Bethe Forum

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Classifying orbifolds – technical details

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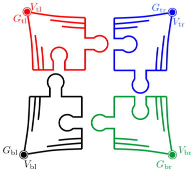
Based on 1209.3906 and 1304.7742

in collaboration with S. Ramos-Sánchez, M. Ratz, J. Torrado and P. Vaudrevange



Outline

Strings and compact dimensions

Heterotic string theory is appealing, because ...

- it is a unified theory of gravity and gauge interactions
- it is mathematically restrained and introduces only one new parameter
- it can preserve exactly $\mathcal{N} = 1$ supersymmetry in four dimensions

However, some mysteries remain ...

- Geometric constructions predict the number of space-time dimensions to be ten
- A simple compactification on a $\mathbb{T}^6$ preserves too much SUSY
- To compactify, one can go for instance either the **Calabi-Yau** or the **Orbifold** way

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Heterotic compactifications

- We need a six-dimensional compact topological space with $\mathrm{SU}(3)$**-holonomy**
- Manifolds with this property are called **Calabi-Yau**
- However, they are very complicated and many properties remain unknown (for some recent progress on the matter, cf. Groot Nibbelink and Ruehle 2014)
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Why orbifolds

- Orbifolds admit an exact CFT description on the world-sheet
- Modular invariance conditions for the partition function can be explicitly stated
- Same for the mass equations

Spectrum

$\Rightarrow$ whole spectrum (in principle) computable

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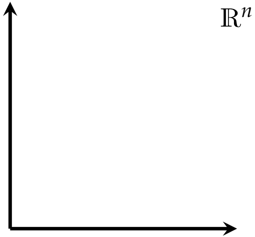

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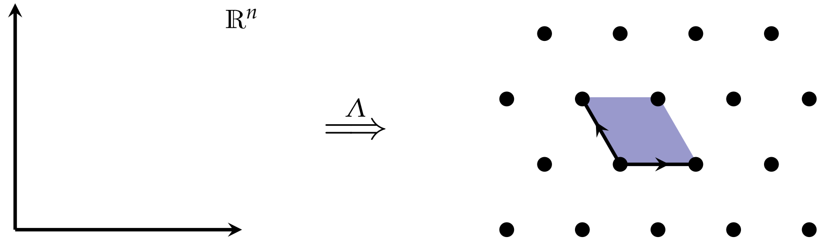

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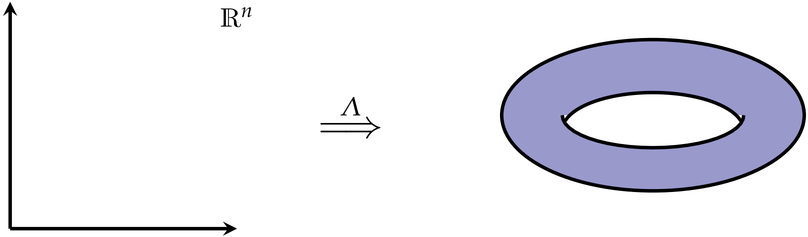

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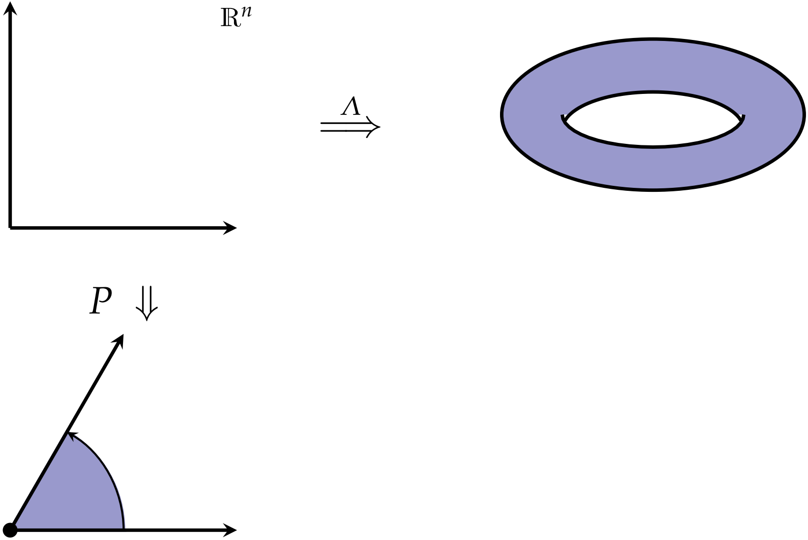

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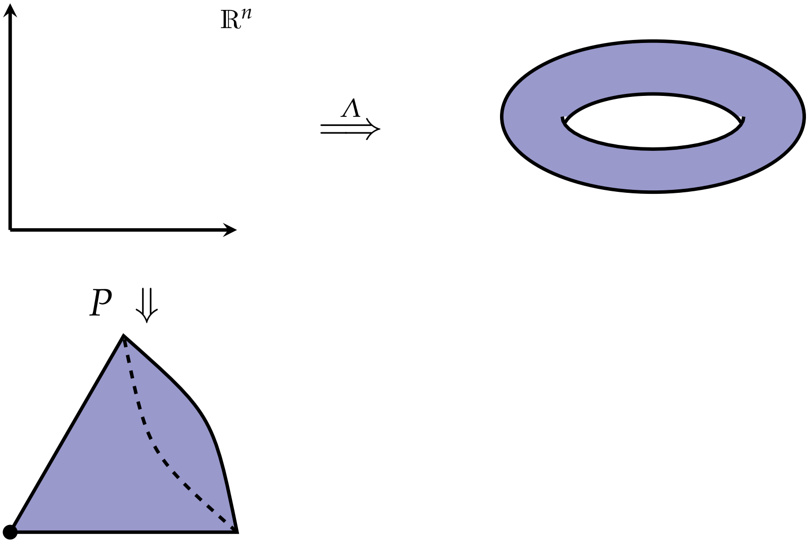

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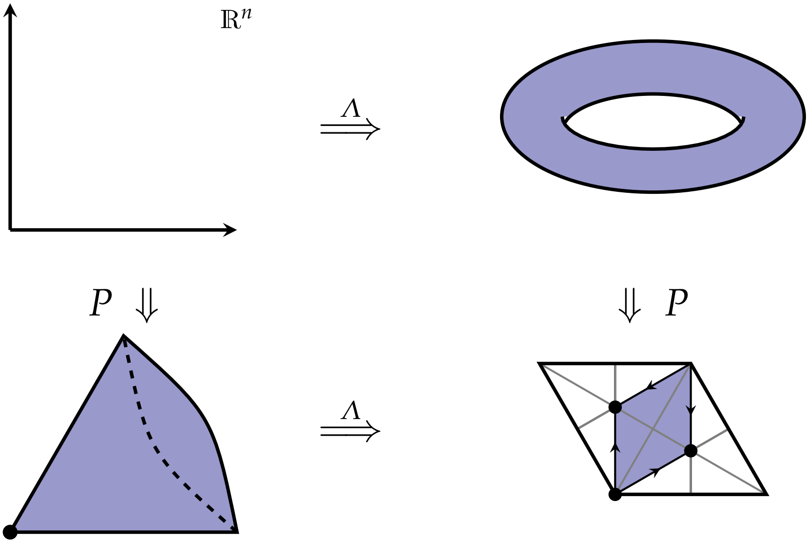

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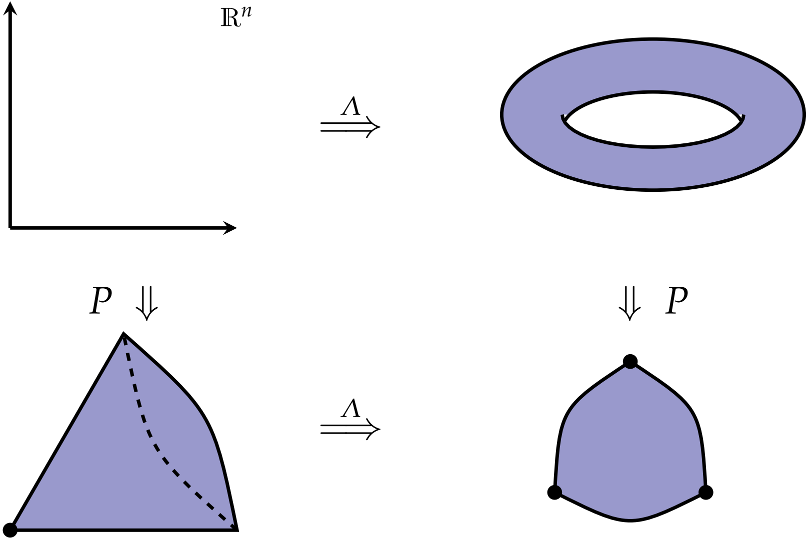

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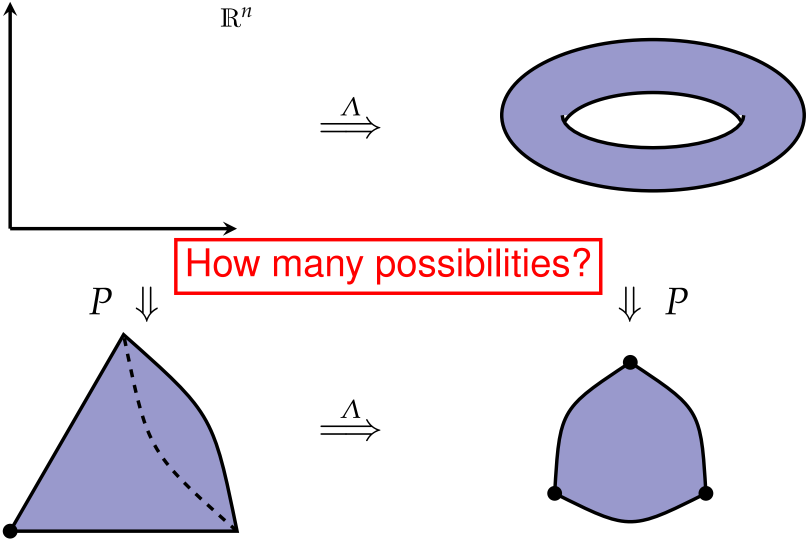

How many possibilities?

The classification story so far

(excerpt)

- The first paper on orbifolds already classified all Abelian point groups which admit $\mathcal{N} = 1$ SUSY in 4D
  Dixon, Harvey, Vafa and Witten 1985b
- Abelian orbifolds had been studied quite well, mostly omitting roto-translations and focussing on Lie root lattices
  ▸ Why is this a bad idea?
  Bailin and Love 1999; Donagi and Faraggi 2004
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ orbifolds had been studied extensively **with** roto-translations – but only $\mathbb{Z}_2 \oplus \mathbb{Z}_2$
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$\Rightarrow$ Yes it is!

Definition

Let M be a manifold and G a discrete group which acts on M. Then, the quotient $\mathcal{O} = M/G$ has the structure of an **orbifold**.

In our cases, M will be $\mathbb{R}^6$ and G will be a **crystallographic space group**.

▸ Complete definition

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▸ Complete definition

Space groups

A **space group** S is a discrete subgroup of the Euclidean group in $\mathbb{R}^n$ which contains n linearly independent translations.

- The elements $g \in S$ have the structure (ϑ, λ), where ϑ is a rotation/reflection and λ a translation: $g \cdot v \equiv \vartheta \cdot v + \lambda$
- Let $h = (\omega, \tau) \in S$; then $h \circ g \; = \; (\omega\vartheta, \omega\lambda + \tau)$

Augmented matrix notation

$$g \; = \; \left(\begin{array}{c|c} \vartheta & \lambda \\ \hline \mathbf{0} & 1 \end{array}\right)$$

$$\left(\begin{array}{c|c} \omega & \tau \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} \vartheta & \lambda \\ \hline \mathbf{0} & 1 \end{array}\right) \; = \; \left(\begin{array}{c|c} \omega\vartheta & \omega\lambda + \tau \\ \hline \mathbf{0} & 1 \end{array}\right)$$

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The lattice

- The subset $\Lambda = \{(\mathrm{id}, \lambda)\} \subseteq S$ is called the lattice of the space group.
- In general, for $g = (\vartheta, \lambda) \in S$, λ needs not to be an element of the lattice. Elements of this form are called roto-translations.
- Every lattice Λ defines an equivalence relation on vectors from $\mathbb{R}^n$: $v \approx w \;:\Leftrightarrow\; v - w \in \Lambda$.
- The fundamental domain is the unit cell of the lattice, with i. e. a torus $\mathbb{T} := \mathbb{R}^n / \Lambda$.

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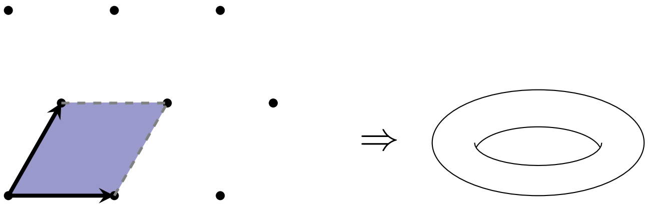

The point group

- For $S = \{(\vartheta, \lambda)\}$, the point group is $P = \{\vartheta\}$.
- In general, P is a discrete subgroup of $\mathrm{O}(6)$.
- Λ always is a normal subgroup of $S \Longrightarrow S$ is a semi-direct product iff P is a subgroup of it. Then, $S = P \ltimes \Lambda$.
- In general, one has **roto-translations** (ϑ, τ) with $\tau \notin \Lambda$!
- In that case, one yields $\mathcal{O} = \mathbb{T}/G$ from the torus by modding out the **orbifolding group**: $G = \langle(\vartheta, n_i e_i)\rangle$ where $(e_i)_{i\in\{1,\ldots,6\}}$ is a basis for Λ and $0 \leq n_i < 1$.

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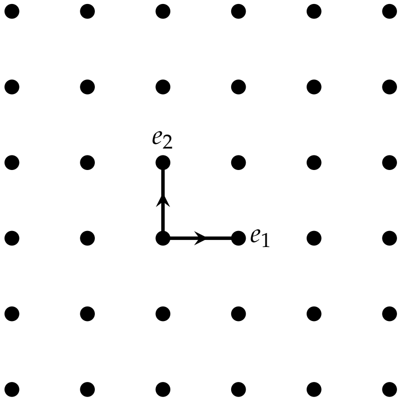

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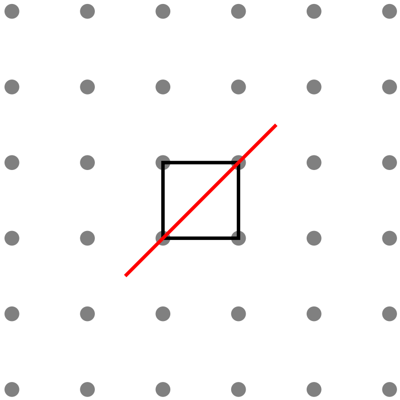

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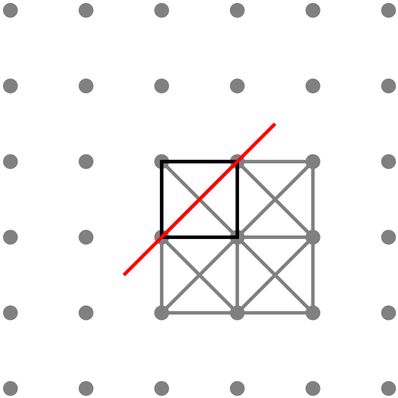

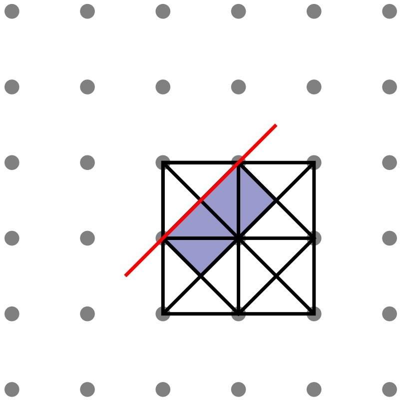

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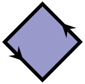

Bringing order to chaos

Alice Krige. Picture: http://de.eonline.com

Bringing order to chaos

- For any S, the short exact sequence $\mathbf{0} \to \Lambda \to S \to P \to \mathbf{1}$ holds. Thus, P maps Λ to itself.
- Consequently, when changing from Euclidean to lattice basis, the point group becomes a subgroup of $\mathrm{GL}(n, \mathbb{Z})$.

Let S and S' be two space groups of the same degree n. Let P and P' be their point groups. They belong to the same ...

1. affine class, iff they are isomorphic, i. e. if there is an affine mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f^{-1}Sf = S'$.
2. $\mathbb{Z}$-class, iff P and P' are conjugate in $\mathrm{GL}(n, \mathbb{Z})$, i. e. if there is a matrix $V \in \mathrm{GL}(n, \mathbb{Z})$ such that $V^{-1}PV = P'$.
3. $\mathbb{Q}$-class, iff P and P' are conjugate in $\mathrm{GL}(n, \mathbb{Q})$, i. e. if there is a matrix $V \in \mathrm{GL}(n, \mathbb{Q})$ such that $V^{-1}PV = P'$.

Plesken and Schulz 2000

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Plesken and Schulz 2000

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- The space of invariant forms, or short the form space of P, is

$$\mathcal{F}(P) \;=\; \left\{ F \in \mathbb{R}^{n\times n}_{\text{sym}} \mid \forall p \in P \;:\; p^T F p = F \right\} \,. \tag{1}$$

- Point groups P and P' belong to the same $\mathbb{Z}$-class, iff $\exists V \in \mathrm{GL}(n,\mathbb{Z})$, such that $V^{-1}PV \;=\; P'$
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The sublattice algorithm

Opgenorth, Plesken and Schulz 1998

Start with a finite unimodular group $G \leq \mathrm{GL}(n, \mathbb{Z})$ and compute G-sublattices of the natural lattice $L_0 = \mathbb{Z}^{n\times 1}$.

- Preprocessing: take the action of G on L_0 modulo a prime p which divides $|G|$
- Save the irreducible constituents U of the resulting representation $G \to \mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$
- Now keep a list of lattices L (starting with L_0) and compute sublattices which are kernels of homomorphisms $\varphi : L \to U$ for each U obtained as above
- This amounts to solving a set of linear equations over $\mathbb{Z}/p\mathbb{Z}$
- Perform LLL reduction on the resulting lattices
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Zassenhaus 1948; Holt and Plesken 1989

Start with a finite unimodular group $G \leq \mathrm{GL}(n, \mathbb{Z})$ and compute affine extensions.

- Compute vector systems $\widetilde{\mathrm{Der}}(G, \mathbb{Q}^n/\mathbb{Z}^n)$ consisting of all $v : G \to \mathbb{Q}^n$ that satisfy $(gh)v = (gv)h + hv \mod \mathbb{Z}^n$ for all $g,\ h \in G$
- Then factor out the submodule $\mathrm{Inn}\widetilde{\mathrm{Der}}(G, \mathbb{Q}^n/\mathbb{Z}^n)$ consisting of all $v_w : G \to \mathbb{Q}^n$ with $gv_w = w(1-g)$ for $g \in G$ which is the biggest $\mathbb{Q}$-subspace of $\widetilde{\mathrm{Der}}(G, \mathbb{Q}^n/\mathbb{Z}^n)$
- Lastly, decide which vector systems are still equivalent. This boils down to a orbit calculation in the normalizer of G in $\mathrm{GL}(n, \mathbb{Z})$

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How to preserve $\mathcal{N} = 1$ SUSY

Theory

- The point group generator is an element of $\mathrm{SO}(6) \approx \mathrm{SU}(4) \supsetneq \mathrm{SU}(3)$
- Demand exactly one surviving spinor $\Rightarrow$ $\mathrm{SU}(3)$ holonomy
- $\Rightarrow$ At most three independent rotations, two in the Abelian case (coming from the Cartan of $\mathrm{SU}(3)$)

An example

$$\begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) & 0 & 0 & 0 & 0 \\ \sin(\pi/3) & \cos(\pi/3) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(2\pi/3) & -\sin(2\pi/3) & 0 & 0 \\ 0 & 0 & \sin(2\pi/3) & \cos(2\pi/3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\pi) & -\sin(\pi) \\ 0 & 0 & 0 & 0 & \sin(\pi) & \cos(\pi) \end{pmatrix}$$

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Take P as a discrete subgroup of the $\mathbf{6}$ of $\mathrm{SO}(6) \cong \mathrm{SU}(4)$ and break to $\mathrm{SU}(3)$:

$$\mathbf{6} \to \boldsymbol{a} \oplus \boldsymbol{b} \oplus \cdots .$$

Group characters

- $\mathbf{6} \to \bigoplus_{i=1}^{c} n_i \boldsymbol{\rho}_i$ with $n_i = \frac{1}{|P|} \sum_{g \in P} \chi_{\boldsymbol{\rho}_i}(g) \overline{\chi_{\mathbf{6}}(g)}$
- Iff $\mathbf{6} \to \boldsymbol{a} \oplus \overline{\boldsymbol{a}}$ plus, possibly, some singlets, then $P \subsetneq \mathrm{U}(3)$.
- To check $P \subsetneq \mathrm{SU}(3)$, produce explicit matrix representations with GAP and check their determinants.

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Hodge numbers

MF, Ramos-Sanchez and Vaudrevange 2013

Untwisted sector

- Use the three-dimensional representation ρ used in the SUSY-checking
- Then, $\rho \otimes \overline{\rho} \to h_{\mathrm{U}}^{(1,1)} \mathbf{1} \oplus \cdots$ and $\rho \otimes \rho \to h_{\mathrm{U}}^{(2,1)} \mathbf{1} \oplus \cdots$

Twisted sectors

- Construct conjugacy classes $[g]$ of constructing elements of space group elements with fundamental domain on the torus
- If the null-space of g is zero-dimensional, this yields one twisted 27-plet and thus 1 to $h_{\mathrm{T}}^{(1,1)}$
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Hodge numbers

MF, Ramos-Sanchez and Vaudrevange 2013

Untwisted sector

- Use the three-dimensional representation ρ used in the SUSY-checking
- Then, $\rho \otimes \overline{\rho} \to h_{\mathrm{U}}^{(1,1)} \mathbf{1} \oplus \cdots$ and $\rho \otimes \rho \to h_{\mathrm{U}}^{(2,1)} \mathbf{1} \oplus \cdots$

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Fundamental groups

Dixon, Harvey, Vafa and Witten 1985a; Dixon, Harvey, Vafa and Witten 1986; Brown and Higgins 2002

- $\pi_1(\mathcal{O})$ measures the "connectedness" of the orbifold
- A non-trivial π_1 is a prerequisite for non-local GUT breaking schemes
- To compute π_1, first generate $\{g \in S | \ \exists x \in \mathbb{R}^6 \ : \ gx = x\} = F \subsetneq S$ of all elements that leave a point fixed
- Then, $\langle F \rangle$ is a normal subgroup of S
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Complete Classification (abelian)

$\mathbb{Q}$-class	$\mathbb{Z}$-cl.	aff. cl.	$\mathbb{Q}$-class	$\mathbb{Z}$-cl.	aff. cl.
$\mathbb{Z}_3$	1	1	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	12	35
$\mathbb{Z}_4$	3	3	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	10	41
$\mathbb{Z}_6 - I$	2	2	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$	2	4
$\mathbb{Z}_6 - II$	4	4	$\mathbb{Z}_2 \oplus \mathbb{Z}_6'$	4	4
$\mathbb{Z}_7$	1	1	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	5	15
$\mathbb{Z}_8 - I$	3	3	$\mathbb{Z}_3 \oplus \mathbb{Z}_6$	2	4
$\mathbb{Z}_8 - II$	2	2	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	5	15
$\mathbb{Z}_{12} - I$	2	2	$\mathbb{Z}_6 \oplus \mathbb{Z}_6$	1	1
$\mathbb{Z}_{12} - II$	1	1			

Previous work

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$\mathbb{Z}_6 - I$	2 (1)	2 (1)	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$	2	4
$\mathbb{Z}_6 - II$	4 (3)	4 (3)	$\mathbb{Z}_2 \oplus \mathbb{Z}_6'$	4	4
$\mathbb{Z}_7$	1	1	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	5	15
$\mathbb{Z}_8 - I$	3 (1)	3 (1)	$\mathbb{Z}_3 \oplus \mathbb{Z}_6$	2	4
$\mathbb{Z}_8 - II$	2	2	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	5	15
$\mathbb{Z}_{12} - I$	2	2	$\mathbb{Z}_6 \oplus \mathbb{Z}_6$	1	1
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Previous work

These are known in the literature, e. g. Bailin and Love 1999

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Previous work

Förste et al. missed four lattices, Donagi & Wendland got almost the correct number of affine classes Förste, Kobayashi, Ohki and Takahashi 2007; Donagi and Wendland 2009

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$\mathbb{Z}_{12} - I$	2	2	$\mathbb{Z}_6 \oplus \mathbb{Z}_6$	1	1
$\mathbb{Z}_{12} - II$	1	1			

Previous work

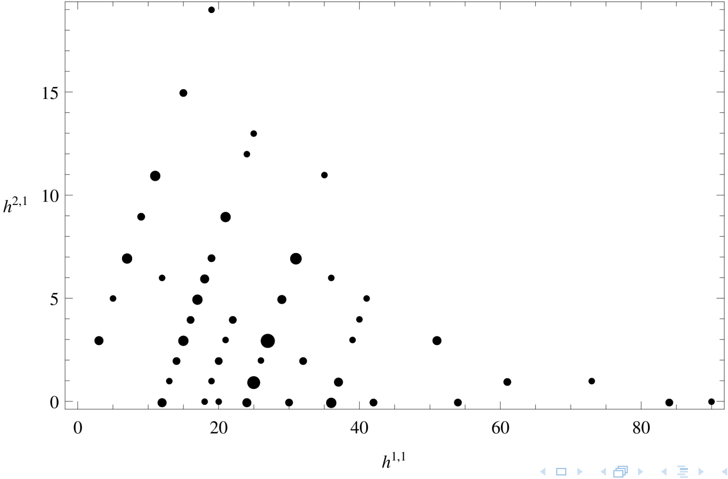
To the best of our knowledge, these are new!

Complete Classification (non-abelian)

$\mathbb{Q}$-class	$\mathbb{Z}$-cl.	aff. cl.	$\mathbb{Q}$-class	$\mathbb{Z}$-cl.	aff. cl.
S_3	6	11	$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	1	1
D_4	9	48	$\mathbb{Z}_3 \times A_4$	3	3
A_4	9	15	$\mathbb{Z}_6 \times S_3$	2	4
D_6	2	8	$\Delta(48)$	4	8
$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$	6	18	$\mathrm{GL}(2,3)$	1	4
QD_{16}	4	14	$\mathrm{SL}(2,3) \rtimes \mathbb{Z}_2$	1	3
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	5	55	$\Delta(54)$	3	10
$\mathbb{Z}_3 \times S_3$	6	16	$\mathbb{Z}_3 \times \mathrm{SL}(2,3)$	1	2
Frobenius T_7	3	3	$\mathbb{Z}_3 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$	1	1
$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	1	1	$\mathbb{Z}_3 \times S_4$	3	3
$\mathrm{SL}(2,3)$–I	4	7	$\Delta(96)$	4	12
$\mathbb{Z}_4 \times S_3$	1	2	$\mathrm{SL}(2,3) \rtimes \mathbb{Z}_4$	1	2
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	2	6	$\Sigma(36\phi)$	2	4
$\mathbb{Z}_3 \times D_4$	2	2	$\Delta(108)$	1	1
$\mathbb{Z}_3 \times Q_8$	2	2	$\mathrm{PSL}(3,2)$	1	3
S_4	6	19	$\Sigma(72\phi)$	2	2
$\Delta(27)$	3	10	$\Delta(216)$	1	1
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	5	30			

Some statistics

	$\mathcal{N}=1$			$\mathcal{N}=2$			$\mathcal{N}=4$			$\sum$		
	$\mathbb{Q}$	$\mathbb{Z}$	aff.	$\mathbb{Q}$	$\mathbb{Z}$	aff.	$\mathbb{Q}$	$\mathbb{Z}$	aff.	$\mathbb{Q}$	$\mathbb{Z}$	aff.
Abelian	17	60	138	4	10	23	1	1	1	22	71	162
Non-Abelian	35	108	331	3	7	27	0	0	0	38	115	358
$\sum$	52	168	469	7	17	50	1	1	1	**60**	**186**	**520**



Outlook

Narain, Sarmadi and Vafa 1987

- In recent years, **asymmetric orbifolds** found heightened interest
- There, right- and left-movers are compactified on different geometries
- In this framework, one matrix describes the whole geometry and all Wilson lines
- A classification of space groups in $n = 22$ dimensions would be desirable

Limitations of CARAT

The i. m. f. groups in $n = 22$ are known. However, the current implementation of CARAT, especially $\mathbb{Q} \to \mathbb{Z}$ is built on the assumption $n \leq 6$ and does not simply generalise.
A new implementation could fix this.

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- Feasibility: the orders of the involved groups explode
- The size of the first $\mathbb{Q}$-class of i. m. f. groups in $n = 22$ is $2^{41} \times 3^{9} \times 5^{4} \times 7^{3} \times 11^{2} \times 13 \times 17 \times 19$ ($\mathcal{O}(10^{27})$)

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Conclusion

- There is only a finite number of affine classes of space groups for a given degree n (cf. CY spaces)
- A complete classification of six-dimensional space groups is readily available through an easy to use computer program, CARAT
- We tested these for $\mathcal{N} \leq 1$ SUSY in four dimensions and compiled a **complete** list
- Hodge numbers and Fundamental groups are available
- A generalisation for asymmetric orbifolds seems out of reach at the moment

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Thank you!

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Untwisted Hodge numbers

Untwisted moduli $(h_{\mathsf{U}}^{(1,1)}, h_{\mathsf{U}}^{(2,1)})$	non-Abelian point groups
(2,2)	S_3, D_4, D_6
(2,1)	QD_{16}, $(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_4 \times S_3$, $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, $\mathrm{GL}(2,3)$, $\mathrm{SL}(2,3) \rtimes \mathbb{Z}_2$
(2,0)	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_3 \times S_3$, $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$, $\mathrm{SL}(2,3){-}\mathrm{I}$, $\mathbb{Z}_3 \times D_4$, $\mathbb{Z}_3 \times Q_8$, $(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$, $\mathbb{Z}_6 \times S_3$, $\mathbb{Z}_3 \times \mathrm{SL}(2,3)$, $\mathbb{Z}_3 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$, $\mathrm{SL}(2,3) \rtimes \mathbb{Z}_4$
(1,1)	A_4, S_4
(1,0)	T_7, $\Delta(27)$, $\mathbb{Z}_3 \times A_4$, $\Delta(48)$, $\Delta(54)$, $\mathbb{Z}_3 \times S_4$, $\Delta(96)$, $\Sigma(36\phi)$, $\Delta(108)$, $\mathrm{PSL}(3,2)$, $\Sigma(72\phi)$, $\Delta(216)$

Strict orbifold definition

Thurston 2002

An **orbifold** $\mathcal{O}$ is a topological Hausdorff space X_0 with the following structure data: $\{U_i, \Gamma_i, \tilde{U}_i, \varphi_i\}_{i \in I}$, such that:

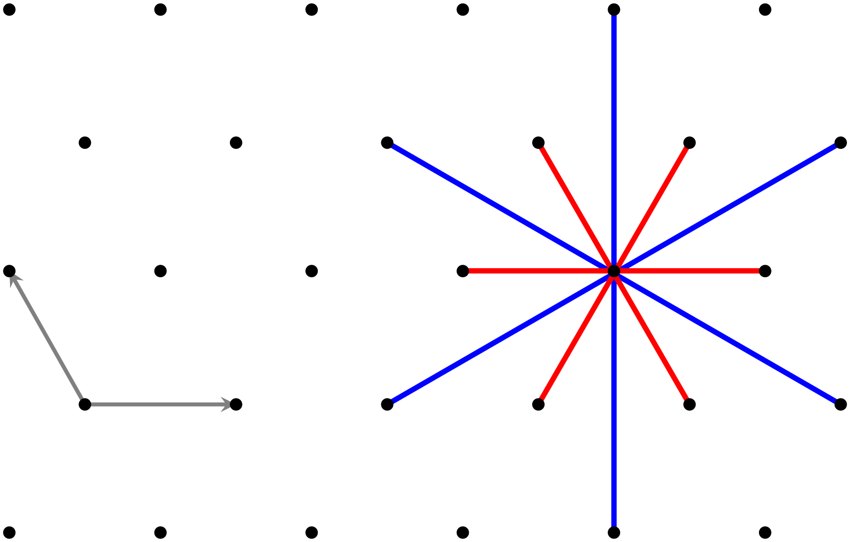
1. $\{U_i\}_{i \in I}$ is an open covering of X_O which is closed under finite intersections,
2. $\forall i \in I$, Γ_i is a discrete group with an action on an open subset $\tilde{U}_i \subseteq \mathbb{R}^n$,
3. $\forall i \in I$, $\varphi_i : \ U_i \to \tilde{U}_i / \Gamma_i$ is a homeomorphism; $\tilde{U}_i / \Gamma_i$ means the set of equivalence classes one gets from identifying each point in U_i with its orbit under the action of Γ_i,
4. $\forall i, j \in I$ with $U_i \subseteq U_j$ there is an injective homomorphism $f_{ij} : \ \Gamma_i \hookrightarrow \Gamma_j$ and an embedding $\tilde{\varphi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j$ such that the following diagram commutes.

Strict orbifold definition

$$
\begin{array}{ccc}
\tilde{U}_i & \xrightarrow{\quad\tilde{\varphi}_{ij}\quad} & \tilde{U}_j \\
\downarrow & & \downarrow \\
\tilde{U}_i/\Gamma_i & \xrightarrow{\quad\varphi_{ij} = \tilde{\varphi}_{ij}/\Gamma_i\quad} & \tilde{U}_j/\Gamma_i \\
\uparrow \varphi_i & & \downarrow f_{ij} \\
 & & \tilde{U}_j/\Gamma_i \\
 & & \uparrow \varphi_j \\
U_i & \subseteq & U_j
\end{array}
$$

Back

The problem with Lie lattices



Back

Bravais groups and crystal families

Let $G \leq \mathrm{GL}(n,\mathbb{Z})$ be a finite unimodular group and

$$\mathcal{F}(G) = \{F \in \mathbb{R}^{n\times n}_{\mathrm{sym}} | \forall g \in G.\ g^T F g = F\}$$

its form space. Then

$$B(\mathcal{F}) = \{g \in \mathrm{GL}(n,\mathbb{Z}) | \forall F \in \mathcal{F}.\ g^T F g = F\}$$

is the **Bravais group of** $\mathcal{F}$.

The **Bravais group of** G is $B(G) = B(\mathcal{F}(G))$.

Two finite subgroups $G, H \leq \mathrm{GL}(n,\mathbb{Z})$ belong to the same crystal family, iff there exist subgroups $G' \leq G$ and $H' \leq H$ with $\mathcal{F}(G') = \mathcal{F}(G)$ and $\mathcal{F}(H') = \mathcal{F}(H)$ and G' and H' $\mathbb{Q}$-equivalent

Normalizers

- The normalizer of U in G is $\{g \in G \ : \ gUg^{-1} = U\}$
- The normalizer of $G \leq \mathrm{GL}(n, \mathbb{Z})$ is the stabilizer of G in the conjugation action of $N(B)$ on the set of subgroups of B
- $N(B)$ is the normalizer of the Bravais group of G
- $N(B) = \{g \in \mathrm{GL}(n, \mathbb{Z}) \ : \ g^T \mathcal{F}(B) g = \mathcal{F}(B)\}$