Lattice investigation of gauge theories in the large N limit

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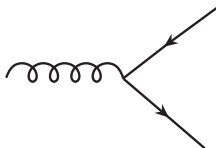
What is the large N limit?

$$\mathbf{QCD} \rightarrow \text{Gauge group SU(3)}$$

$$\mathcal{L} = -\frac{1}{2}\text{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \sum_{f=1}^{n_f} \bar{\Psi}_f\left(i\gamma^\mu D_\mu - m_f\right)\Psi_f$$

$$D_\mu = \partial_\mu - igA_\mu^\alpha T^\alpha \qquad F_{\mu\nu} = (i/g)[D_\mu, D_\nu] \qquad T^\alpha \in \mathfrak{su}(3)$$

In general: $T^\alpha \in \mathfrak{su}(N) \rightarrow$ Gauge group SU(N)

The t'Hooft limit I

$$\beta(g) = \frac{\partial g}{\partial \log(\mu)} = -\frac{1}{(4\pi)^2}\left(\frac{11N - 2N_f}{3}\right)g^3 + O(g^5)$$

What if then $N \to \infty$

t'Hooft, A planar diagram for strong interactions (1974)

In order to make sense of a large N expansion, the original idea from t'Hooft was to keep the parameter $\lambda = g^2 N$ fixed.

$$\beta(\lambda) = \frac{\partial \lambda}{\partial \log(\mu)} = -\frac{11}{24\pi^2}\lambda^2 + O(\lambda^3)$$

In the original t'Hooft proposal, N_f is kept fixed, however, one can have different approaches to the infinite N limit.

Motivation

Why study the large N limit of gauge theories?

A very simple answer: In many ways it is simpler!!! (**???**)

Original idea:

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle^{N=\infty} + O(1/N)$$

Calculating $\langle \mathcal{O} \rangle^{N=\infty}$ should be simpler (???).

- It provides a natural expansion parameter $1/N$ (natural approach to QCD?).
- Since the appeareance of the gauge/gravity duality (AdS/CFT), there has been greater interest in studying the large N limit.

The t'Hooft limit II

Using the double line convention, one has for example, for the gluon propagator at three loop order:

$$g^6 N^5 = N^2 \lambda^3 \qquad g^6 N^4 = N \lambda^3 \qquad g^6 N^3 = \lambda^3$$
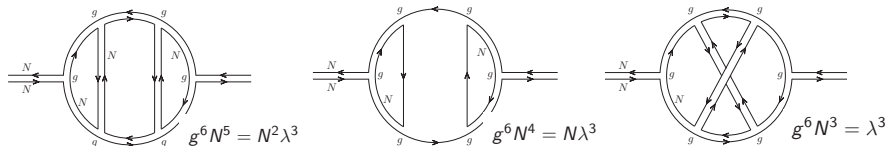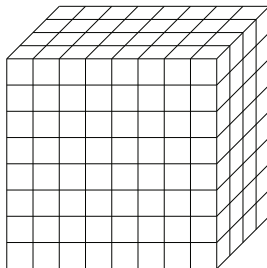
In a general case, the topological expansion is given by the Euler characteristic of the surface in which the diagram can be embedded.

$$\mathcal{A} \propto N^\chi \qquad\qquad \chi = F - E + V$$
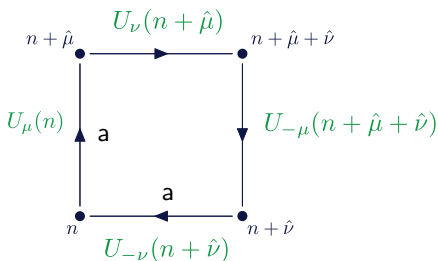
SU(N) on the lattice I

Why lattice to study SU(N)?

- Provides a way to study the theory in a non-perturbative way (and test the gauge/gravity duality tools).
- Can help to guide the analytical approaches to constructing a QCD string dual (???).

SU(N) on the lattice II

The idea is to discretize spacetime which defines a regularization scheme dependent on the lattice spacing a.
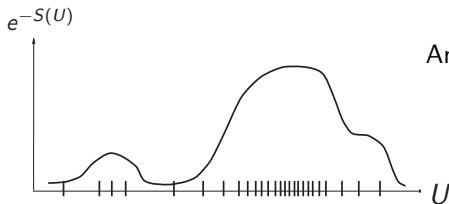

$$U_\mu(n) = e^{igaA_\mu(n)}$$

SU(N) on the lattice II

The basic idea to obtain predictions in lattige gauge theory is based on Markov chains and importance sampling to calculate:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D[U] \mathcal{O}(U) e^{-S(U)}$$

In practice, one generates a chain of configurations $\{U_1, U_2, U_3, \cdots, U_M\}$ such that the fields are randomly distributed with a probability density $D[U] e^{-S(U)}$.

And for a given operator one estimates:

$$\langle \mathcal{O} \rangle = \frac{1}{M} \sum_i^M \mathcal{O}(U_i) + O(M^{-1/2})$$

Hybrid Montecarlo algorithm (HMC)

One defines an algebra valued field $\pi(x, \mu) = \pi^a(x, \mu) T^a$, which is to be interpreted as the canonical momentum of the gauge field U.

Then, for a given operator $\mathcal{O}$ one has:

$$\int D[U]\mathcal{O}(U)e^{-S(U)} \propto \int D[\pi]D[U]\mathcal{O}(U)e^{-H(\pi,U)}$$
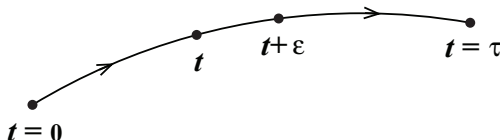
where $H(\pi, U) = \frac{1}{2}(\pi, \pi) + S(U)$

And one can use the Hamilton equations of motion:

$$\dot{\pi}(x, \mu) = -F(x, \mu), \qquad F^a(x, \mu) = \frac{\partial S(e^\omega U)}{\partial \omega^a(x, \mu)}\bigg|_{\omega=0}$$

$$\dot{U}(x, \mu) = \pi(x, \mu)U(x, \mu)$$

HMC

After integrating the Hamilton equations up to time τ one makes an acceptance rejection test (Metropolis)

$$\Delta H(\pi, U) = H(\pi_\tau, U_\tau) - H(\pi_0, U_0)$$

And one accepts the new configuration U_τ with probability
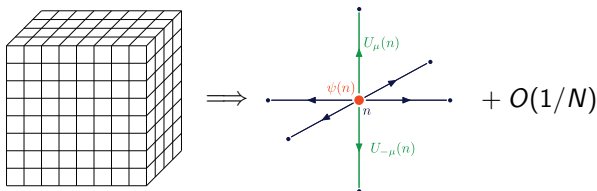$P_{acc}(\pi, U) = min\{1, e^{-\Delta H(\pi, U)}\}$

In this way one constructs a set of link variables $\{U_1, U_2, ..., U_M\}$ to be used for computations.

Eguchi-Kawai and Volume reduction

$SU(N)$ lattice gauge theory in $L^d \equiv SU(N)$ lattice gauge theory in $1^d + O(1/N)$

Eguchi and Kawai, Reduction of dynamical degrees of freedom in the Large N Gauge Theory (1982)

The original proposal doesn't work $\rightarrow$ "center symmetry" is spontaneously broken in the weak coupling regime.

Twisted Eguchi-Kawai I

One way to cure for the breaking of center symmetry is using Twisted Boundary Conditions.

$$S_{EK} = 2bN \sum_{\mu\nu} Tr \left(\mathbb{I} - \frac{1}{2} \left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + h.c. \right) \right)$$

Turns into:

$$S_{TEK} = 2bN \sum_{\mu\nu} Tr \left(\mathbb{I} - \frac{1}{2} z_{\mu\nu} \left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + h.c. \right) \right)$$

$$z_{\mu\nu} = exp\{2\pi\imath \frac{n_{\mu\nu}}{N}\} \in Z_N$$

Gonzalez-Arroyo and Okawa, Twisted Eguchi-Kawai model: a reduced model for large N lattice gauge theory. (1983)

$$n_{\mu\nu} = kL\,\epsilon_{\mu\nu} \qquad \text{with} \quad k = O(L) \quad \text{and} \quad L^2 = N$$

Twisted Eguchi-Kawai II
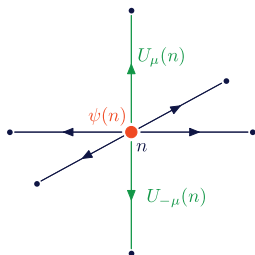
Practical advantages of volume reduction:

- Single site lattice instead of L^4 lattice.
- One can consider (much) larger values of N and match to the corresponding L value ($L^2 = N$).

Twisted Eguchi-Kawai III

Gonzalez-Arroyo and Okawa, Phys.Lett. B718 (2013)
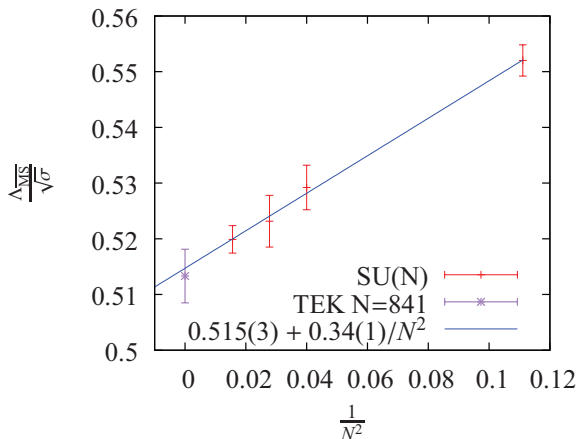
Project

- SU(N) HMC algorithm.
- Link updates pure gauge SU(N) algorithm.
- Twisted Eguchi-Kawai volume reduction.
- Physical observables (Static Potential, String Tension, "Gradient Flow").

Project

First Task: SU(N) pure gauge openQCD

Partial implementation of the changes in openQCD to work with the SU(N) gauge group.

Problem: openQCD hard coded for standard SU(3) calculations.

```
typedef struct
{
    complex c11,c12,c13,c21,c22,c23,c31,c32,c33;
} su3;

typedef struct
{
    complex c1,c2,c3;
} su3_vector;

/*
* r.c1+=c*s.c1 (c real)
* r.c2+=c*s.c2
* r.c3+=c*s.c3
*/

#define _vector_mulr_assign(r,c,s) \
    (r).c1.re+=(c)*(s).c1.re; \
    (r).c1.im+=(c)*(s).c1.im; \
    (r).c2.re+=(c)*(s).c2.re; \
    (r).c2.im+=(c)*(s).c2.im; \
    (r).c3.re+=(c)*(s).c3.re; \
    (r).c3.im+=(c)*(s).c3.im
```

```
#define NCOL 4

typedef struct
{
complex_dble mSUN[NCOL][NCOL];
} suN;

typedef struct
{
complex_dble vSUN[NCOL];
} suN_vector;

/*
* r.vSUN[i]+=c*s.vSUN[i] (c real)
*/

#define _Nvector_mulr_assign(r,c,s){\
    int _i = 0;\
    for(_i<NCOL){\
        r.vSUN[_i].re+=(c)*(s).vSUN[_i].re;\
        r.vSUN[_i].im+=(c)*(s).vSUN[_i].im;\
    };
```

Project

And one has to deal with inline assembly type instructions in order to efficiently use the hardware (future work).

```
/*
 * Computes s+c*r assuming r is stored in xmm0,xmm1,xmm2 and that c
 * has been loaded to xmm6,xmm7 by _sse_load_real_dble(z). The result
 * appears in xmm0,xmm1,xmm2 all other xmm registers are unchanged
 */

#define _sse_mulr_vector_add_dble(s) \
__asm__ __volatile__ ("mulpd %%xmm6, %%xmm0 \n\t" \
                      "mulpd %%xmm7, %%xmm1 \n\t" \
                      "mulpd %%xmm6, %%xmm2 \n\t" \
                      "addpd %0, %%xmm0 \n\t" \
                      "addpd %1, %%xmm1 \n\t" \
                      "addpd %2, %%xmm2" \
                      : \
                      : \
                      "m" ((s).c1), \
                      "m" ((s).c2), \
                      "m" ((s).c3) \
                      : \
                      "xmm0", "xmm1", "xmm2")
```

Project

Current work in dealing with arbitrary N.

- Random SU(N) matrix generator.
- Exponential of an $\mathfrak{su}(N)$ matrix.
- Projection to SU(N) group.
- Random $\mathfrak{su}(N)$ matrix generator.

One strategy is to use SU(2) subgroups of SU(N).

Project

Example: exponential function.

Luscher, Schwarz-preconditioned HMC algorithm for two-flavor lattice QCD. (2005)

Given $X \in \mathfrak{su}(N)$, one can define:

$$
Y_1 = \begin{pmatrix} y_{11}^{22} & x_{12} & 0 & \cdots \\ x_{21} & -y_{11}^{22} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}
Y_m = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & & \\ 0 & \cdots & y_{ii}^{jj} & \cdots & x_{ij} & 0 & \cdots \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & x_{ji} & \cdots & -y_{ii}^{jj} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & & & \\ \vdots & \vdots & \vdots & \vdots & & & \end{pmatrix}
$$

$y_{ii}^{jj} = \left(x_{ii} - x_{jj} \right) / N$

$m = 1, \cdots, N(N-1)/2 = m_{max}$

Project

Then one can define:

$$U_m = \left(1 + \frac{1}{4} Y_m\right)\left(1 - \frac{1}{4} Y_m\right)^{-1}$$

$$U_{m_{max}} = \left(1 + \frac{1}{2} Y_{m_{max}}\right)\left(1 - \frac{1}{2} Y_{m_{max}}\right)^{-1}$$

And compute:

$$\exp(\epsilon X) = U_1 U_2 \cdots U_{m_{max}-1} U_{m_{max}} U_{m_{max}-1} \cdots U_2 U_1 + O(\epsilon^3) \qquad \epsilon \ll 1$$

Conclusions

- Studying gauge theories in the large N limit can help us to understand better some features from QCD.
- Lattice provides a well defined framework to study non-perturbative physics of large N gauge theories.
- One has to deal with technical details to build algorithms for general N SU(N) lattice gauge theories.
- Eguchi-Kawai volume reduction (with twisted boundary conditions) seems to be the most practical way to study really large N gauge theories on the lattice.

THANK YOU FOR YOUR ATTENTION

Large N expansions

There exist a number of different ways to construct a sensible physical theory in the large N limit. Amongst those, one can cite:

- t'Hooft limit: $N \to \infty$ with $\lambda = g^2 N$ and N_f kept fixed.
- Veneziano limit: $N \to \infty$ with N/N_f kept fixed.
- Ramond limit: $N \to \infty$ with fermions in the antisymmetric 2 index representation of the group.

Link-updates algorithms (local updates)

Basic algorithm:

1. Pick a link variable $U(x, \mu)$ and $X \in \mathfrak{su}(N)$ randomly.
2. Accept the new value $U'(x, \mu) = e^X U(x, \mu)$ with probability
   $P_{acc} = min\{1, e^{S_g(U) - S_g(U')}\}$

Works for local actions, otherwise it is too time consuming.

One can consider certain changes for faster exploration of the gauge configuration space like microcanonical moves and heathbaths.

EK model variants

Options:

- Quenched EK: the Eigenvalues are forced to satisfy the center symmetry (Bhanot, Heller, Neuberger).
- Twisted EK: one uses Twisted boundary conditions instead of periodic (Gonzalez-Arroyo, Okawa).
- Adjoint EK: adding fermions in the adjoint representation of SU(N) (Kovtun, Unsal, Yaffe).
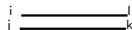- Partial volume reduction (Neuberger, Narayanan, Kiskis).

The t'Hooft limit II

- In the t'Hooft limit one can make a double expansion, in terms of the gauge coupling but also in terms of the topology of the diagrams, which results in a expansion in terms of $1/N$.

- One can see the origin of this topological expansion when considering a double line representation of Feynman diagrams.

For fermions: $\langle \Psi^i \Psi^j \rangle \propto \delta^{ij}$

$$i \underline{\hspace{2cm}} j$$

For the gauge fields: $\langle A^i_{\mu\,j} A^k_{\nu\,l} \rangle \propto (\delta^i_l \delta^k_j - \delta^i_j \delta^k_l / N)$

$$\begin{matrix} i \\ j \end{matrix} \underline{\overline{\hspace{2cm}}} \begin{matrix} l \\ k \end{matrix}$$

Eguchi-Kawai and Volume reduction I

One more simplification in the large N limit!!

Expectation values of products of physical operator factorize up to $\mathcal{O}(1/N)$ corrections.

$$\langle G_1 G_2 ... G_n \rangle = \langle G_1 \rangle \langle G_2 \rangle ... \langle G_n \rangle + O(1/N^{2n-2})$$

If in addition:

1. Center symmetry $(\mathbb{Z}_N)$ is not spontaneously broken

$SU(N)$ lattice gauge theory in $L^d \equiv SU(N)$ lattice gauge theory in 1^d

Eguchi and Kawai, Reduction of dynamical degrees of freedom in the Large N Gauge Theory (1982)