Origin of non–Abelian discrete symmetries

Michael Ratz

Bethe workshop, Bonn, June 05, 2014

Based on:

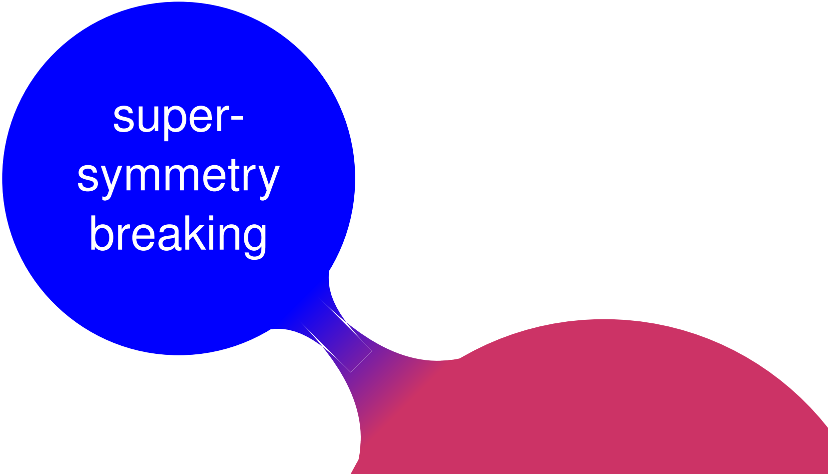
- T. Kobayashi, H.P. Nilles, F. Plöger, S. Raby & M.R.. Nucl. Phys. B768, 135
- H.P. Nilles, M.R. & P. Vaudrevange, Fortsch. Phys. 61, 493
- M.-C. Chen, M.R. & A. Trautner, JHEP 1309, 096
- H.P. Nilles, S. Ramos–Sánchez, M.R. & P. Vaudrevange, Phys. Lett. B726, 876
- M.-C. Chen, M. Fallbacher, K.T. Mahanthappa, M.R. & A. Trautner, Nucl. Phys. B883, 267

Outline

non–Abelian discrete R and non–R symmetries

Outline

because it is
an R symmetry



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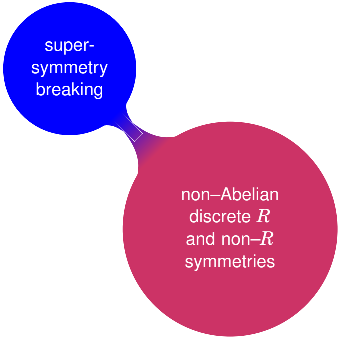
super-
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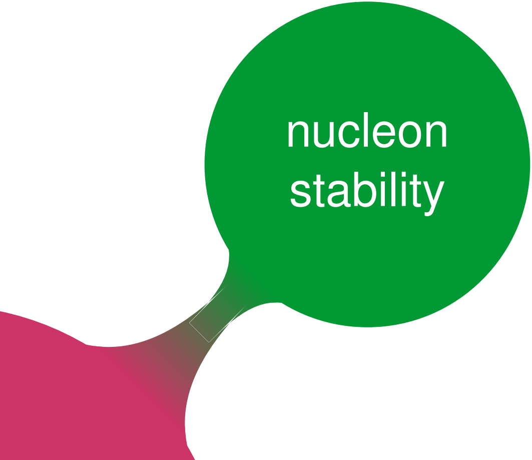
Outline

e.g. $\mathbb{Z}_4^R$ symmetry



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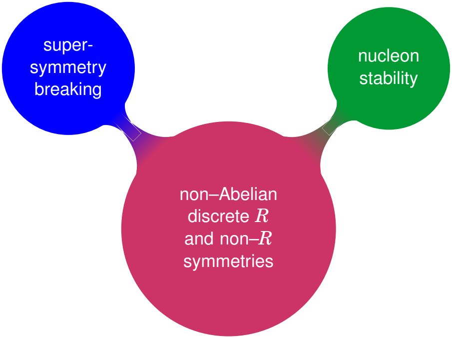
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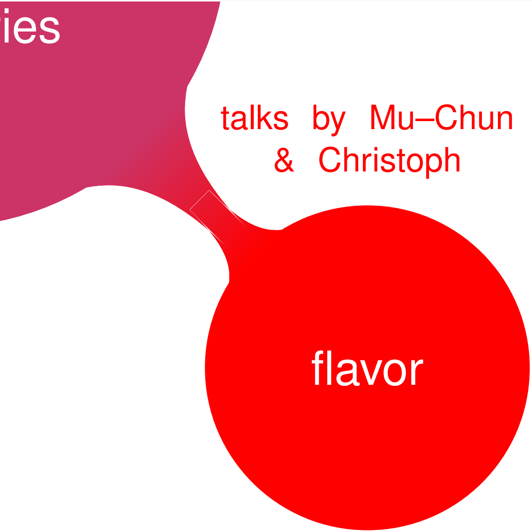

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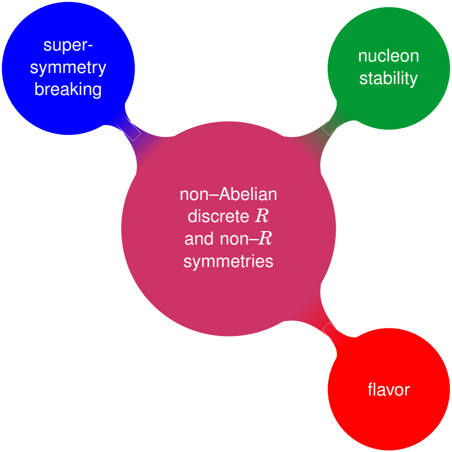
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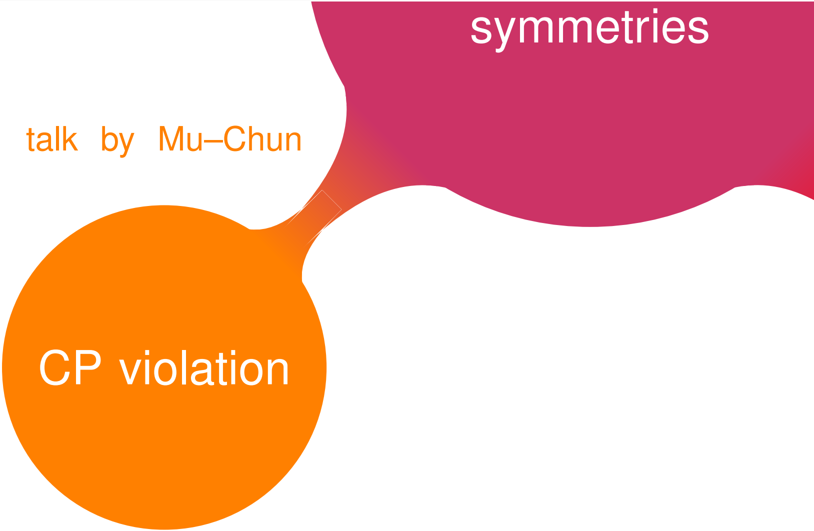

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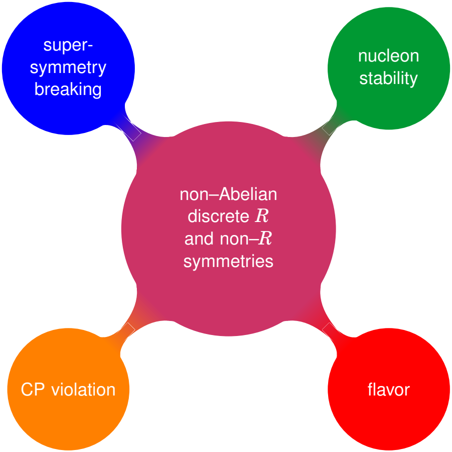
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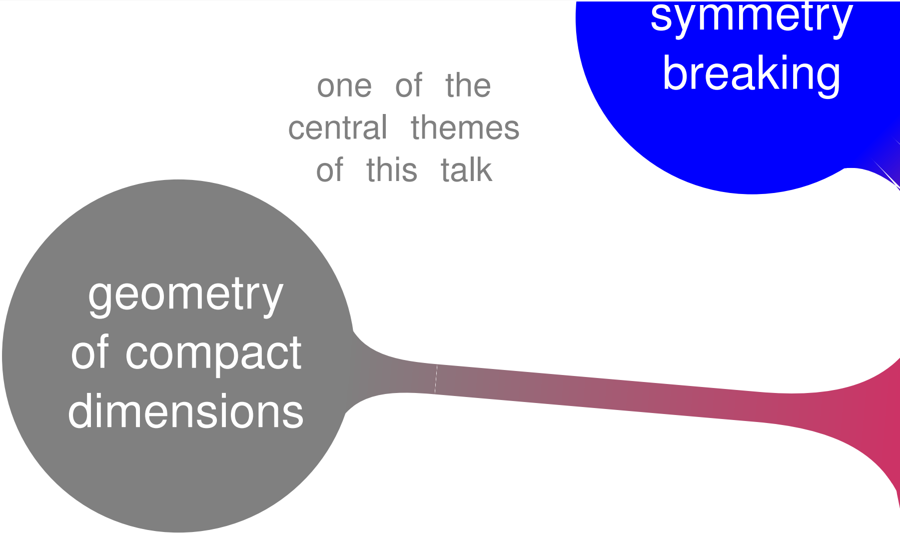
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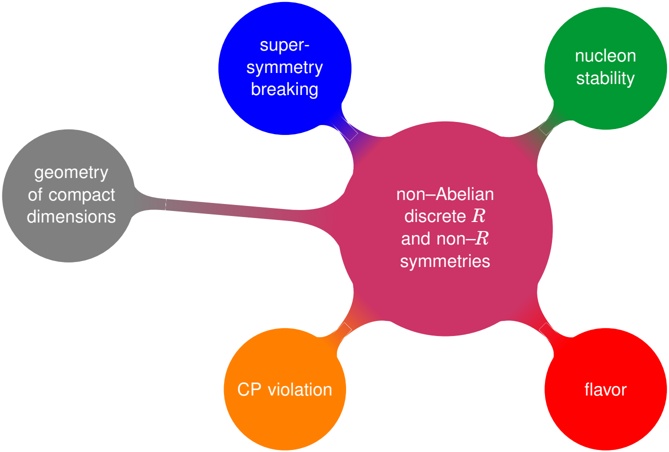
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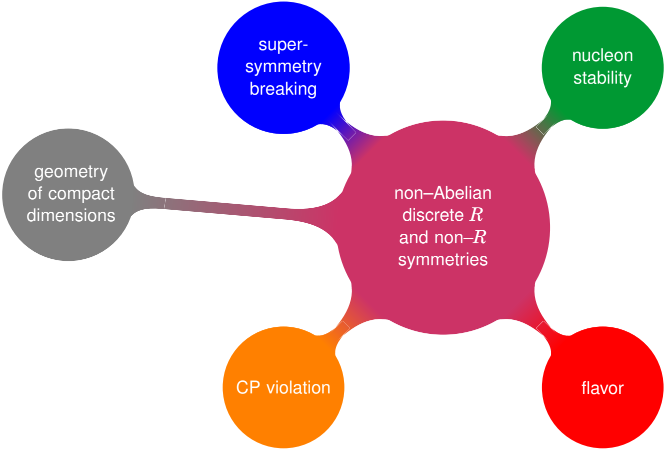
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Non–Abelian discrete R symmetries

Textbook knowledge:

1. Maximal R symmetry of $\mathcal{N} = 1$ supersymmetry is Abelian, i.e. $\mathrm{U}(1)_R$

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Non–Abelian discrete R symmetries

Reminder: Abelian discrete R symmetries

- Superpotential transforms as

$$\mathscr{W} \;\to\; \mathrm{e}^{2\pi\,\mathrm{i}\,q_{\mathscr{W}}/M}\,\mathscr{W}$$

$$q_{\mathscr{W}} = 2q_\theta$$

Reminder: Abelian discrete R symmetries

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- Superfields $\Phi^{(f)} = \phi^{(f)} + \sqrt{2}\,\theta\psi^{(f)} + \theta\theta\,F^{(f)}$ transform as

$$\Phi^{(f)} \;\to\; \mathrm{e}^{2\pi\,\mathrm{i}\,q^{(f)}/M}\,\Phi^{(f)}$$

Reminder: Anomalies in Abelian discrete symmetries

Krauss & Wilczek (1989); Ibáñez & Ross (1991, 1992); Banks & Dine (1992)

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- Most convenient way to compute anomalies: path integral approach

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Fujikawa (1979)

- ☞ Most convenient way to compute anomalies: path integral approach

Araki (2007); Araki, Kobayashi, Kubo, Ramos-Sánchez, M.R. & Vaudrevange (2008)

- ☞ Works both for Abelian and non–Abelian discrete symmetries

Anomaly coefficients for Abelian $\mathbb{Z}_M^{(R)}$ symmetries

☞ Consider the action of one generator of the discrete group

Anomaly coefficients for Abelian $\mathbb{Z}_M^{(R)}$ symmetries

- Consider the action of one generator of the discrete group
- Fermions acquire a $\mathbb{Z}_M^{(R)}$ phase: $\psi^{(f)} \;\to\; \mathrm{e}^{2\pi\,\mathrm{i}\,(q^{(f)}-q_\theta)/M}\,\psi^{(f)}$

$\mathbb{Z}_M^{(R)}$ charge of superfield

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- ➥ Non–trivial transformation of the path integral measure

$$\prod_f \mathcal{D}\psi^{(f)}\,\mathcal{D}\overline{\psi}^{(f)} \;\to\; J^{-2}\prod_f \mathcal{D}\psi^{(f)}\,\mathcal{D}\overline{\psi}^{(f)}$$

with $J^{-2} \;=\; \exp\left\{\mathrm{i}\,\frac{2\pi}{M}\,A_{G-G-\mathbb{Z}_M^R}\int \mathrm{d}^4x\,\frac{1}{32\pi^2}\,F^{b,\mu\nu}\widetilde{F}^b_{\mu\nu}\right\}$

and $A_{G-G-\mathbb{Z}_M^{(R)}} \;=\; \sum_f \ell\left(\boldsymbol{r}^{(f)}\right)\cdot q_{\psi^{(f)}} + q_\theta\,\ell(\mathrm{adj}\,G)$

representation of $\psi^{(f)}$

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$q_{\psi^{(f)}} \;=\; \left(q^{(f)} - q_\theta\right)$ with $q^{(f)}$ R charge of superfield

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Dynkin index : $\delta_{ab}\,\ell(\boldsymbol{r}) \;=\; \mathrm{tr}\,[\mathrm{t}_a(\boldsymbol{r})\,\mathrm{t}_b(\boldsymbol{r})]$

Discrete Green–Schwarz anomaly cancellation

☞ Coupling of ‘axion’ a to field strength of the continuous gauge symmetry

$$\mathscr{L}_{\text{axion}} \supset -\frac{a}{8} F^b \widetilde{F}^b$$

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- Relation between $\Delta^{(\mathsf{u})}$ and $A_{G-G-\mathbb{Z}_M}$

$$A_{G-G-\mathbb{Z}_M} = 2\pi M_{\mathsf{u}} \Delta^{(\mathsf{u})} \mod \frac{M_{\mathsf{u}}}{2}$$

order of u : $\mathsf{u}^{M_{\mathsf{u}}} = \mathbb{1}$

Comment on settings with more than one axions

☞ One can have several axions a_α

$$\mathscr{L}_{\text{axion}} \supset -F^b\widetilde{F}^b \sum_\alpha \frac{c_\alpha}{8} a_\alpha$$

real coefficients

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- One can also have more than one gauge factor, i.e. $G = \prod_i G^{(i)}$

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Lüdeling, Ruehle & Wieck (2012)

- This allows one to cancel abritrary discrete anomalies

Anomaly (non-)universality

- However, in supersymmetric theories the axions are always accompanied by a superpartner ‘saxion’ field

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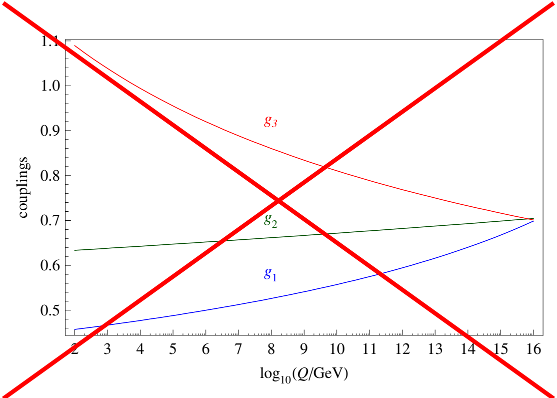
Non–universal λ_i coefficients for the SM gauge factors will spoil the picture of MSSM gauge coupling unification



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Can be avoided by demanding anomaly universality

$$A_{G^{(i)}-G^{(i)}-\mathbb{Z}_M^{(R)}} = \rho \mod \frac{M}{2} \quad \forall\, G^{(i)}$$

Non–Abelian discrete R symmetries

☞ Action of u on representation $\boldsymbol{d}$

$$U_{\mathsf{u}}(\boldsymbol{d}) \;=\; \exp\left(2\pi\,\mathsf{i}\,\lambda_{\mathsf{u}}(\boldsymbol{d})\,/\,M_{\mathsf{u}}\right)$$

order of u

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matrix w/ integer eigenvalues

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- Transformation of fermions

$$\psi^{(f)} \to U_{\mathsf{u}}\left(\boldsymbol{d}^{(f)}\right)\,\psi^{(f)} = \exp\left[2\pi\,\mathrm{i}\,\lambda_{\mathsf{u}}\left(\boldsymbol{d}^{(f)}\right)\,/\,M_{\mathsf{u}}\right]\,\psi^{(f)}$$

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☞ Effective $\mathbb{Z}_{M_{\mathsf{u}}}$ charges

$$\delta_{\mathsf{u}}^{(f)} \;:=\; \mathrm{tr}\left[\lambda_{\mathsf{u}}\left(\boldsymbol{d}^{(f)}\right)\right] \;=\; \frac{M_{\mathsf{u}}}{2\pi\,\mathrm{i}}\,\ln\det U_{\mathsf{u}}\left(\boldsymbol{d}^{(f)}\right)$$

Anomaly coefficients for non–Abelian discrete R symmetries

- Relation between the transformation behavior of a superfield Φ and the corresponding fermion ψ

$$\boldsymbol{d}^{(\Phi)} = \boldsymbol{d}^{(\theta)} \otimes \boldsymbol{d}^{(\psi)}$$

1–dimensional representation

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$$\boldsymbol{d}^{(\Phi)} = \boldsymbol{d}^{(\theta)} \otimes \boldsymbol{d}^{(\psi)}$$

- Relation between fermion and superfield anomaly contributions

$$\delta^{(\psi)} = \delta^{(\Phi)} - \dim\left(\boldsymbol{d}^{(\Phi)}\right) \cdot \delta^{(\theta)}$$

Anomaly coefficients for non–Abelian discrete R symmetries (cont'd)

☞ Anomaly coefficients for transformation u

$$A_{G-G-\mathbb{Z}^R_{M_\mathsf{u}}} = \sum_s \ell(\boldsymbol{r}^{(s)}) \cdot \left[\delta^{(s)} - \dim\left(\boldsymbol{d}^{(s)}\right)\ \delta^{(\theta)}\right] + \ell\left(\mathrm{adj}\, G\right) \cdot \delta^{(\theta)}$$

superfield charges

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$$A_{\mathrm{U}(1)-\mathrm{U}(1)-\mathbb{Z}^R_{M_\mathsf{u}}} = \sum_s \left(Q^{(s)}\right)^2\ \dim\left(\boldsymbol{r}^{(s)}\right) \cdot \left[\delta^{(s)} - \dim\left(\boldsymbol{d}^{(s)}\right)\ \delta^{(\theta)}\right]$$

Anomaly coefficients for non–Abelian discrete R symmetries (cont'd)

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$$
\begin{aligned}
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A_{\mathrm{U}(1)-\mathrm{U}(1)-\mathbb{Z}^R_{M_\mathsf{u}}} &= \sum_s \left(Q^{(s)}\right)^2\,\dim\left(\boldsymbol{r}^{(s)}\right) \cdot \left[\delta^{(s)} - \dim\left(\boldsymbol{d}^{(s)}\right)\,\delta^{(\theta)}\right] \\
A_{\mathrm{grav}-\mathrm{grav}-\mathbb{Z}^R_{M_\mathsf{u}}} &= -21\,\delta^{(\theta)} + \delta^{(\theta)} \sum_G \dim(\mathrm{adj}\,G) \\
&\quad + \sum_s \dim\left(\boldsymbol{r}^{(s)}\right) \cdot \left[\delta^{(s)} - \dim\left(\boldsymbol{d}^{(s)}\right)\,\delta^{(\theta)}\right]
\end{aligned}
$$

Anomaly relations

☞ Anomaly coefficients for two group elements u of order M_u and v of order M_v

$$A_\mathsf{u} = \rho \mod \frac{M_\mathsf{u}}{2} \quad \text{and} \quad A_\mathsf{v} = \sigma \mod \frac{M_\mathsf{v}}{2}$$

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- Anomaly coefficient of group element $\mathsf{w} = \mathsf{u} \cdot \mathsf{v}$ of order M_w

$$\begin{aligned} A_\mathsf{w} &= \sum_f \ell\left(\boldsymbol{r}^{(f)}\right) \delta_\mathsf{w}^{(f)} + \ell\left(\mathrm{adj}\, G\right) \delta_\mathsf{w}^{(\theta)} \\ &= \sum_f \ell\left(\boldsymbol{r}^{(f)}\right) \cdot \left[\frac{M_\mathsf{w}}{M_\mathsf{u}} \delta_\mathsf{u}^{(f)} + \frac{M_\mathsf{w}}{M_\mathsf{v}} \delta_\mathsf{v}^{(f)}\right] + \ell\left(\mathrm{adj}\, G\right) \cdot \left[\frac{M_\mathsf{w}}{M_\mathsf{u}} \delta_\mathsf{u}^{(\theta)} + \frac{M_\mathsf{w}}{M_\mathsf{v}} \delta_\mathsf{v}^{(\theta)}\right] \\ &= \frac{M_\mathsf{w}}{M_\mathsf{u}} \left(\rho \mod \frac{M_\mathsf{u}}{2}\right) + \frac{M_\mathsf{w}}{M_\mathsf{v}} \left(\sigma \mod \frac{M_\mathsf{v}}{2}\right) \end{aligned}$$

Anomaly relations (cont'd)

Three cases:

1. Neither u nor v generates an anomalous symmetry , i.e. $\rho = \sigma = 0$
   ↷ symmetry generated by {u, v} is anomaly–free

Araki, Kobayashi, Kubo, Ramos-Sánchez, M.R. & Vaudrevange (2008)

Anomaly relations (cont'd)

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2. Only one element, say u, generates an anomalous symmetry, i.e. $\rho \neq 0 = \sigma$
   ↷ w = u · v is anomalous with an anomaly coefficient
   $A_{\mathsf{w}} = M_{\mathsf{w}} \left(\frac{\rho}{M_{\mathsf{u}}} \mod \frac{1}{2}\right)$

Anomaly relations (cont'd)

Three cases:

1. Neither u nor v generates an anomalous symmetry , i.e. $\rho = \sigma = 0$
   $\curvearrowright$ symmetry generated by $\{\mathsf{u}, \mathsf{v}\}$ is anomaly–free

   Araki, Kobayashi, Kubo, Ramos-Sánchez, M.R. & Vaudrevange (2008)

2. Only one element, say u, generates an anomalous symmetry, i.e. $\rho \neq 0 = \sigma$
   $\curvearrowright$ $\mathsf{w} = \mathsf{u} \cdot \mathsf{v}$ is anomalous with an anomaly coefficient
   $A_{\mathsf{w}} \;=\; M_{\mathsf{w}} \left(\frac{\rho}{M_{\mathsf{u}}} \mod \frac{1}{2}\right)$

3. Both u and v generate anomalous symmetries
   $\curvearrowright$ anomaly coefficient for w is $A_{\mathsf{w}} \;=\; M_{\mathsf{w}} \cdot \left[\left(\frac{\rho}{M_{\mathsf{u}}} + \frac{\sigma}{M_{\mathsf{v}}}\right) \mod \frac{1}{2}\right]$

GS mechanism for non–Abelian discrete symmetries

- Two operations u and v induce shifts of the axion

$$\mathsf{u} \;:\; a \;\to\; a + \Delta^{(\mathsf{u})} \quad \text{and} \quad \mathsf{v} \;:\; a \;\to\; a + \Delta^{(\mathsf{v})}$$

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- Perfect groups are always anomaly–free

> a perfect group equals
> its commutator subgroup

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- Perfect groups are always anomaly–free

- Simple (finite) non–Abelian groups are always perfect

Chen, Fallbacher, M.R., Trautner & Vaudrevange (in preparation)

GS cancellation of anomalies

☞ Two generating elements u and v

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$$\begin{aligned} A_{\mathsf{u}\cdot\mathsf{v}} &= 2\pi M_{\mathsf{w}} \left(\Delta^{(\mathsf{u})} + \Delta^{(\mathsf{v})}\right) \mod \frac{M_{\mathsf{w}}}{2} \\ &= \frac{M_{\mathsf{w}}}{M_{\mathsf{u}}} \left(\rho \mod \frac{M_{\mathsf{u}}}{2}\right) + \frac{M_{\mathsf{w}}}{M_{\mathsf{v}}} \left(\sigma \mod \frac{M_{\mathsf{v}}}{2}\right) \end{aligned}$$

Discrete symmetries from orbifolds

Orbifolds

① start with some $\mathbb{R}^d$

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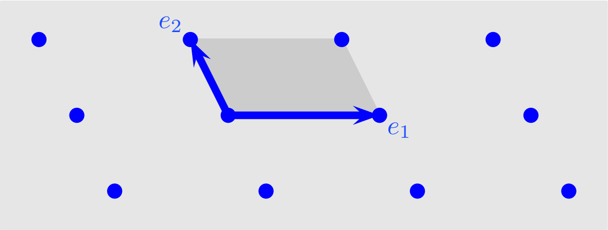
- choose basis vectors e_a



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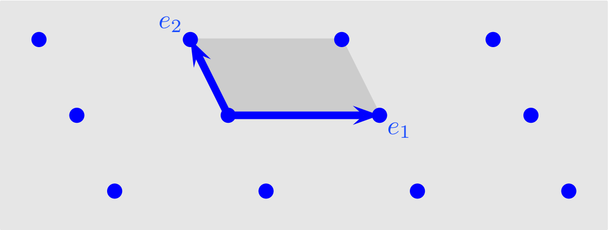
- choose basis vectors e_a
- define torus lattice $\Lambda = \{m_\alpha e_\alpha;\ m_\alpha \in \mathbb{Z}\}$



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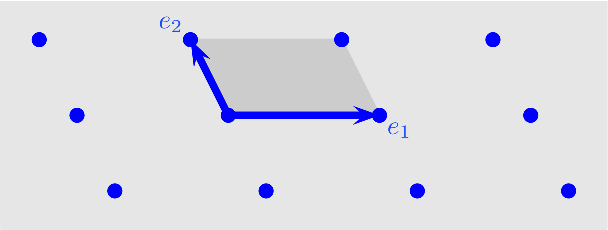
- identify points differing by lattice vectors $\ell \in \Lambda$



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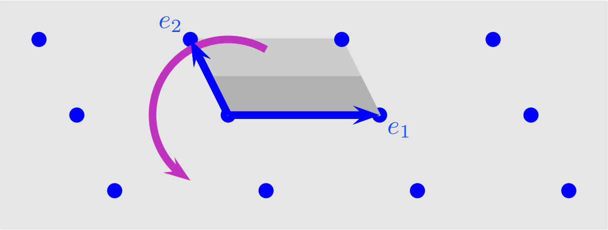

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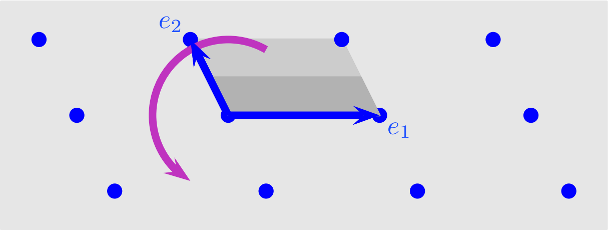

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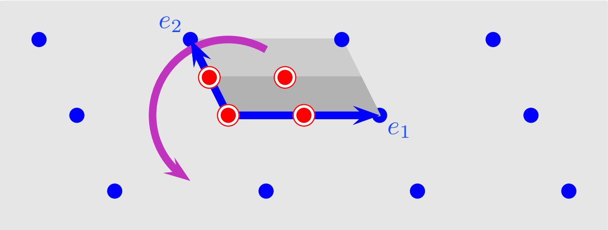

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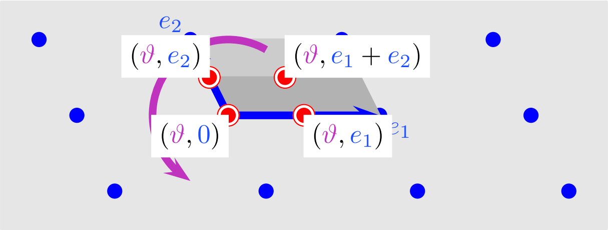
- ℓ is only determined up to translations $\lambda \in (\mathbb{1} - \vartheta)\,\Lambda$



Orbifold and space group

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basis vectors of the torus lattice $\Lambda = \Lambda_{\mathrm{G}_2} \oplus \Lambda_{\mathrm{SU}(3)} \oplus \Lambda_{\mathrm{SO}(4)}$

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$z \;\overset{g}{\mapsto}\; \vartheta^k\, z + n_\alpha\, e_\alpha$ and $X \;\overset{g}{\mapsto}\; X + \pi\, \left(k\, V + n_\alpha\, W_\alpha\right)$

16–dimensional shift vector

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"Wilson lines"

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Groot Nibbelink, Hillenbach, Kobayashi & Walter (2004)

- $g = \left(\vartheta^k, n_\alpha\,e_\alpha\right) \;\leftrightarrow\; \begin{cases} \text{local twist} & : \quad v_g = k\,v \\ \text{local shift} & : \quad V_g = k\,V + n_\alpha\,W_\alpha \end{cases}$

Massless closed (twisted) string

- ☞ Boundary condition: $\boldsymbol{Z}(\tau, \sigma + \pi) = g\,\boldsymbol{Z}(\tau, \sigma)$

$$g = \left(\vartheta^k, n_\alpha\, e_\alpha\right) \in \mathbb{S}$$

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- Label states by boundary conditions

$$\left|p_\text{sh}, q_\text{sh}, \widetilde{N}, \widetilde{N}^*, g\right\rangle = |q_\text{sh}\rangle_\text{R} \otimes \left(\widetilde{\alpha}^i_{-\omega_i}\right)^{\widetilde{N}^i} \left(\widetilde{\alpha}^{\bar{\imath}}_{-1+\omega_i}\right)^{\widetilde{N}^{*i}} |p_\text{sh}\rangle_\text{L} \otimes |g\rangle$$

shifted left–mover momentum $p_\text{sh} = p + V_g$ with $p \in \Lambda_{\mathrm{E}_8 \times \mathrm{E}_8}$

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shifted right–mover
momentum $q_{\mathrm{sh}} = q + v_g$ with $q \in \Lambda_{\mathrm{SO}(8)}$
& $q_{\mathrm{sh}}(\text{boson}) = q_{\mathrm{sh}}(\text{fermion}) + (1/2, -1/2, -1/2, -1/2)$

Massless closed (twisted) string

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oscillator operators

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- State is created by the vertex operator (in -1 ghost picture)

$$\boldsymbol{V}_{-1}^{(g)} \;=\; \mathrm{e}^{-\boldsymbol{\phi}}\,\mathrm{e}^{2\mathrm{i}\,q_\mathrm{sh}\cdot\boldsymbol{H}}\,\mathrm{e}^{2\mathrm{i}\,p_\mathrm{sh}\cdot\boldsymbol{X}} \prod_{i=1}^{3} \left(\partial\boldsymbol{Z}^i\right)^{\widetilde{N}^i} \left(\partial\boldsymbol{Z}^{*\,i}\right)^{\widetilde{N}^{*i}} \boldsymbol{\sigma}_g$$

(bosonized) right–moving coordinates

Massless closed (twisted) string

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bosonized superconformal ghost

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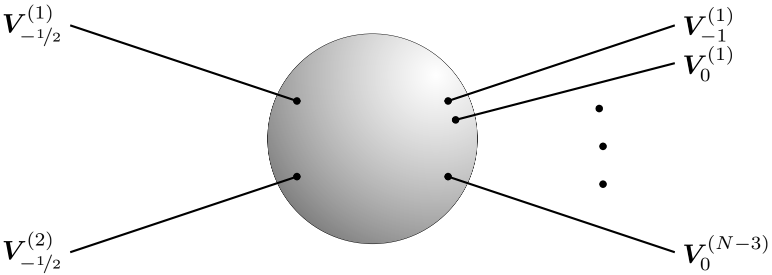
twist field

Selection rules

Hamidi & Vafa (1987); Dixon, Friedan, Martinec & Shenker (1987)
Font, Ibáñez, Nilles & Quevedo (1988b, a); Font, Ibáñez, Quevedo & Sierra (1990)

☞ Superpotential from correlators of vertex operators

$$\mathcal{A} \;=\; \left\langle \boldsymbol{V}_{-1/2}^{(g_1)}\,\boldsymbol{V}_{-1/2}^{(g_2)}\,\boldsymbol{V}_{-1}^{(g_3)}\,\boldsymbol{V}_{0}^{(g_4)}\dots\boldsymbol{V}_{0}^{(g_L)}\right\rangle$$



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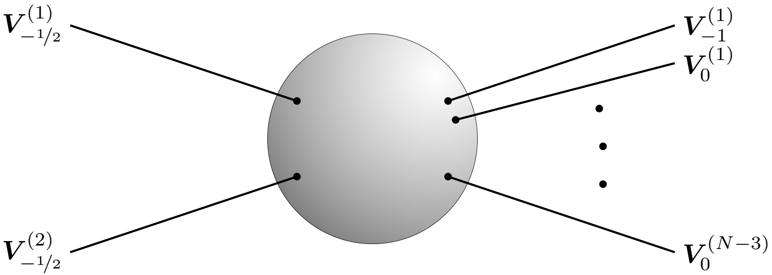

- Correlation function factorizes into correlators involving separately the fields $\boldsymbol{\phi}$, $\boldsymbol{X}^I$, $\boldsymbol{\sigma_g}$, $\boldsymbol{H}$ and $\boldsymbol{Z}^i$

The $\mathbb{Z}_6$–II orbifold

- Generator of $\mathbb{Z}_6$ is represented by the twist vector $v \;=\; \left(0, \frac{1}{6}, \frac{1}{3}, -\frac{1}{2}\right)$

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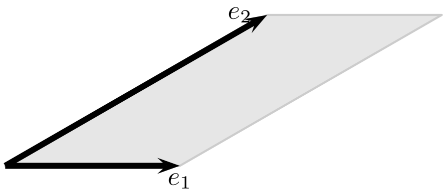

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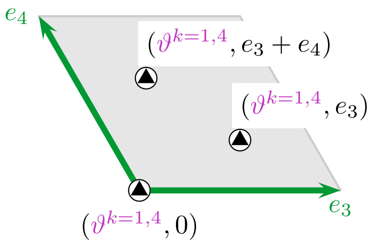

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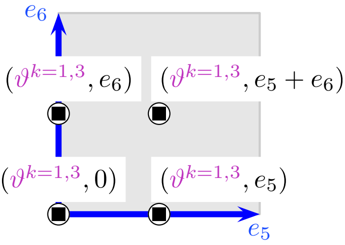

Discrete R symmetries and sublattice rotations

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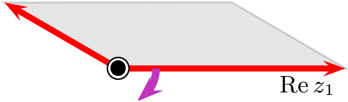
$$\boldsymbol{Z}^j \xrightarrow{\vartheta^{(i)}} \mathrm{e}^{2\pi\,\mathrm{i}\,(r_i)^j}\,\boldsymbol{Z}^j \quad \text{for } i = 1, 2, 3$$

$$r_1 = \left(0, \tfrac{1}{6}, 0, 0\right)$$



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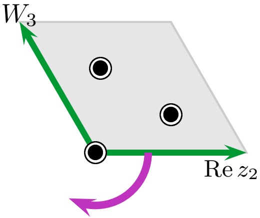

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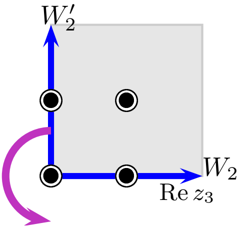

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- More explicitly

$$\begin{pmatrix} \boldsymbol{Z}^1 \\ \boldsymbol{Z}^2 \\ \boldsymbol{Z}^3 \end{pmatrix} \xmapsto{\vartheta} \begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}/6} & 0 & 0 \\ 0 & \mathrm{e}^{2\pi\,\mathrm{i}/3} & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi\,\mathrm{i}/2} \end{pmatrix} \begin{pmatrix} \boldsymbol{Z}^1 \\ \boldsymbol{Z}^2 \\ \boldsymbol{Z}^3 \end{pmatrix}$$

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$$\Delta\widetilde{N}^j = \widetilde{N}^{*j} - \widetilde{N}^j$$

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crucial:

$\vartheta \in \mathrm{SU}(3)_\mathrm{hol}$ while $\vartheta^{(i)} \notin \mathrm{SU}(3)_\mathrm{hol}$ $\curvearrowright$ superspace coordinate θ transforms non–trivially under $\vartheta^{(i)}$

R charges and γ phases

☞ 'Old' R charges

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$$T_j \sim |q_{\mathrm{sh}}\rangle_{\mathsf{R}} \otimes \widetilde{\alpha}^{\bar{j}}_{-1}\,|0\rangle_{\mathsf{L}} \otimes |(\mathbb{1},0)\rangle$$

$$q_{\mathrm{sh}} = \begin{cases} (0,-1,0,0) & \text{for} \quad \bar{j} = \bar{1} \\ (0,0,-1,0) & \text{for} \quad \bar{j} = \bar{2} \\ (0,0,0,-1) & \text{for} \quad \bar{j} = \bar{3} \end{cases}$$

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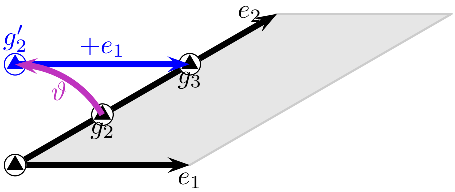
- For example, the constructing elements g_2 and g_3 belong to the same conjugacy class



The "geometrical eigenstate" $|[g]\rangle$

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$\gamma(g,h) \equiv 0$ if $g\cdot h = h\cdot g$
'≡' means 'modulo 1'

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➥ Redefinition of the twist fields σ_g

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Some properties of the γ phases

☞ For fixed $g \in \mathbb{S}$, $\gamma(g,h)$ is a homomorphism from the space group $\mathbb{S}$ to $\mathbb{Z}_6$

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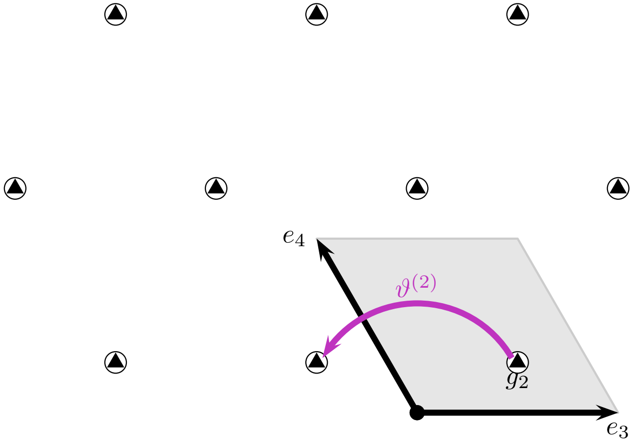
γ charges for sublattice rotations

- It turns out that, in its action on $|[g]\rangle$, $\vartheta^{(j)}$ is equivalent to an appropriate space–group transformation $h \in \mathbb{S}$



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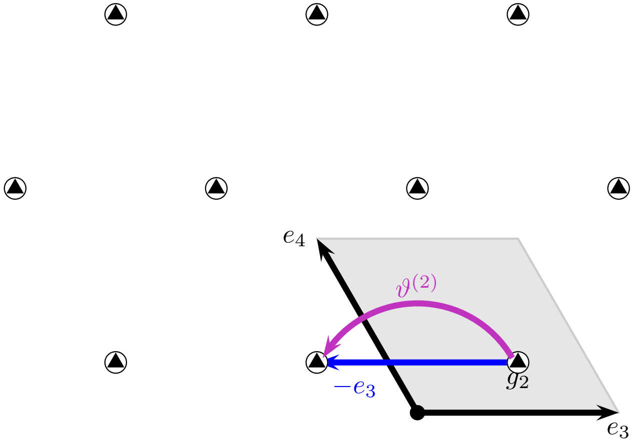

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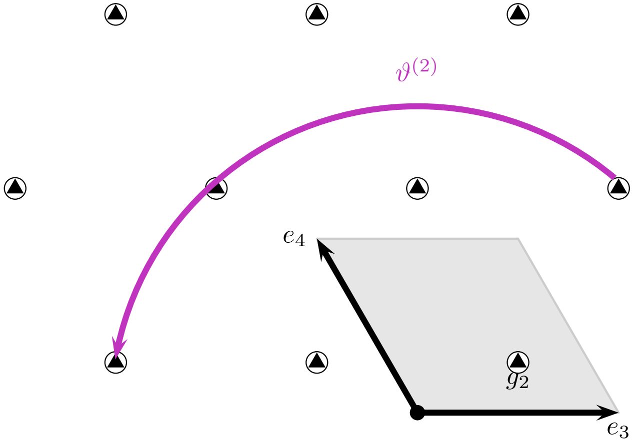

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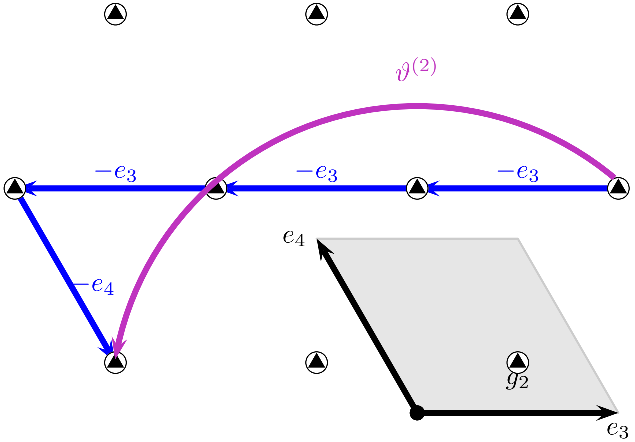

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bottom–line:

$\vartheta^{(j)}$ are conjugacy–class preserving outer automorphisms of the space group $\mathbb{S}$

R charges for twisted fields

- Proper R charges

Nilles, Ramos-Sánchez, M.R. & Vaudrevange (2013)

$$R^j = q^j_{\mathrm{sh}} + \Delta\widetilde{N}^j - N^j\,\gamma(g,\vartheta^{(j)})$$

order of the sublattice rotation

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$$p_{\text{sh}} \cdot V_h - \left(q_{\text{sh}} + \Delta\widetilde{N}\right) \cdot v_h - \frac{1}{2}\left(V_g \cdot V_h - v_g \cdot v_h\right) + \gamma(g,h) \stackrel{!}{\equiv} 0$$

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☞ This allows us to compute, for a given $g\in\mathbb{S}$, the γ phases $\gamma(g,h)$ for all $h\in\mathbb{S}$

R charges for twisted fields: example

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☞ E.g. second two–torus (ϑ acts as $\mathbb{Z}_3$)

$$|[g_a]\rangle = \sum_{m_3,m_4} \mathrm{e}^{-2\pi\,\mathrm{i}\,(m_3+m_4)\,\gamma(g_a,e_3)} \left|\left(\vartheta^k, (n_3+m_3+m_4)\,e_3 + (n_4+2m_4-m_3)\,e_4\right)\right\rangle$$

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☞ Compare

$$|[g_a]\rangle \xrightarrow{h=(\mathbb{1},s_3\,e_3+s_4\,e_4)} \mathrm{e}^{2\pi\,\mathrm{i}\,(s_3+s_4)\,\gamma(g_a,e_3)}\,|[g_a]\rangle$$

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$$|[g_a]\rangle \xrightarrow{(\vartheta^{(2)},0)} \mathrm{e}^{-2\pi\,\mathrm{i}\,(n_3+n_4)\,\gamma(g_a,e_3)}\,|[g_a]\rangle$$

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➥ $\gamma\left(g_a,\vartheta^{(2)}\right) \;\equiv\; -k\,(n_3+n_4)\,\gamma(g_a,e_3)$

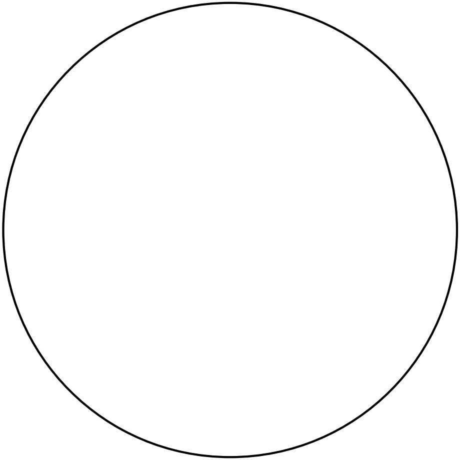
R charges for $\mathbb{Z}_6$–II

- Effective R charges

$$
\begin{aligned}
R^1 &= -6\Big[q_{\mathsf{sh}}^1 + \Delta\widetilde{N}^1 - 6\,\gamma(g,\theta) \\
&\qquad - 6\,k\,(n_3 + n_4)\,\gamma(g,e_3) + 6\,\big(n_5\,\gamma(g,e_5) + n_6\,\gamma(g,e_6)\big)\Big] \\
R^2 &= -6\Big[q_{\mathsf{sh}}^2 + \Delta\widetilde{N}^2 + 3\,k\,(n_3 + n_4)\,\gamma(g,e_3)\Big] \\
R^3 &= -2\Big[q_{\mathsf{sh}}^3 + \Delta\widetilde{N}^3 - 2\,\big(n_5\,\gamma(g,e_5) + n_6\,\gamma(g,e_6)\big)\Big]
\end{aligned}
$$

Flavor symmetries from orbifolds

Example: $\mathbb{S}^1/\mathbb{Z}_2$



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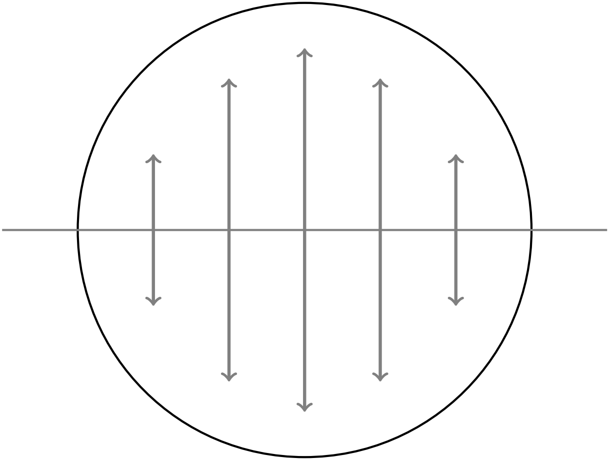

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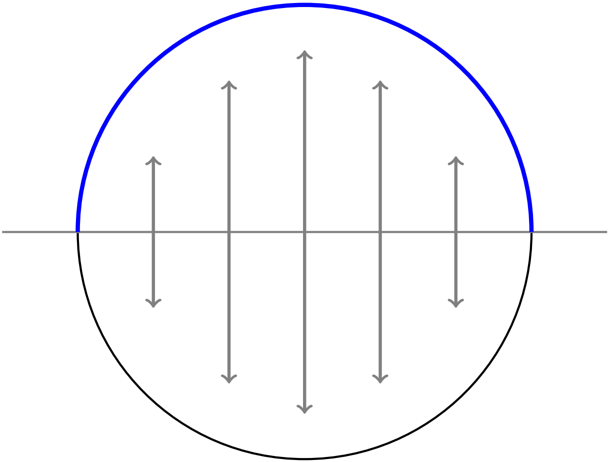

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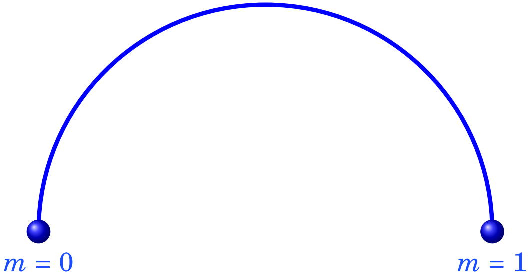

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$\in (\mathbb{1} - \vartheta)\ \Lambda$

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bottom–line:

couplings need to be invariant under $|\Psi_\mathrm{loc}\rangle \;\to\; T\,|\Psi_\mathrm{loc}\rangle$ where $T\in\{-\mathbb{1},\sigma_3,\sigma_1\}$

Example: $\mathbb{S}^1/\mathbb{Z}_2$

☞ Flavor symmetry arising from the space group rule is the multiplicative closure of an S_2 permutation symmetry with $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$G_{\text{flavor}} = S_2 \cup (\mathbb{Z}_2 \times \mathbb{Z}_2) = S_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2) = D_4$$

$$D_4 = \{\pm\mathbb{1} \, , \, \pm\sigma_1 \, , \, \pm\mathrm{i}\sigma_2 \, , \, \pm\sigma_3\}$$

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☞ Other orbifolds: same conclusions

Character table for D_4

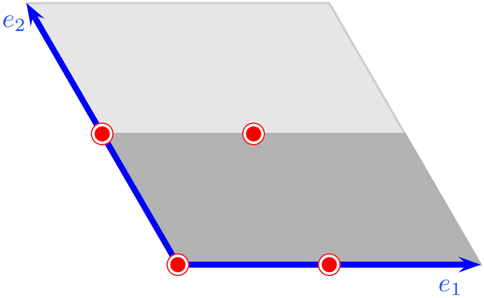
representation	$\mathbb{1}$	$-\mathbb{1}$	$\pm\sigma_1$	$\pm\sigma_3$	$\mp\mathrm{i}\sigma_2$
doublet D	2	–2	0	0	0
singlet A_1	1	1	1	1	1
singlet B_1	1	1	1	–1	–1
singlet B_2	1	1	–1	1	–1
singlet A_2	1	1	–1	–1	1

$$D_1\,\overline{D}_1 + D_2\,\overline{D}_2 \sim A_1 \qquad D_1\,\overline{D}_2 + D_2\,\overline{D}_1 \sim B_1$$

$$D_1\,\overline{D}_1 - D_2\,\overline{D}_2 \sim B_2 \qquad D_1\,\overline{D}_2 - D_2\,\overline{D}_1 \sim A_2$$

Symmetry enhancement (I)

☞ Consider $\mathbb{Z}_2$ plane $\mathbb{T}^2/\mathbb{Z}_2$ with special symmetries: e_1 and e_2 have the same length and enclose an angle of 120°



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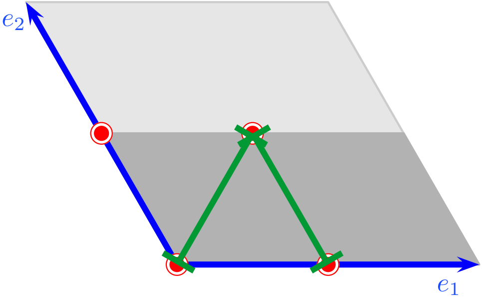

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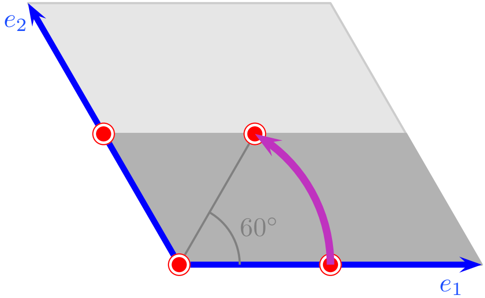

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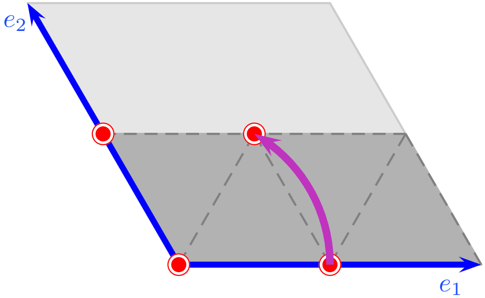

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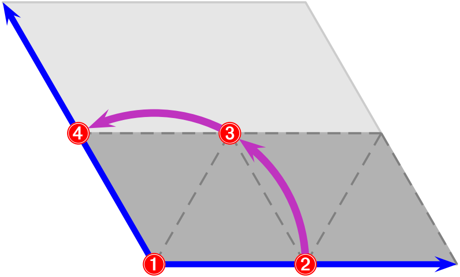
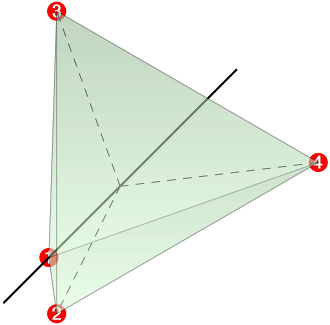
☞ Orbifold is a regular tetrahedron

Tetrahedron





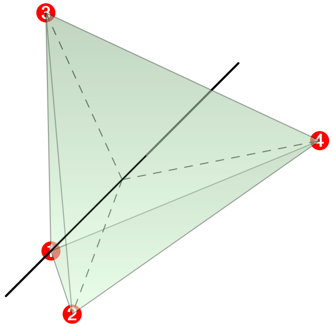
Tetrahedron

The tetrahedron is invariant under 120° rotations around an axis that goes through one of its vertices and hits the center of the opposite face, corresponding to

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

acting on

$$\begin{pmatrix} \text{❶} \\ \text{❷} \\ \text{❸} \\ \text{❹} \end{pmatrix}$$



Tetrahedron

The tetrahedron is invariant under $180°$ rotations around an axis that hits to opposite edges in their middle, corresponding to

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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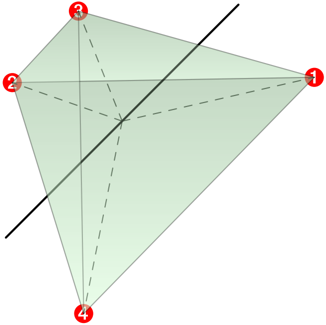

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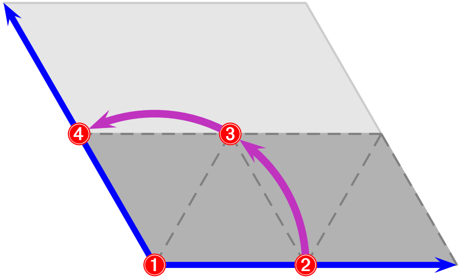

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- A_4 arises as multiplicative closure of the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ groups with elements $\{\mathbb{1}, S\}$ and $\{\mathbb{1}, T, T^2\}$

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- Angle and ratio are parametrized by a field Z
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- In other words, the fluctuations of Z around the critical value furnish a non–trivial representation under the symmetry

Full flavor symmetry SG(192, 1493)

- Character table

$\mathbf{1}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}'$	1	−1	1	−1	1	−1	−1	1	1	−1	1	1	1
$\mathbf{2}$	2	0	2	0	−1	0	0	2	2	0	−1	2	2
$\mathbf{3}$	3	−1	−1	1	0	1	−1	3	−1	−1	0	−1	3
$\overline{\mathbf{3}}$	3	−1	3	−1	0	1	1	−1	−1	−1	0	−1	3
$\mathbf{3}'$	3	1	−1	−1	0	−1	1	3	−1	1	0	−1	3
$\overline{\mathbf{3}}'$	3	1	3	1	0	−1	−1	−1	−1	1	0	−1	3
$\mathbf{3}''$	3	−1	−1	1	0	−1	1	−1	3	−1	0	−1	3
$\overline{\mathbf{3}}''$	3	1	−1	−1	0	1	−1	−1	3	1	0	−1	3
$\mathbf{4}$	4	2	0	0	1	0	0	0	0	−2	−1	0	−4
$\overline{\mathbf{4}}$	4	−2	0	0	1	0	0	0	0	2	−1	0	−4
$\mathbf{6}$	6	0	−2	0	0	0	0	−2	−2	0	0	2	6
$\mathbf{8}$	8	0	0	0	−1	0	0	0	0	0	1	0	−8

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$\mathbb{Z}_{12}$	0	1	2	3	4	5	6	7	8	9	10	11
$\mathbb{Z}_4$	0	3	2	1	0	3	2	1	0	3	2	1
$\mathbb{Z}_3$	0	1	2	0	1	2	0	1	2	0	1	2

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bottom–line:

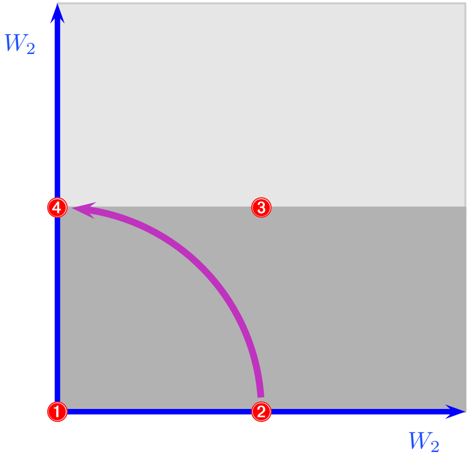
non–Abelian discrete R symmetries can arise from Abelian orbifolds

Symmetry enhancement (V)

- Consider a torus where e_1 and e_2 have the same length and enclose 90°

Symmetry enhancement (V)

- Consider a torus where e_1 and e_2 have the same length and enclose $90°$
- Switch on two identical Wilson lines



Symmetry enhancement (V)

- ☞ Consider a torus where e_1 and e_2 have the same length and enclose $90°$
- ☞ Switch on two identical Wilson lines

➥ Two pairs of equivalent fixed points:

$$\begin{pmatrix} ❶ \\ ❸ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} ❷ \\ ❹ \end{pmatrix}$$

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- ➥ Setting can give rise to models with $2 + 1$ generations

Summary

Summary

R symmetries can be non–Abelian even in $\mathcal{N} = 1$ SUSY

- superspace coordinate transforms in non–trivial 1–dimensional representation

Summary

- R symmetries can be non–Abelian even in $\mathcal{N} = 1$ SUSY

- Green–Schwarz anomaly cancellation also available for non–Abelian symmetries
  - GS axion transforms in non–trivial 1–dimensional representation
  - Perfect groups are always anomaly–free

Summary

- R symmetries can be non–Abelian even in $\mathcal{N} = 1$ SUSY
- Green–Schwarz anomaly cancellation also available for non–Abelian symmetries
- Non–Abelian discrete R symmetries can emerge from Abelian orbifolds

Summary

R symmetries can be non–Abelian even in $\mathcal{N} = 1$ SUSY

Green–Schwarz anomaly cancellation also available for non–Abelian symmetries

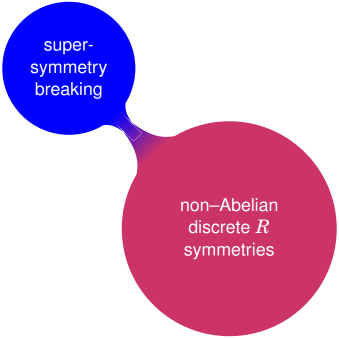
Non–Abelian discrete R symmetries can emerge from Abelian orbifolds

Applications to model building appear to be quite rich
One single symmetry to

- explain flavor structure
- solve μ & proton decay problems
- flavon VEV alignment

Summary

Summary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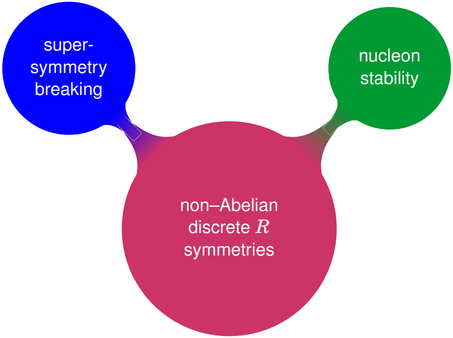
symmetries

Summary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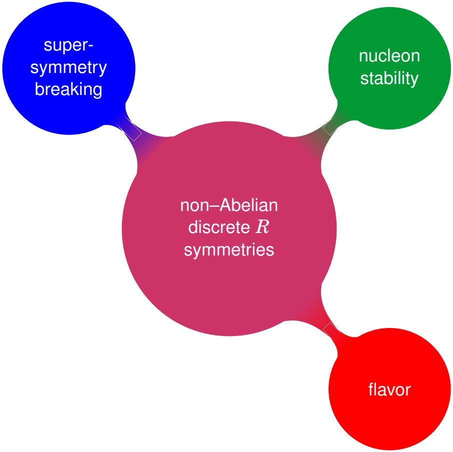
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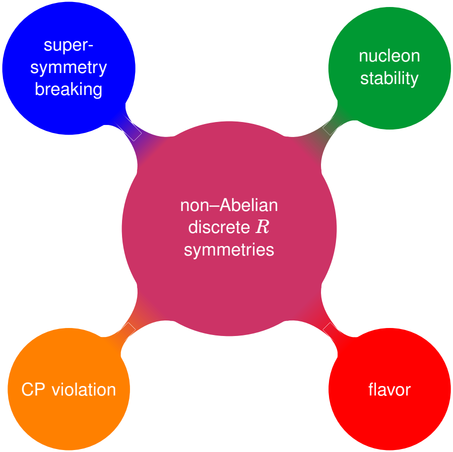
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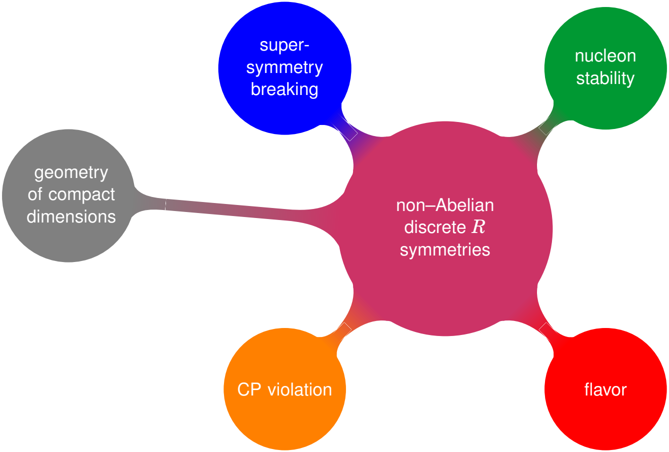
Summary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symmetries
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flavor

Aspen Summer 2014: August 3- 31, 2014

Model Building in the LHC Era

Organizers:

Mu-Chun Chen, Stuart Raby, Michael Ratz, Carlos Wagner

MIAPP Programs 2015

MIAPP Munich Institute for Astro- and Particle Physics

- **Anticipating 14 TeV: Insights into Matter from the LHC and Beyond**
  (June 29 – July 24, 2015) Csaba Csaki, Lisa Randall, Michael Ratz, Andreas Weiler

Vielen Dank!

Complete classification of symmetric toroidal orbifolds

Fischer, M.R., Torrado & Vaudrevange (2013)
cf. talk by M. Fischer

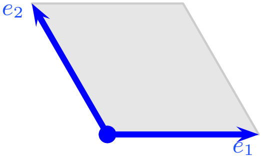
# of generators	# of SUSY	Abelian	non–Abelian
1	$\mathcal{N}=4$	1	0
	$\mathcal{N}=2$	4	0
	$\mathcal{N}=1$	9	0
		14	0
2	$\mathcal{N}=4$	0	0
	$\mathcal{N}=2$	0	3
	$\mathcal{N}=1$	8	32
		8	35
3	$\mathcal{N}=4$	0	0
	$\mathcal{N}=2$	0	0
	$\mathcal{N}=1$	0	3
		0	3
total:	$\mathcal{N}=4$	1	0
	$\mathcal{N}=2$	4	3
	$\mathcal{N}=1$	17	35
		22	38

Abelian orbifolds with $\mathcal{N} = 1$ SUSY

Fischer et al. (2013)
cf. talk by M. Fischer

label of $\mathbb{Q}$–class	twist vector(s)	# of $\mathbb{Z}$–classes	# of affine classes
$\mathbb{Z}_3$	$\frac{1}{3}(1,1,-2)$	1	1
$\mathbb{Z}_4$	$\frac{1}{4}(1,1,-2)$	3	3
$\mathbb{Z}_6$–I	$\frac{1}{6}(1,1,-2)$	2	2
$\mathbb{Z}_6$–II	$\frac{1}{6}(1,2,-3)$	4	4
$\mathbb{Z}_7$	$\frac{1}{7}(1,2,-3)$	1	1
$\mathbb{Z}_8$–I	$\frac{1}{8}(1,2,-3)$	3	3
$\mathbb{Z}_8$–II	$\frac{1}{8}(1,3,-4)$	2	2
$\mathbb{Z}_{12}$–I	$\frac{1}{12}(1,4,-5)$	2	2
$\mathbb{Z}_{12}$–II	$\frac{1}{12}(1,5,-6)$	1	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{1}{2}(0,1,-1)$, $\frac{1}{2}(1,0,-1)$	12	35
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\frac{1}{2}(0,1,-1)$, $\frac{1}{4}(1,0,-1)$	10	41
$\mathbb{Z}_2 \times \mathbb{Z}_6$–I	$\frac{1}{2}(0,1,-1)$, $\frac{1}{6}(1,0,-1)$	2	4
$\mathbb{Z}_2 \times \mathbb{Z}_6$–II	$\frac{1}{2}(0,1,-1)$, $\frac{1}{6}(1,1,-2)$	4	4
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\frac{1}{3}(0,1,-1)$, $\frac{1}{3}(1,0,-1)$	5	15
$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\frac{1}{3}(0,1,-1)$, $\frac{1}{6}(1,0,-1)$	2	4
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\frac{1}{4}(0,1,-1)$, $\frac{1}{4}(1,0,-1)$	5	15
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$\frac{1}{6}(0,1,-1)$, $\frac{1}{6}(1,0,-1)$	1	1
# of Abelian $\mathcal{N} = 1$		60	138

$\mathbb{T}^2/\mathbb{Z}_3$ orbifold



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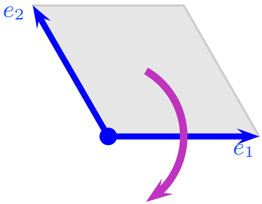
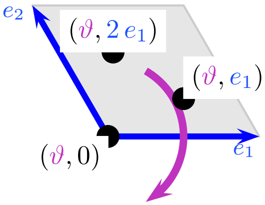


$\mathbb{T}^2/\mathbb{Z}_3$ orbifold



☞ Coupling between n localized states $|(\vartheta, m^{(j)}\,e_1)\rangle$ only allowed if

$$n \;=\; 3\times(\text{integer}) \qquad \wedge \qquad \sum_{j=1}^{n} m_1^{(j)} \;=\; 0 \mod 3$$

$\mathbb{T}^2/\mathbb{Z}_3$ orbifold

$$\begin{pmatrix} |(\vartheta,0)\rangle \\ |(\vartheta,e_1)\rangle \\ |(\vartheta,2e_1)\rangle \end{pmatrix} \rightarrow \begin{pmatrix} |(\vartheta,0)\rangle \\ |(\vartheta,e_1)\rangle \\ |(\vartheta,2e_1)\rangle \end{pmatrix}$$

☞ Coupling between n localized states $\left|(\vartheta,m^{(j)}\,e_1)\right\rangle$ only allowed if

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$\mathbb{T}^2/\mathbb{Z}_3$ orbifold

$$\begin{pmatrix} |(\vartheta,0)\rangle \\ |(\vartheta,e_1)\rangle \\ |(\vartheta,2e_1)\rangle \end{pmatrix} \rightarrow \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} |(\vartheta,0)\rangle \\ |(\vartheta,e_1)\rangle \\ |(\vartheta,2e_1)\rangle \end{pmatrix}$$

$\omega = \mathrm{e}^{2\pi\,\mathrm{i}/3}$

☞ Coupling between n localized states $\left|(\vartheta, m^{(j)}\,e_1)\right\rangle$ only allowed if

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$\mathbb{T}^2/\mathbb{Z}_3$ orbifold

$$\begin{pmatrix} |(\vartheta,0)\rangle \\ |(\vartheta,e_1)\rangle \\ |(\vartheta,2e_1)\rangle \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \begin{pmatrix} |(\vartheta,0)\rangle \\ |(\vartheta,e_1)\rangle \\ |(\vartheta,2e_1)\rangle \end{pmatrix}$$

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➥ Flavor symmetry

$$S_3 \cup (\mathbb{Z}_3 \times \mathbb{Z}_3) = S_3 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3) = \Delta(54)$$

☞ **Note**: $\Delta(54)$ is a 'type I' group

talk by Mu–Chun

Chen, Fallbacher, Mahanthappa, M.R. & Trautner (2014)

Character table of $\Delta(54)$

irrep	1a (1)	6a (9)	6b (9)	3a (6)	3b (6)	3c (6)	2a (9)	3d (6)	3e (1)	3f (1)
$\mathbf{1}_1$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_2$	1	−1	−1	1	1	1	−1	1	1	1
$\mathbf{2}_1$	2	0	0	2	−1	−1	0	−1	2	2
$\mathbf{2}_2$	2	0	0	−1	−1	−1	0	2	2	2
$\mathbf{2}_3$	2	0	0	−1	−1	2	0	−1	2	2
$\mathbf{2}_4$	2	0	0	−1	2	−1	0	−1	2	2
$\mathbf{3}'$	3	$-\overline{\omega}$	$-\omega$	0	0	0	−1	0	$3\overline{\omega}$	3ω
$\overline{\mathbf{3}'}$	3	$-\omega$	$-\overline{\omega}$	0	0	0	−1	0	3ω	$3\overline{\omega}$
$\overline{\mathbf{3}}$	3	ω	$\overline{\omega}$	0	0	0	1	0	3ω	$3\overline{\omega}$
$\mathbf{3}$	3	$\overline{\omega}$	ω	0	0	0	1	0	$3\overline{\omega}$	3ω

Survey of flavor symmetries

orbifold	flavor symmetry	sector	string fundamental states
$\mathbb{S}^1/\mathbb{Z}_2$	D_4	U	$\mathbf{1}$
		T_1	$\mathbf{2}$
$\mathbb{T}^2/\mathbb{Z}_2$	$(D_4 \times D_4)/\mathbb{Z}_2$	U	$\mathbf{1}$
		T_1	$\mathbf{4}$
$\mathbb{T}^2/\mathbb{Z}_3$	$\Delta(54)$	U	$\mathbf{1}$
		T_1	$\mathbf{3}$
		T_2	$\overline{\mathbf{3}}$
$\mathbb{T}^2/\mathbb{Z}_4$	$(D_4 \times \mathbb{Z}_4)/\mathbb{Z}_2$	U	$\mathbf{1}$
		T_1	$\mathbf{2}$
		T_2	$\mathbf{1}_{A_1} + \mathbf{1}_{B_1} + \mathbf{1}_{B_2} + \mathbf{1}_{A_2}$
$\mathbb{T}^2/\mathbb{Z}_6$	trivial		

Survey of flavor symmetries (cont'd)

orbifold	flavor symmetry	sector	string fundamental states
$\mathbb{T}^4/\mathbb{Z}_8$	$(D_4 \times \mathbb{Z}_8)/\mathbb{Z}_2$	U	$\mathbf{1}$
		T_1	$\mathbf{2}$
		T_2	$\mathbf{1}_{A_1} + \mathbf{1}_{B_1} + \mathbf{1}_{B_2} + \mathbf{1}_{A_2}$
		T_3	$\mathbf{2}$
		T_4	$4 \times (\mathbf{1}_{A_1} + \mathbf{1}_{B_1} + \mathbf{1}_{B_2} + \mathbf{1}_{A_2})$
$\mathbb{T}^4/\mathbb{Z}_{12}$	trivial		
$\mathbb{T}^6/\mathbb{Z}_7$	$S_7 \ltimes (\mathbb{Z}_7)^6$	U	$\mathbf{1}$
		T_k	$\mathbf{7}$
		T_{7-k}	$\overline{\mathbf{7}}$

back

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