Background-independent renormalization in quantum gravity models

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- Motivation: Why quantum gravity?

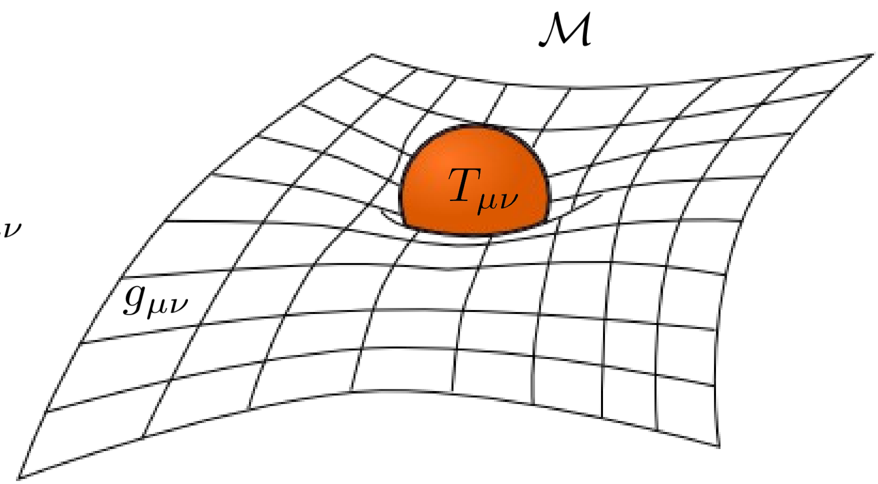
- Spin Foam Models:
  i. Plebański action & constrained BF theory
  ii. discretization and quantisation
  iii. successes and shortcomings

- Renormalization without length scale:
  i. general setup
  ii. diffeomorphism symmetry
  iii. easy example

- How to do it in practice: Approximation methods

- Summary

Classical General Relativity



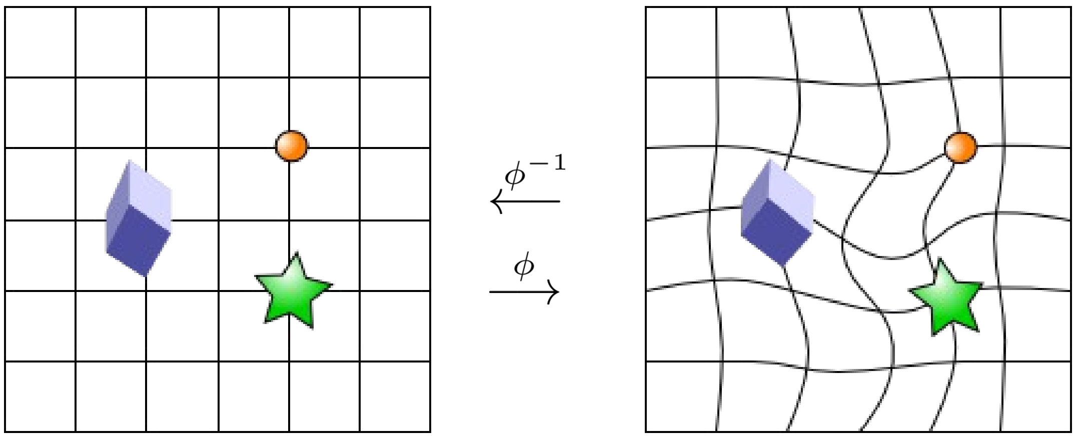
Gravity $\Longleftrightarrow$ Curvature of space-time metric $g_{\mu\nu}$

$$S_{\rm EH}[g_{\mu\nu}] = \frac{1}{16\pi G_{\rm N}} \int d^4x \, \sqrt{-g} \; R$$

Diffeomorphisms: $\phi : \mathcal{M} \to \mathcal{M}$

$$g_{\mu\nu}(x) \to \frac{\partial \phi^{\mu'}}{\partial x^{\mu}} \frac{\partial \phi^{\nu'}}{\partial x^{\nu}} g_{\mu'\nu'}(\phi(x))$$

$$T_{\mu\nu}(x) \to \frac{\partial \phi^{\mu'}}{\partial x^{\mu}} \frac{\partial \phi^{\nu'}}{\partial x^{\nu}} T_{\mu'\nu'}(\phi(x))$$



Einstein's „hole argument“:

$$\boxed{\text{diffeomorphisms = gauge symmetry of GR}}$$

Quantum General Relativity

Quantum gravity: make sense of

$$Z = \int \mathcal{D}g_{\mu\nu} e^{iS_{\mathrm{EH}}[g_{\mu\nu}]}$$

or rather:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}g_{\mu\nu} \, \mathcal{O}[g_{\mu\nu}] \, e^{iS_{\mathrm{EH}}[g_{\mu\nu}]}$$

Perturbative ansatz: $g_{\mu\nu} = \eta_{\mu\nu} + \underline{h_{\mu\nu}} + \ldots$

renormalization: scale-dependent couplings $g_i(k)$ and effective action

$$S_{\mathrm{EH}} \quad \rightarrow \quad S_{\mathrm{eff}}^{(k)}[h_{\mu\nu}] = S_{\mathrm{EH}}[\eta_{\mu\nu} + h_{\mu\nu}] + \sum_i g_i(k) \mathcal{O}_i[h_{\mu\nu}]$$

non-renormalizable [Goroff, Sagnotti '85]

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GR as topological theory with constraints:

Vierbein formalism: Vierbein e^I_μ, spin connection ω^{IJ}_μ $\qquad g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$

$$S_{\text{Pal}}[\omega, e] = \frac{1}{2} \int \epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL}[\omega] + \frac{1}{2\gamma} \int e_I \wedge e_J \wedge F^{IJ}[\omega]$$

$$S_{\text{BF}}[\omega, B] = \int B_{IJ} \wedge F^{IJ}[\omega]$$

[Horowicz '89]

+ simplicity constraints:

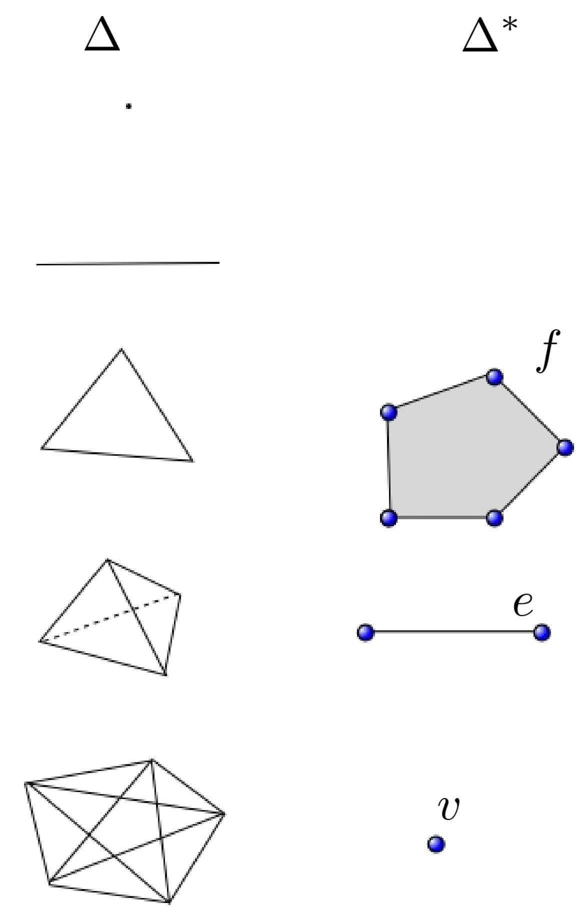
$$B \stackrel{!}{=} *(e \wedge e) + \frac{1}{\gamma} e \wedge e$$

[Plebański '77, Capovilla, Jacobson, Dell, Mason '91]

Strategy: quantize BF theory (easy), and impose simplicity constraints on path integral

In the following:
- D = 4
- Riemannian signature
- fixed $\mathcal{M}$ (no sum over topologies)
- vacuum GR (no matter)

Quantizing (discretized) BF theory:



Triangulation Δ of manifold $\mathcal{M}$

Variables: $h_e \; = \; \mathcal{P} \exp \int_e \omega \; \in \; Spin(4) \; \simeq SU(2) \times SU(2)$

$$B_f \; = \; \int_{f^*} B \; \in \; \mathfrak{spin}(4) \; \simeq \; \mathfrak{su}_2 \times \mathfrak{su}_2$$

Action: $\quad S_{BF,\mathrm{disc}}[g_e, B_f] \; = \; \sum_f \mathrm{tr}(B_f H_f)$

Path integral:

$$Z_{\mathrm{BF}} \; = \; \int dg_e dB_f \; \exp\left(i \sum_f \mathrm{tr}(B_f H_f) \right)$$

$$= \sum_{j_f^\pm, \iota_e^\pm} \prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v$$

e.g. [Ponzano, Regge, '69, Baez '99]

$$j_f^\pm = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \qquad\qquad \iota_e^\pm \quad SU(2) \text{ -invariant tensors}$$

Constraining quantized BF theory:

[Reisenberger '94, Barrett, Crane '99, Livine, Speziale '07, Engle, Pereira, Rovelli, Livine '07, Freidel, Krasnov '07, Oriti Baratin '11, ...]

Sate sum (path integral) for BF theory:

$$Z_{\mathrm{BF}} = \sum_{j_f^\pm, \iota_e^\pm} \prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v$$

Classical simplicity constraints:

$$B \stackrel{!}{=} *(e \wedge e) + \frac{1}{\gamma} e \wedge e$$

Restriction of spins / intertwiners in quantum theory:

$$j_f^\pm = \frac{|1 \pm \gamma|}{2} k_f$$ for some half-integer k_f

Spin Foam model:

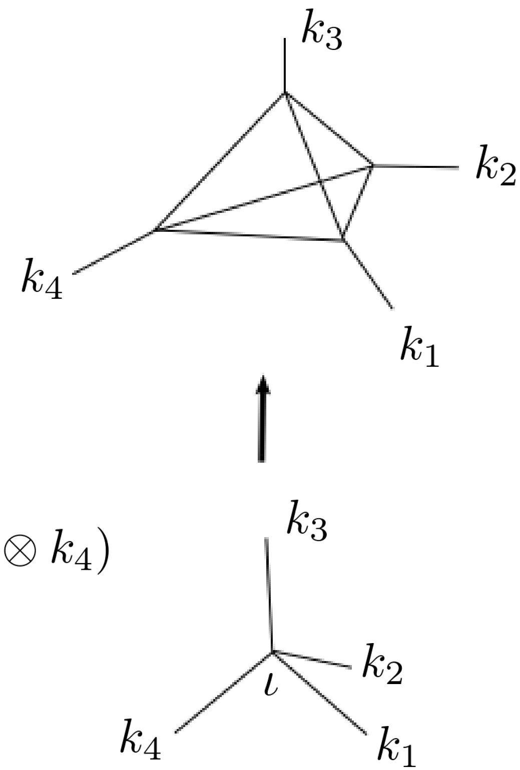
$$Z_{\mathrm{EPRL}} = \sum_{k_f, \iota_e} \prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v$$

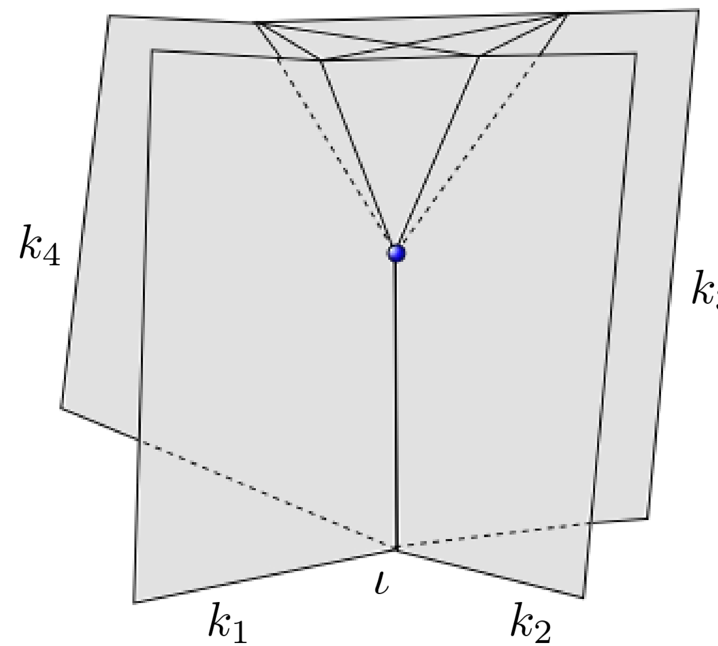
Geometrical interpretation of states

2-complex dual to triangulation, with spins and intertwiners

Spatial slice: graph

$\iota \in Inv(k_1 \otimes k_2 \otimes k_3 \otimes k_4)$





(spatial) boundary states: spin networks

Shape space of quantized tetrahedra with fixed areas given by $k_1, \ldots, k_4$

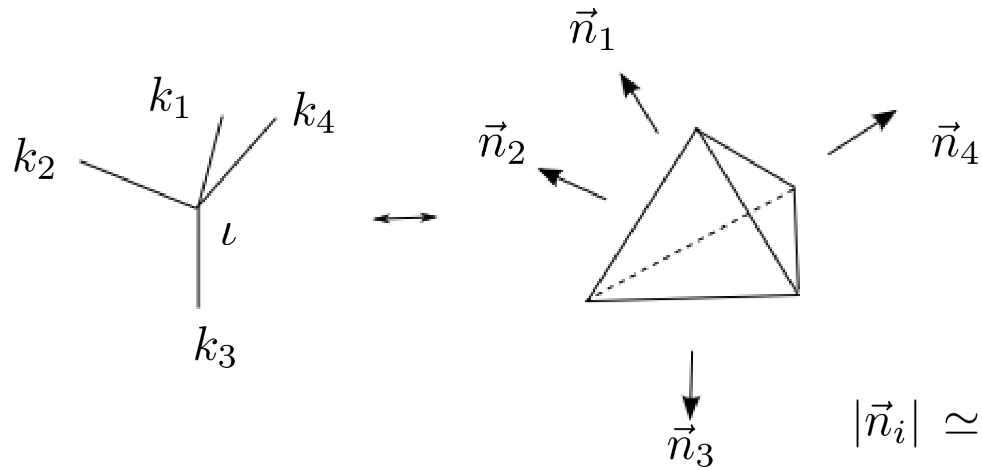
Properties of the EPRL Spin Foam Model:

[Barrett, Dowdall, Fairbairn, Gomez, Hellmann, '07]

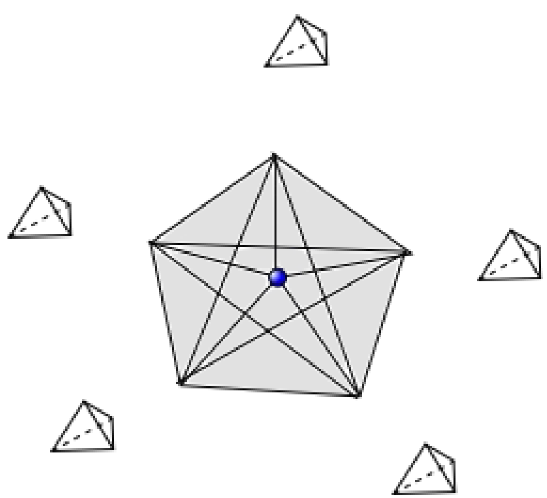
Geometrical interpretation of large quantum number limit:

Coherent states: approximate spatial geometry (tetrahedra)

[Livine, Speziale '07]



$$|\vec{n}_i| \simeq k_i$$



$$k_f \to \infty$$

$$\sim e^{i \sum_f a_f \theta_f} + \dots$$

Regge action (discrete GR)

[Regge, '61]

Properties of the Spin Foam Model ctd:

Gap in geometric quantities: e.g. (spatial) area observables have discrete spectrum

$$\hat{A}_f|\psi\rangle \;=\; \gamma \ell^2_{\text{Planck}} \sqrt{k_f(k_f+1)}|\psi\rangle$$

[Ashtekar, Lewandowski, '93, Rovelli, Smolin, '95]

3d: Ponzano-Regge model [Ponzano, Regge, '69]

3d + cosm. const: Turaev-Viro model [Witten '89, Reshetikhin, Turaev, '91,Turaev, Viro, '92]

Tensor Field Theories: QFTs on Lie groups, spin foam amplitudes as Feynman graphs

$$Z_{\text{TFT}} \;=\; \sum_{\Delta} \frac{\lambda^{|\Delta|}}{|\Delta|!} Z_{\Delta}$$

[DePietri, Freidel, Kransov, Rovelli '00, Oriti '06, Gurau '09, Gurau, Rivasseau '11, ...]

In 2d: matrix model for quantum gravity

Symmetry reduced case: cosmological condensate [Gielen, Oriti, Sindoni, '13]

Problems with the EPRL model:

classical diffeo symmery broken in EPRL spin foam model [Dittrich '08, BB, Dittrich, '09]

continuum limit, perfect discretization? [Oeckl et al '04, BB, Dittrich, Steinhaus, '09, Hellmann, Kamiński '13, Hoehn '13]

renormalization without background structure? [BB '11, BB, Dittrich, Hellmann, Kamiński '12, Dittrich, Seinhaus, '13]

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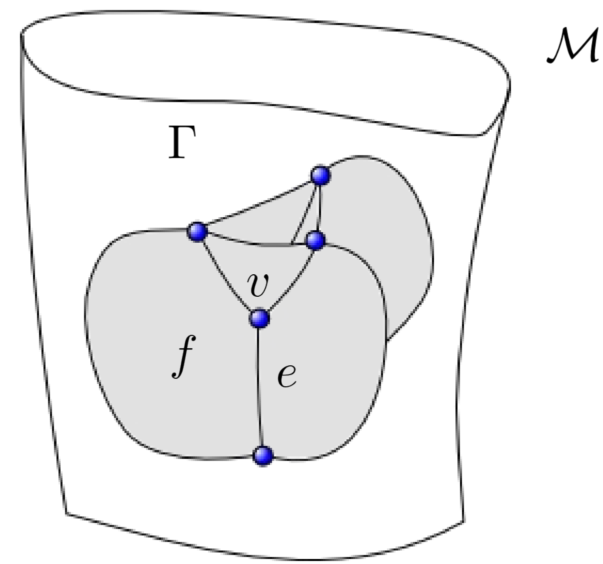
Generalized spin foam models

[Pfeiffer '01, Pfeiffer, Oeckl '02, Magliaro, Perini, '10, BB, Dittrich, Hellmann, Kamiński '12]

Start with manifold $\mathcal{M}$ and consider embedded, oriented 2-complexes $\Gamma \subset \mathcal{M}$

variables: group elements h_{ef} for $e \subset f$

$h_{ef} \in G$ compact Lie group

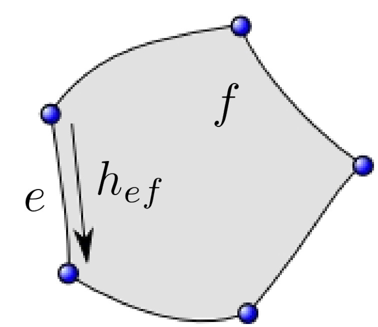


configuration space: $\mathcal{A}_\Gamma \simeq G^n$

integration measure: μ_Γ

observables: $\mathcal{O}_\Gamma \in C^0(\mathcal{A}_\Gamma)$

$$\langle \mathcal{O} \rangle_\Gamma = \int_{G^n} d\mu_\Gamma(h_{ef})\, \mathcal{O}(h_{ef})$$



Generalized spin foam models

special cases: lattice gauge theory:

$G = SU(N),\ \Gamma$ hypercubic lattice

$$d\mu_\Gamma(h_{ef}) \;=\; dh_{ef} \prod_f e^{-S_{\mathrm{Wilson}}[H_f]} \prod_e \int_G dg_e \prod_{f \subset e} \delta\big(g_e, h_{ef}\big)$$

EPRL spin foam model:

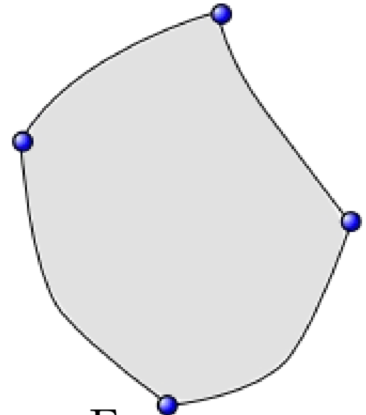
$G = SU(2) \times SU(2),\ \Gamma$ dual to 4d triangulation

$$d\mu_\Gamma(h_{ef}) \;=\; dh_{ef}\ \delta(H_f) \int dg_{ve} \prod_{ef} E(g_{ve}^{-1} h_{ef} g_{we})$$

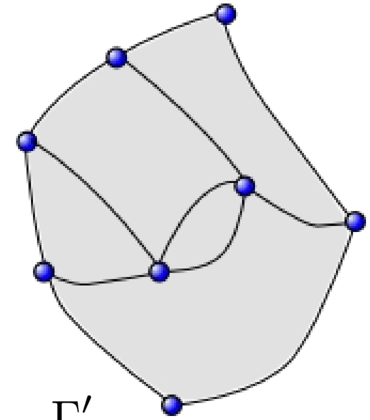
Continuum limit

Natural partial ordering of Γ's (in semi-analytic category)

$$\Gamma \;\leq\; \Gamma'$$



Γ



Γ'

Not every two Γ 's can be compared, but: for each two $\Gamma,\,\Gamma'$ there is a finer one:

projection („coarse graining map“) $\quad \pi_{\Gamma'\Gamma}\;:\;\mathcal{A}_{\Gamma'}\;\to\;\mathcal{A}_{\Gamma}$

$$\pi_{\Gamma'\Gamma}(\{h_{e'f'}\})_{ef} \;=\; \overrightarrow{\prod_{e'\subset e,\, f'\subset f}} h_{e'f'}$$

Continuum limit

[Ashtekar, Isham '92,
Ashtekar, Lewandowski '94]

Consider all Γ (or sufficiently many) at the same time:

projective limit: $\overline{\mathcal{A}} := \lim\limits_{\Gamma \leftarrow} \mathcal{A}_\Gamma = \big\{ \{a_\Gamma\}_\Gamma \,\big|\, a_\Gamma \in \mathcal{A}_\Gamma, \pi_{\Gamma'\Gamma} a_{\Gamma'} = a_\Gamma \big\}$

space of (generalized) continuum connections: compact Hausdorff space

$$\mathcal{A} \subset \overline{\mathcal{A}}$$

condition for continuum measure μ on $\overline{\mathcal{A}}$:

$$\langle \mathcal{O}_\Gamma \rangle_\Gamma = \langle \mathcal{O}_\Gamma \circ \pi_{\Gamma'\Gamma} \rangle_{\Gamma'}$$

for all observables $\mathcal{O}_\Gamma \in C^0(\mathcal{A}_\Gamma)$ „cylindrical consistency“

$\big\{ \mu_\Gamma \big\}_\Gamma \rightarrow \mu$ Radon measure on $\overline{\mathcal{A}}$

Where is the physical intuition?

configuration space: $\mathcal{A}_\Gamma$ are finite-dim „slices“ through $\overline{\mathcal{A}}$

Γ provides cut-off: only finitely many holonomies

The two-complexes are the scales!

observables: all $\mathcal{O}_\Gamma \in C^0(\mathcal{A}_\Gamma)$ for all Γ

only finitely many holonomies measurable at a time

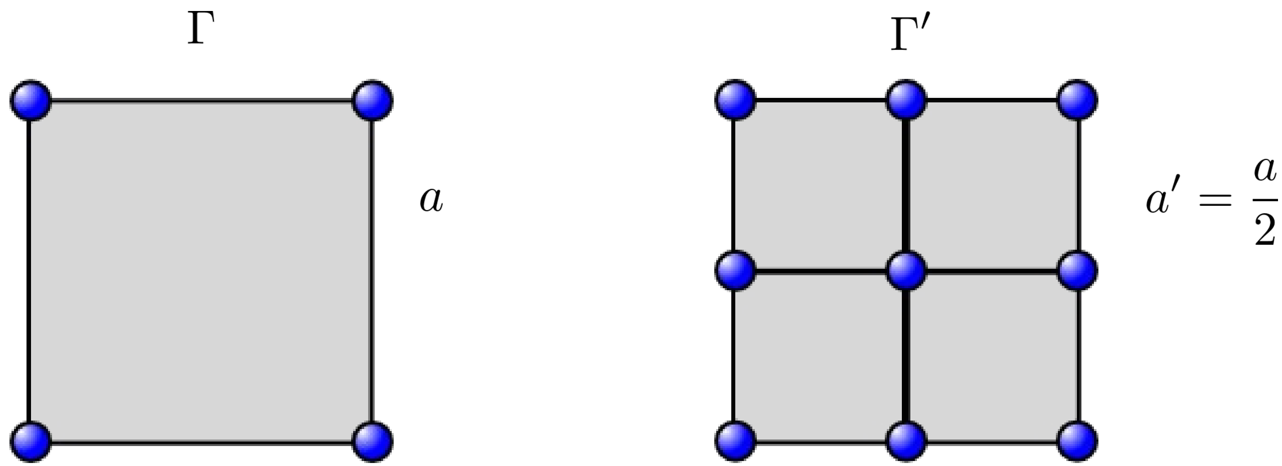
expectation values of observables: $\langle \mathcal{O}_\Gamma \rangle = \int_{\overline{\mathcal{A}}} d\mu \, \mathcal{O}_\Gamma := \langle \mathcal{O}_\Gamma \circ \pi_{\Gamma'\Gamma} \rangle_{\Gamma'}$

for any $\Gamma' \geq \Gamma$

Where is the physical intuition?

cylindrical consistency: $\langle \mathcal{O} \rangle_\Gamma \ = \ \langle \mathcal{O} \circ \pi_{\Gamma'\Gamma} \rangle_{\Gamma'}$

is precisely the idea of Wilsonian RG flow!



$$d\mu \ = \ e^{-S[h_e, g_i(a)]} \, dh_e$$

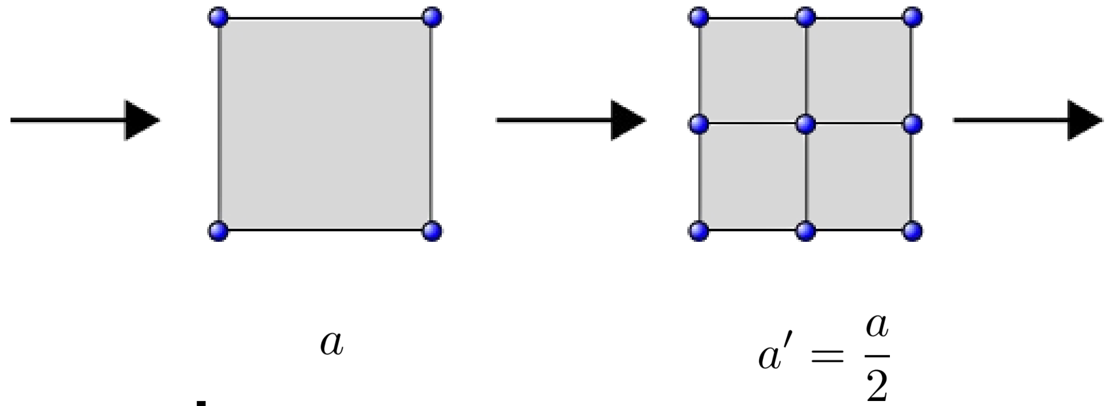
measure („action“) parametrised by parameters $g_i(a)$ which depend on a

$$e^{-S[h_e,\, g_i(a)]} \ = \ \int dh_{e'} \; e^{-S[h_{e'},\, g_i(a')]} \, \delta\left(h_e \ = \ \overrightarrow{\prod}_{e' \subset e} h_{e'}\right)$$

„scale“ with background:

$g_i(a)$

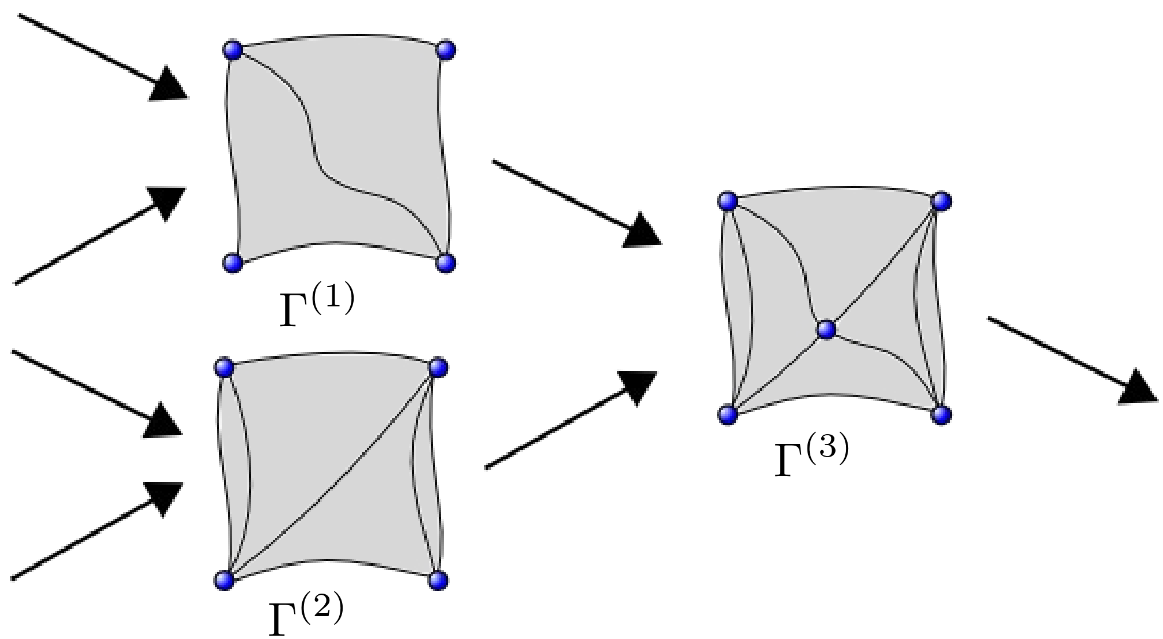
convergence
in the sense of sequence:



„scale“ without background:

$\{\mu_\Gamma\}_\Gamma$

convergence
in sense of filters:



Diffeomorphism group action

Group of (semi-analytic) diffeomorphisms act on classical connections $\mathcal{A} \subset \overline{\mathcal{A}}$

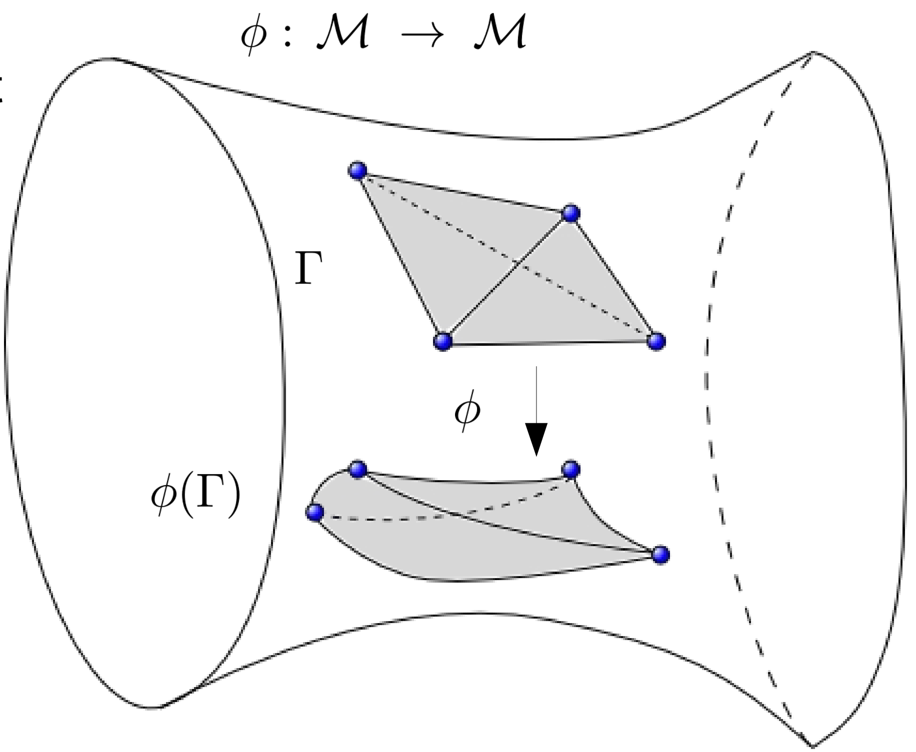
$$A \longmapsto (\phi^{-1})^* A$$

induced action on generalized connections via action on 2-complexes.

$$\phi \,:\, \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$$

invariance of continuum measure under ϕ equivalent to

$$\langle \mathcal{O} \rangle_\Gamma \;=\; \langle \phi^* \mathcal{O} \rangle_{\phi(\Gamma)}$$



for all Γ supporting $\mathcal{O}$

$$\Leftrightarrow \quad \phi_* \mu \;=\; \mu$$

Diffeomorphism group action

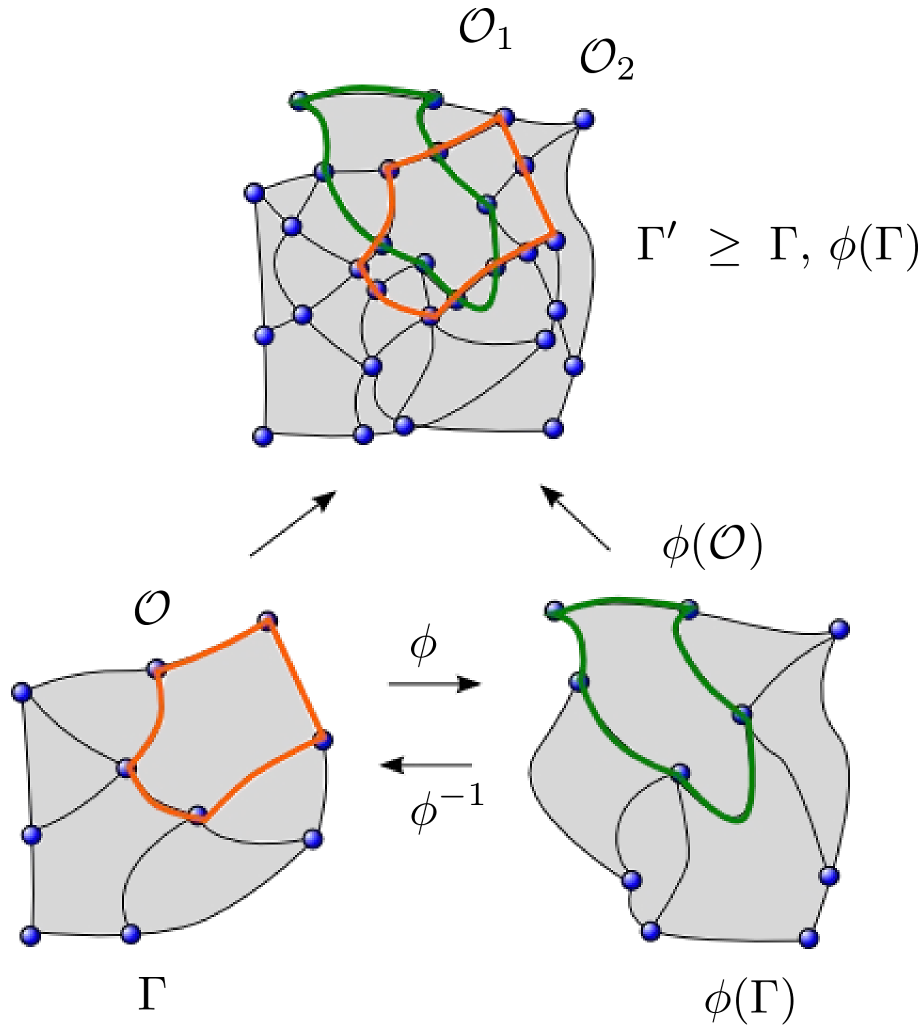
Diffeomorphism-invariance of partial measures μ_Γ

$$\langle \mathcal{O} \rangle_\Gamma \;=\; \langle \phi^* \mathcal{O} \rangle_{\phi(\Gamma)}$$

together with cylindrical consistency
this is a very strong condition:

$$\mathcal{O}_1 \;=\; \mathcal{O} \circ \pi_{\Gamma'\Gamma}$$
$$\mathcal{O}_2 \;=\; \phi(\mathcal{O}) \circ \pi_{\Gamma'\phi(\Gamma)}$$

$$\langle \mathcal{O}_1 \rangle_{\Gamma'} \;\stackrel{!}{=}\; \langle \mathcal{O}_2 \rangle_{\Gamma'}$$



Example:

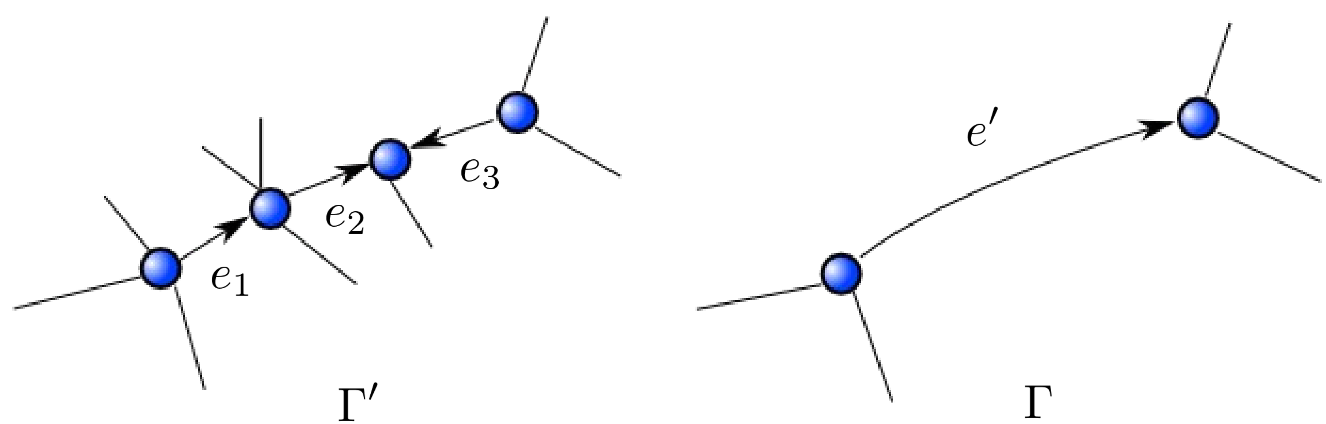
Near trivial example: $\dim \mathcal{M} \;=\; 2, \qquad G = U(1) \qquad \mathcal{A}_\Gamma \;=\; U(1)^E$

„charge-network functions“ $\mathcal{O}(h_e) \;=\; \prod_e h_e^{m_e}$

$$\langle \mathcal{O} \rangle_\Gamma \;:=\; \frac{1}{Z} \int_{U(1)^E} dh_e\, \mathcal{O}(h_e) \prod_f \sum_{n_f \in \mathbb{Z}} \exp\left(-n_f^2/2a_f + i\theta_f n_f\right) \left(\prod_{e \subset f} h_e^{[e,f]}\right)^{n_f}$$

$$= \int_{U(1)^E} d\mu_\Gamma^{\vec{a},\vec{\theta}}(h_e)\, \mathcal{O}(h_e) \qquad\qquad a_f > 0,\; \theta_f \in \mathbb{R}$$

coarse graining map:



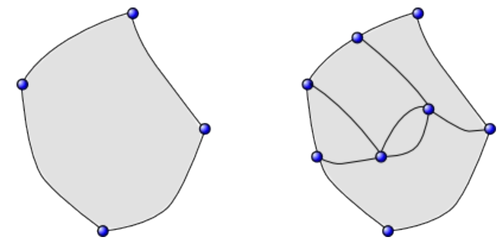
$$\pi_{\Gamma'\Gamma}(h_e) \;=\; (\cdots,\; \underbrace{h_{e1} h_{e_2} h_{e_3}^{-1}}_{=h_{e'}}, \cdots)$$

Example:

RG equations:

$$a_f^\Gamma = \sum_{f'\subset f} a_{f'}^{\Gamma'}$$

$$\theta_f^\Gamma = \sum_{f'\subset f} [f', f]\theta_{f'}^{\Gamma'}$$



$\Gamma \leq \Gamma'$

Obvious solution:

$g \in \mathrm{Sym}^2 T^* M$ (area) metric

$\theta \in \Omega^2(M)$ 2-form

$$a_f = \int_f \mathrm{dvol}_g \qquad \theta_f = \int_f \theta$$

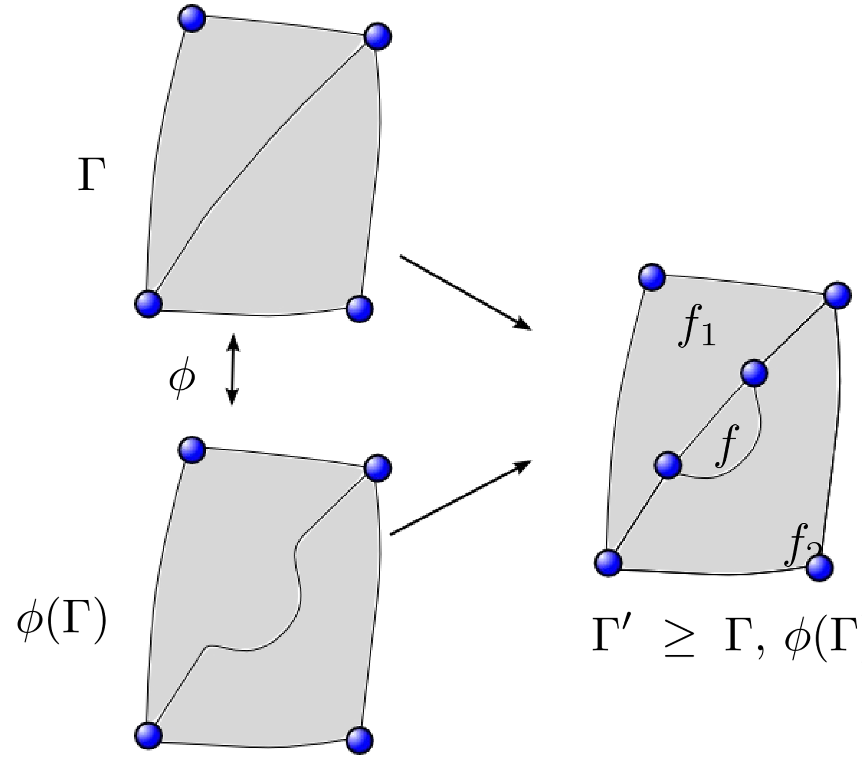
Limit solutions:

$\theta = 0$ 2d Yang-Mills

$$g \to 0 \qquad \int_{\mathcal{A}} \mathcal{D}A\, \delta\big(F[A] - \theta\big)$$

Example:

Diffeomorphism-invariance:



$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{\Gamma} \stackrel{!}{=} \langle \phi_* \mathcal{O} \rangle_{\phi(\Gamma)} = \langle \phi_* \mathcal{O} \rangle$$

$$\Rightarrow \quad \langle \mathcal{O} \circ \pi_{\Gamma'\Gamma} \rangle_{\Gamma'} = \langle \phi_* \mathcal{O} \circ \pi_{\Gamma'\phi(\Gamma)} \rangle_{\Gamma'}$$

$$\Rightarrow \quad a_{f_1} + a_f + a_{f_2} = a_{f_1} - a_f + a_{f_2}$$

$$\theta_{f_1} + \theta_f + \theta_{f_2} = \theta_{f_1} - \theta_f + \theta_{f_2}$$

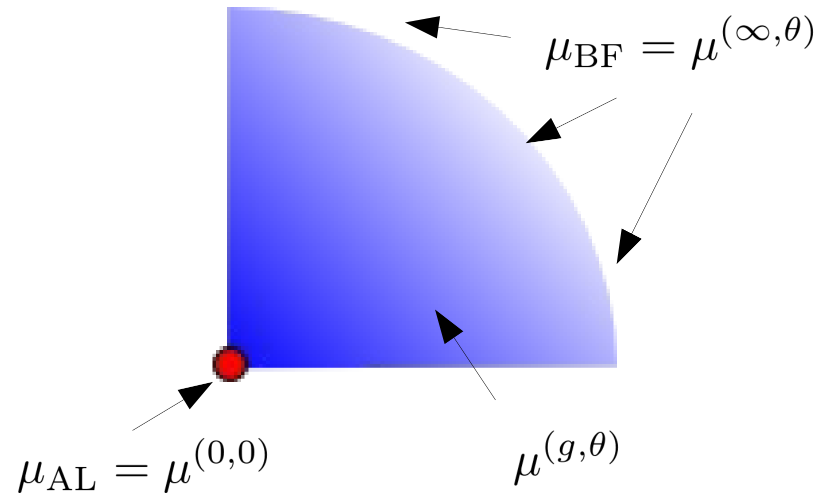
Only solutions: $g \to 0, \infty$ (limits exist as measures on $\overline{\mathcal{A}}$)

$\theta = 0$

Could have been guessed from $\phi_* \mu_{g,\theta} = \mu_{\phi_* g, \phi_* \theta}$

Example:

Space of solutions to GR equations:



$$d\mu^{(g,\theta)} = \frac{1}{Z}\prod_e dh_e \prod_f \sum_{n_f\in\mathbb{Z}} \exp\left(-n_f^2/2a_f + i\theta_f n_f\right)\left(\prod_{e\subset f} h_e^{[e,f]}\right)^{n_f}$$

$$a_f = \int_f \mathrm{dvol}_g \qquad\qquad \theta_f = \int_f \theta$$

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Approximations:

RG flow equations: conditions of cylindrical consistency of partial measures

$$\langle \mathcal{O} \rangle_\Gamma \;=\; \langle \mathcal{O} \circ \pi_{\Gamma'\Gamma} \rangle_{\Gamma'}$$

for all $\Gamma \leq \Gamma'$ and all $\mathcal{O} \in C^0(\mathcal{A}_\Gamma)$ $\Rightarrow$ In most cases impossible to check.

Possible approximations:

- Don't check for all 2-complexes Γ, just for subset (e.g. lattices, or hose dual to triangulations)

- Don't check on all observables $\mathcal{O}$, just on few interesting ones

- Search for solutions in subset of all measures, e.g.

$$d\mu(h_{ef}) \;=\; \prod_{ef} dh_{ef}\;\exp\Big(-S^{(g_i)}[h_{ef}]\Big)$$

truncation to few parameters g_i and minimize the error.

Example:

[BB, Dittrich, Hellmann, Kamiński '12]

Use finite group:

$$G = S_3 \; = \; \{1,\, (12),\, (13),\, (23),\, (123),\, (132)\}$$

has three irreps: trivial $\mathbf{0}_{S_3}$, sign $\mathbf{1}_{S_3}$, and $\mathbf{2}_{S_3}$

$$\rho_{\mathbf{2}_{S_3}}(12) \; = \; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho_{\mathbf{2}_{S_3}}(123) \; = \; \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \qquad \begin{array}{l} c = \cos(2\pi/3), \\ s = \sin(2\pi/3) \end{array}$$

Truncation of measure space (mimics EPRL model):

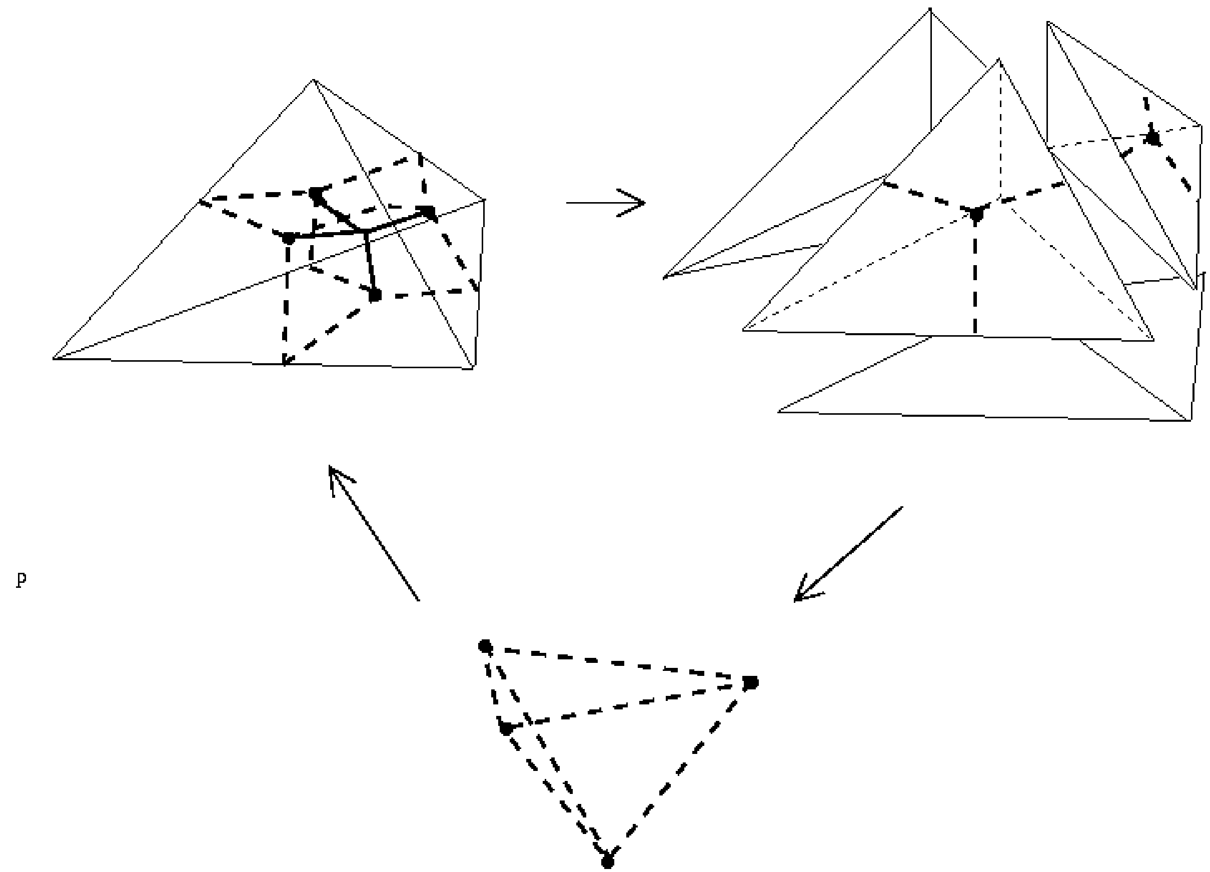
$$d\mu_\Gamma \; = \; dh_{ef} \prod_f \delta(H_f) \int dg_{ve} \prod_{e \subset f} \Big(e_0 + e_1 \rho_{\mathbf{1}_{S_3}}(\tilde{h}_{ef}) + e_{01} \rho_{\mathbf{1}_{S_3}}(\tilde{h}_{ef})_{\uparrow\uparrow} + e_{02} \rho_{\mathbf{1}_{S_3}}(\tilde{h}_{ef})_{\downarrow\downarrow} \Big)$$

$$\tilde{h}_{ef} \; = \; g_{ve}^{-1} h_{ef} g_{we}$$

Example:

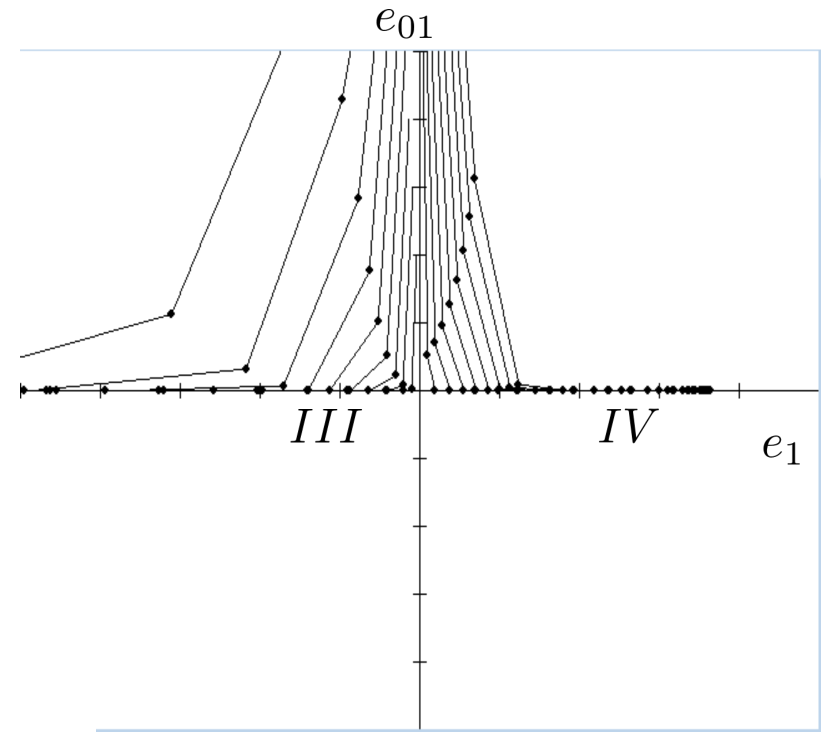
3d: hierarchical lattics made from tetrahedra, only consider observables at boundary

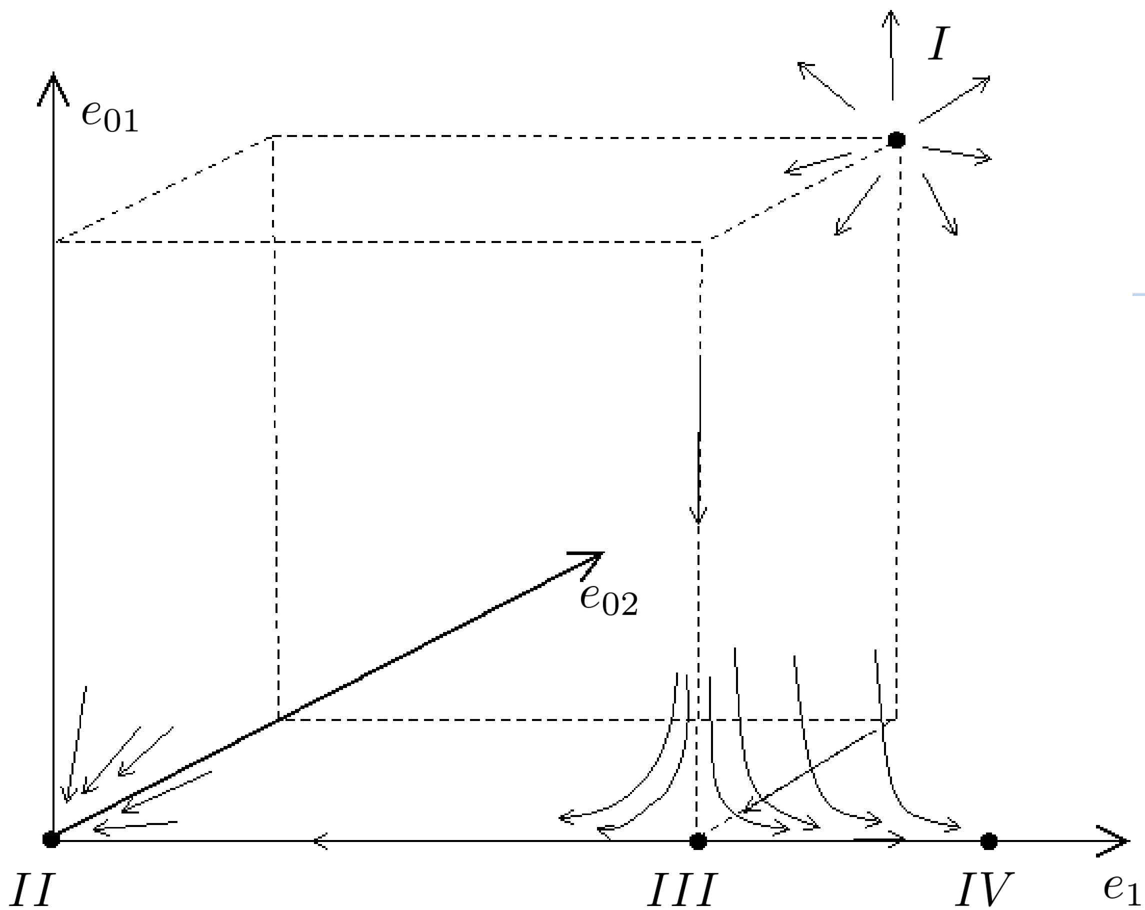
RG step:



Example:

Numerical flow:





I: S_3 BF theory
II: HTF
III: $\mathbf{Z}_2$ BF theory
IV: nontrivial

- Motivation: Why quantum gravity?
- Spin Foam Models:
  1. GR as constrained BF theory
  2. discretization and quantisation
  3. successes and shortcomings
- Renormalization without length scale:
  1. general setup
  2. diffeomorphism symmetry
  3. easy example
- How to do it in practice: Approximation methods
- Summary

Summary:

- Spin foam models are interesting models for background-independent QGR
- EPRL model nice candidate with interesting properties
- Renormalization of spin foam models in principle understood
  (flow along partially ordered set rather than line)
- Diffeomorphism-invariance strong condition!
- Approximation methods are being developed

Open questions:

- How to incorporate Lorentzian signature? (non-compact groups)
- Other interesting diff-invariant models (lower-dimensional, Seiberg-Witten?)
- How good are approximation methods (e.g. compared to Migdal-Kadanoff)?
- Interesting coupling constants for QGR: R-terms, etc.
- Asymptotic safety? Renormalizability?