

## 4. Recent developments

### 4.1. $c \geq a$ -theorem

The  $c$ -theorem for 2dim cft has been proven by Zamolodchikov in 1986

There has been a long history of attempts to get similar statements also in higher dimensions, which succeeded only in 2011 (Komargodski/Schwimmer and followers)

We start with the discussion of the old  $c$ -theorem:

① For 2dim. unitary renormalizable cft exists a function

$c(g_1, g_2, \dots)$  of the couplings with

$$\mu \frac{\partial}{\partial \mu} c = -\beta_j(g) \frac{\partial}{\partial g_j} c(g) \leq 0, \quad \mu \text{ RG scale}$$

i.e. monotony of RG flow.

- ② At RG fixpoints  $c(g)$  is a constant, then the theory is conformal and  $c(g)$  equal to the corresponding central charge
- ③ If the RG flow connects a UV and and IR stable fixpoint one then has  $c_{\text{UV}} > c_{\text{IR}}$

Proof of c-theorem (adapted from Cardy PRL 60 (88) 2706)

Standard technical input from 2 dim cft:

Use Euclidean version

$$z = x^1 + ix^2, \bar{z} = x^1 - ix^2, \partial = \frac{1}{2} (\partial_1 - i\partial_2), \partial_1 = \partial + \bar{\partial}$$

$$\bar{\partial} = \frac{1}{2} (\partial_1 + i\partial_2), \partial_2 = i(\partial - \bar{\partial})$$

$$ds^2 = (dx^1)^2 + (dx^2)^2 = dz d\bar{z} \quad \text{i.e. } \gamma_{++} = \gamma_{--} = 0, \quad \gamma_{+-} = \frac{1}{2} \quad (\Rightarrow \eta^{+-} = 2)$$

$$T_{++} = \frac{1}{2} (T_{1+} - iT_{2+}) = \frac{1}{4} (T_{11} - iT_{12} - iT_{21} - T_{22}) = \frac{1}{4} (T_{11} - T_{22} - 2iT_{12})$$

$$T_{--} = \frac{1}{4} (T_{11} - T_{22} + 2iT_{12})$$

(93)

$$T_{+-} = \frac{1}{4} (T_{11} + T_{22}) \quad (\partial_+ = \partial, \partial_- = \bar{\partial})$$

i.e. conformal invariance  $\Leftrightarrow T_{+-} = 0$

conservation of energy-momentum  $\partial^\mu T_{\mu\nu} = 0$  i.e.  $\partial_+ \eta^{+-} T_{-+} + \partial_- \eta^{-+} T_{++} = 0$

$$\bar{\partial} T_{++}(z, \bar{z}) = 0$$

$$\partial_- T_{--}(z, \bar{z}) = 0 \quad \Rightarrow \text{with } T := T_{++}, \bar{T} = T_{--} \\ T = T(z), \bar{T} = \bar{T}(\bar{z})$$

genomic qft:

$$\text{Trace } \Theta := 4 T_{+-}(z, \bar{z})$$

conservation of energy momentum:

$$\boxed{\bar{\partial} T + \frac{1}{4} \partial \Theta = 0}$$

Define functions  $F(z\bar{z}), G(z\bar{z}), H(z\bar{z})$  (depending only on  $|z|^2 = z\bar{z}$ )

by

$$\frac{1}{z^4} F(z\bar{z}) := \langle T(z, \bar{z}) T(0, 0) \rangle,$$

$$\frac{1}{z^3 \bar{z}} G(z\bar{z}) := \langle \Theta(z, \bar{z}) T(0, 0) \rangle, \quad \frac{1}{z^2 \bar{z}^2} H(z\bar{z}) := \langle \Theta(z, \bar{z}) \Theta(0, 0) \rangle$$

Consider now derivatives (t dimensionless scalar)

$$\dot{F} := \frac{d}{dt} F(t, z\bar{z}) \Big|_{t=1} = z\bar{z} F'(z\bar{z}) \text{ etc.}$$

$$\begin{aligned} \text{Then } F(tz\bar{z}) &= z^4 \langle T(z, t\bar{z}) T(0, 0) \rangle \Rightarrow \overset{\circ}{F} = z^4 \bar{z} \langle \partial T(z, \bar{z}) T(0, 0) \rangle \\ &= -\frac{1}{4} z^4 \bar{z} \langle \partial \Theta(z, \bar{z}) T(0, 0) \rangle \end{aligned}$$

$$\begin{aligned} \overset{\circ}{F} &= -\frac{1}{4} z^4 \bar{z} \partial \left( \frac{1}{z^3 \bar{z}} G \right) = -\frac{1}{4} z^4 \bar{z} \left( -3 \frac{1}{z^4 \bar{z}} G + \frac{1}{z^3 \bar{z}} \partial G \right) \\ &= \frac{3}{4} G - \frac{1}{4} z \partial G, \text{ with } \partial G = \bar{z} G' = \frac{\bar{z}}{z\bar{z}} \overset{\circ}{G} = \frac{1}{2} \overset{\circ}{G} \end{aligned}$$

finally  $\boxed{\overset{\circ}{F} = \frac{3}{4} G - \frac{1}{4} \overset{\circ}{G}}$  similarly one finds  $\boxed{\overset{\circ}{G} = G - \frac{1}{4} \overset{\circ}{H} + \frac{1}{2} H}$

with  $C := 2F - G - \frac{3}{8}H \Rightarrow \boxed{\overset{\circ}{C} = -\frac{3}{4}H}$

Remember now trace anomaly  $\Theta = \sum_j \beta_j \Theta_j$  i.e.  $H = z^2 \bar{z}^2 \beta_j \beta_k \langle \Theta_j(z, \bar{z}) \Theta_k(0, 0) \rangle$

Define  $G_{j\ell}(z\bar{z}) := z^2 \bar{z}^2 \langle \Theta_j(z, \bar{z}) \Theta_\ell(0, 0) \rangle$

Reflection positivity of Euclidean qft ( $\Leftrightarrow$  unitarity of Minkowski version)  $\Rightarrow G_{j\ell}$  positive definite

$$\Rightarrow \boxed{\dot{C} = z\bar{z} \frac{\partial}{\partial(z\bar{z})} C = -\frac{3}{4} \beta_j \beta_e G_{je}(z\bar{z})}$$

Consequences for RG flow:

$F, G, H$  by definition naive dimensionless, so far dependence on  $\mu$  suppressed.

$$F = F(z\bar{z}, \mu, g) = \hat{F}(\mu^2 z\bar{z}, g), \text{ analogous for } G, H.$$

$$\text{i.e. } C = \hat{C}(\mu^2 z\bar{z}, g) \Rightarrow z\bar{z} \frac{\partial}{\partial(z\bar{z})} C = \mu^2 \frac{\partial}{\partial \mu} C = \cancel{\mu \frac{\partial}{\partial \mu} C} = \dot{C}$$

$$\text{i.e. } \frac{1}{2} \mu \frac{\partial}{\partial \mu} C(z\bar{z}, \mu, g) = -\frac{3}{4} \beta_j \beta_e G_{je}(z\bar{z}, \mu, g)$$

Since  $T_{\mu\nu}$  has anomalous dimension zero  $\Rightarrow \mu \frac{\partial}{\partial \mu} C + \beta_j \frac{\partial}{\partial g_j} C = 0$

$$\Rightarrow \boxed{\beta_j \frac{\partial}{\partial g_j} C = \frac{3}{4} \beta_k \beta_e G_{ke}} \quad \& \quad \boxed{\mu \frac{\partial}{\partial \mu} C = -\frac{3}{4} \beta_k \beta_e G_{ke}}$$

Due to positive definiteness of  $G_{ke}$   $\mu \frac{\partial}{\partial \mu} C < 0$  as long as  $\beta_j \neq 0$

at a fixpoint  $\beta_j = 0 \Rightarrow$  conformal invariance  $\Rightarrow \theta = 0 \Rightarrow G, H = 0$

$$\Rightarrow C = 2F = 2z^4 \langle TT \rangle \quad \text{i.e. } \langle TT \rangle = \frac{C}{z^4} \quad \square$$

Comment:

We had on page (66), (67)  $\theta = \frac{C}{24\pi} R$

C seems to be a matter of theories  
in non flat backgrounds only

but:  $\frac{\delta R}{\delta g_{\mu\nu}} \Big|_{g \text{ flat}} \neq 0 \Rightarrow C \text{ defined via trace anomaly}$   
appears also in flat 2point  
function of T !!

Note: Standard 2dim CFT uses some different  
normalization for  $T_{\mu\nu}$  !!

## 4 dimensional gft's

The anomaly terms discussed on page (61) can be grouped into combinations of the Riemann tensor:

$$E = R_{\mu\nu\sigma\tau}^2 - 4R_{\mu\nu}^2 + R^2, \quad W^2 = R_{\mu\nu\sigma\tau}^2 - 2R_{\mu\nu}^2 + \frac{1}{3}R^2$$

↑  
Euler density, topological

↑  
square of Weyl tensor, Mf conformally flat  $\Leftrightarrow W_{\mu\nu\sigma\tau} = 0$

i.e.  $\langle T_{\mu}^{\mu} \rangle = cW^2 - aE$  in conformal case

Conjecture: Cardy (PLB 215 (1988) 749)

Formulate gft on a sphere, consider  $C \propto \int_{S^N} \langle T_{\mu}^{\mu} \rangle \sqrt{g} d^N x$ .

In 2 dim. case at conformal points it gives back the central charge  $c$   
 $(\int R \sqrt{g} d^2 x$  topological, Gauß-Bonnet)

In 4 dim case for cft's the  $c \cdot W^2$  term does not contribute  
(sphere  $S^4$  is conformally flat)  $\Rightarrow$   $a$ -term is picked up.

$\Rightarrow$  people started trying to prove "a-theorem" for 4 dim qft.

Sketch of the Komargodski-Schwimmer arguments:

Start with a cft with action  $S_{\text{UV}}$  and trace anomaly, parametrized by  $a_{\text{UV}}$  &  $c_{\text{UV}}$ . Let  $e^{iW}$  denote the partition function, then

$$W_{\text{UV}}[e^{2\phi} g_{\text{UV}}] = W_{\text{UV}}[g_{\text{UV}}] + S_{\text{WZ}}[g_{\text{UV}}, \phi, a_{\text{UV}}, c_{\text{UV}}],$$

$S_{\text{WZ}}$ : Wess-Zumino  
action  $\equiv$  integrated  
anomaly term

To model a situation, where  $W_{\text{UV}}$  is the UV end of an RG flow, we have to add to  $S_{\text{UV}}$  relevant or marginal perturbations. Such terms break Weyl invariance explicitly.

This breaking can be compensated by coupling to a dilaton field  $\tau(x)$  by replacing each dimensionful parameter (masses and/or cutoffs, RG scales)

$M \rightarrow M e^{-\mathcal{I}(x)}$ , and use Weyl rescalings in the form  $\boxed{g_{\mu\nu} \rightarrow e^{2\tilde{\sigma}} g_{\mu\nu}, \mathcal{I} \rightarrow \mathcal{I} + \delta}$  | (99)

Then the only Weyl invariance breaking contribution, also of the perturbed theory, comes from the trace anomaly related to  $S_{UV}$ .

$$e^{iW} = \int d^D \varphi e^{i(S_{UV}[\varphi, g_{\mu\nu}] + \text{perturbations}(\varphi, g_{\mu\nu}, \mathcal{I}))}$$

The same anomaly as in the UV has to be present also in the IR

(analog to 't Hooft's anomaly matching for chiral anomalies)

If in IR we would have only

$$W[g_{\mu\nu}, \mathcal{I}] = W_{IR}[g_{\mu\nu}] + \text{pert}$$

$$\text{then } W[e^{2\tilde{\sigma}} g_{\mu\nu}, \mathcal{I} + \delta] = W[g_{\mu\nu}, \mathcal{I}] + S_{IR}[g_{\mu\nu}, \delta, a_{IR}, C_{IR}]$$

this would violate anomaly matching

Then the only possibility is, that  $W$  in the IR contains an explicit  $\tau$ -dependent  $W_2$ -term, i.e.

$$W[g_{\mu\nu}, \tau] = W_{IR}[g_{\mu\nu}] + S_{W_2}[g_{\mu\nu}, \tau, a_{\mu\nu - AIR}, C_{\mu\nu - AIR}]$$

Interpreting the  $\tau$ -dependence via  $S_{W_2}$  as dilaton interaction, a careful analysis of the properties of the forward  $2 \rightarrow 2$  scattering of dilatons in the infrared yields  $a_{\mu\nu - AIR} > 0$ .

#### 4.2. Scale invariance $\rightarrow$ conformal invariance?

The techniques used in the proof of the  $\alpha$ -theorem have also given a new boost to the long lasting question, whether always scale inv. implies already full conformal invariance.

Polchinski 88: In two dimensions, for unitary theories with discrete spectrum of scaling dimensions: scale invariance  $\rightarrow$  full conformal inv.  
 (note: there are counterexamples in non-unitary theories)

Luty, Polchinski, Rattazzi 1204.5221

In 4dim. unitary theories  
 scale  $\rightarrow$  conformal up to some technical assumption  
 concerning the 4-point fct. of the  
 trace  $T^{\mu}_{\mu}$ .

Dymarski, Komargodski, Schwimmer, Theisen 1309.2921

4dim. again yes or  $T^{\mu}_{\mu}$  behaves like a generalized free field  
 unitary

See also: Review by Nakayama, 1302.0884  
 Yonekura, 1403.4545 } and references  
 Baume, Keven-Zur, Rattazzi, Vitale 1401.5983  
 Jack, Osborn 1312.0428 } herein !!

### 4.3. Conformal bootstrap

to be completed