{TS}, g_{TS})$ and $(L, g_2) \cong (M_{ST}, g_{ST})$ are not related by an automorphism of $L$.

# $\epsilon$-complex geometry

| Almost complex structure | Almost para-complex structure |
|---|---|
| $J^2 = -1$ | $J^2 = 1$, balanced eigenvalues |
| $J \cong \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes \mathbb{1}$ | $J \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbb{1}$ |

- $J$ integrable: (para-)complex structure
- $J$ skew wrt metric: (para-)Hermitian structure
- $J$ parallel wrt Levi-Civita connection: (para-)Kähler structure

# $\epsilon$-quaternionic geometry

| Almost quaternionic structure | Almost para-quaternionic structure |
|---|---|
| $J_\alpha J_\beta = -J_\beta J_\alpha = J_\gamma$, $\alpha, \beta, \gamma$ cyclic. | |
| $J_\alpha^2 = J_\beta^2 = J_\gamma^2 = -1$ | $J_\alpha^2 = J_\beta^2 = -J_\gamma^2 = 1$ |

$Q = \text{span}\{J_1, J_2, J_3\}$ $\epsilon$-quaternionic-Kähler structure:

- $J_1, J_2, J_3$ skew wrt the metric.
- $Q$ is parallel wrt the Levi-Civita connection

$J_1, J_2, J_3$ are not separately parallel.
$J_1, J_2, J_3$ are, in general, not integrable!

# Para-quaternionic Kähler geometry of $M_{ST}$ and $M_{TS}$

Define the following endomorphisms on $(\mathfrak{l}, \langle \cdot, \cdot \rangle_{\epsilon_1, \epsilon_2})$:

$$
\begin{aligned}
J_1 &= \epsilon_2 U^2 \wedge V_2 - B \wedge A + \epsilon \frac{\sqrt{3}}{2} U^1 \wedge U^0 - \epsilon_2 \frac{1}{2} U^1 \wedge V_1 \\
&\quad + \epsilon_2 \frac{1}{2} U^0 \wedge V_0 + \epsilon \frac{\sqrt{3}}{2} V_1 \wedge V_0 \,, \\
J_2 &= \epsilon_2 \frac{\sqrt{3}}{2} U^1 \wedge V_2 + \epsilon \frac{1}{2} V_0 \wedge V_2 - \frac{1}{2} U^0 \wedge U^2 - \epsilon_1 \frac{\sqrt{3}}{2} V_1 \wedge U^2 \\
&\quad - \frac{1}{2} U^1 \wedge A - \epsilon_1 \frac{\sqrt{3}}{2} V_0 \wedge A - \frac{\sqrt{3}}{2} U^0 \wedge B + \epsilon_1 \frac{1}{2} V_1 \wedge B \,, \\
J_3 &= \epsilon_2 \frac{1}{2} U^0 \wedge V_2 - \epsilon \frac{\sqrt{3}}{2} V_1 \wedge V_2 - \epsilon_1 \frac{\sqrt{3}}{2} U^1 \wedge U^2 + \frac{1}{2} V_0 \wedge U^2 \\
&\quad - \frac{\sqrt{3}}{2} U^0 \wedge A + \epsilon_1 \frac{1}{2} V_1 \wedge A - \epsilon_1 \frac{1}{2} U^1 \wedge B - \frac{\sqrt{3}}{2} V_0 \wedge B \,,
\end{aligned}
$$

Here we write endomorphisms as bivectors:

$$
(u \wedge v)(w) = u \langle v, w \rangle - \langle u, w \rangle v \,, \quad u, v, w, \in \mathfrak{l} \,.
$$

# Result 1

An expected result:

$Q = \text{span}\{J_1, J_2, J_3\}$ is a left-invariant $\epsilon$-quaternionic structure on $L$

$\Rightarrow (L, g_1) \cong (M_{TS}, g_{TS})$ and $(L, g_2) \cong (M_{ST}, g_{ST})$ are para-quaternionic Kähler manifolds.

# Result 2

An unexpected extra feature:

$J_1$ is an **integrable** left-invariant skew-symmetric $\epsilon_1$-complex structure on $L$

$$J_1^2 = \epsilon_1 \mathbb{1} \; ,$$

where $\epsilon_1 = -1$ for ST (and SS) and $\epsilon_1 = 1$ for TS reduction.

$\Rightarrow (L, g_1) \cong (M_{TS}, g_{TS})$ is a **para-complex** manifold, while $(L, g_2) \cong (M_{ST}, g_{ST})$ is a **complex** manifold.

TS and ST reduction lead to distinct geometrical structures on the respective scalar manifolds.

# General properties of the c-map

- Temporal *c*-map: Time-like reduction of 4D $N = 2$ supergravity with vector multiplets gives scalar manifold which is para-quaternionic Kähler with an induced integrable complex structure.
- Euclidean *c*-map: Reduction of Euclidean 4D $N = 2$ supergravity with vector multiplets gives a scalar manifold which is para-quaternionic Kähler with an induced integrable para-complex structure.

V. Cortés, P. Dempster, T.M., O. Vaughan, to appear.

# c-map

1+3 N=2 Supergravity

$N$ special Kähler

$J_1^{(N)}$

$\downarrow$ temporal c-map

0+3

$M$ para-quaternionic-Kähler

$J_1^{(M)}$ integrable complex structure

0+4 N=2 Supergravity

$N'$ special para-Kähler

$J_1^{N'}$

$\downarrow$ Euclidean c-map

0+3

$M'$ para-quaternionic-Kähler

$J_1^{M'}$ integrable para-complex structure

# The Black String submanifold

$M_{ST}$ and $M_{TS}$ 'share' (even when including vector multiplets) the totally geodesic submanifold $N_{pK} \times \mathbb{R}$, which supports static magnetically charged black string solutions (BPS, non-BPS and non-extremal). $N_{pK}$ is a para-Kähler submanifold of maximal dimension.

P. Dempster and T.M., Class. Quantum Grav. **31** (2014) 045019.

# q-map

1+4 supergravity, special real

1+3
special Kähler

0+4
special para-Kähler

0+3
para-quat.-K.

0+3
para quat.-K.

Black String submanifold

$\mathbb{R} \times$ para-Kähler

# Further remarks on time-like reductions

- $(M_{ST}, g_{ST})$, $(M_{TS}, g_{TS})$ are not geodesically complete
- Regular black hole solutions contained in 'solv-patches', W. Chemissany, P. Fré, J. Rosseel, A. S. Sorin, M. Trigiante and T. Van Rief, JHEP **1009** (2010) 080
- Action of duality group in time-like reductions, G. Moore hep-th/9305139, G. Bossard, H. Nicolai and K.S. Stelle, JHEP **0907** (2009) 003

# Related ongoing work

- Work to appear/in progress on time-like reductions $4 \to 3$ (c-map) and $5 \to 3$ (q-map), with V. Cortés, P. Dempster and O. Vaughan
- Work on non-BPS and non-extremal solutions for non-symmetric target spaces, with P. Dempster, D. Errington and O. Vaughan