Symplectic supermanifolds and the gauge algebra of double field theory

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Reminder on double field theory

Once upon a time...

- Sigma model $X: \ \Sigma \to M = T^d$

$$S = \int_\Sigma h^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij}\, d\mu_\Sigma + \int_\Sigma X^* B \ ,$$

where $h \in \Gamma(\otimes^2 T^*\Sigma)$, $G \in \Gamma(\otimes^2 TM)$, $B \in \Gamma(\wedge^2 T^* M)$.

- Classical solutions to e.o.m. (take *closed* string $\Sigma = \mathbb{R} \times S^1$)

$$X^i_R = x^i_{0R} + \alpha^i_0(\tau - \sigma) + i \sum_{n \neq 0} \frac{1}{n} \alpha^i_n e^{-in(\tau-\sigma)} \ , \quad X^i_L = \dots \ ,$$

$$\alpha^i_0 = \frac{1}{\sqrt{2}} G^{ij}\Big(p_j - (G_{jk} + B_{jk}) w^k\Big) \ ,$$

  - p_j: Canonical momentum zero modes
  - w^k: *Winding* zero modes, $w^k := \frac{1}{2\pi}\int_0^{2\pi} \partial_\sigma X^k d\sigma$.

Reminder on double field theory

The idea of "doubling" in field theory   Siegel, Tseytlin, Hull, Zwiebach, Kugo, Hohm

- Two different sets of momenta in α_0^i: Introduce coordinates:

$$p_i \simeq \frac{1}{i}\frac{\partial}{\partial x^i}\ , \quad w^i \simeq \frac{1}{i}\frac{\partial}{\partial \tilde{x}_i}\ . \tag{1}$$

  i.e. "doubled configuration space". Observables $\phi(x, \tilde{x})$.
- In addition: Observables in string theory obey the level matching constraint:

$$\partial_i \tilde{\partial}^i \phi + \tilde{\partial}^i \partial_i \phi = 0\ . \tag{2}$$

- Problem: If ϕ, ψ obey (2), $\phi\psi$ doesn't! Solution: Strong constraint:

$$\partial_i \phi\, \tilde{\partial}^i \psi + \tilde{\partial}^i \phi\, \partial_i \psi = 0\ , \tag{3}$$

  for all elements ϕ, ψ of the algebra of observables.

Reminder on double field theory

A word about notation and generalized geometry Hitchin, Gualtieri

- $O(d,d)$-transformations: $A \in \mathrm{Mat}(d,d)$,

$$A\eta A^t = \eta\ , \quad \eta_{MN} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

- "Double vectors" reduce to generalized vectors:

$$V^M = (V^m(x,\tilde{x}), V_m(x,\tilde{x})) \xrightarrow{\tilde{\partial}^i=0} (V^m(x), V_m(x)) \in TM \oplus T^*M\ .$$

- Contractions as in generalized geometry:

$$V^M W_M = V^M \eta_{MN} W^N = V^i W_i + V_i W^i\ ,$$

and similarly for $\partial_M = (\partial_m, \tilde{\partial}^m)$.

Reminder on double field theory

Gauge algebra and C-bracket

Hull, Zwiebach, arXiv: 0908.1792

- Action principle for generalized metric $\mathcal{H}(G, B)$ and scalar dilaton $\mathfrak{D}$, both depending on $(x, \tilde{x})$. Features:
  - Reduces to bosonic type IIA string by $\tilde{\partial}^i = 0$ (e.g. strong constraint)
  - Inv. under global $O(d, d)$ transformations (T-duality)
  - Gauge symmetry: Let $V = (V^i, V_i)$, then

$$\delta_V \phi = V^M \partial_M \phi = V^i \partial_i \phi + V_i \tilde{\partial}^i \phi \,,$$
$$(\delta_V W)_M = V^K \partial_K W_M + (\partial_M V^K - \partial^K V_M) W_K \,,$$

- Commutator of gauge trafos: C-bracket

$$[\delta_V, \delta_W] = \delta_{[V,W]_C} \,, \quad [\cdot,\cdot]_C : \text{C-bracket}$$
$$\Big([V, W]_C\Big)^M = V^K \partial_K W^M - W^K \partial_K V^M - \frac{1}{2}\Big(V^K \partial^M W_K - W^K \partial^M V_K\Big) \,. \qquad (4)$$

Question

On which space are the objects $V^M(x, \tilde{x})$ defined, and does this space have a structure which gives the C-bracket and the strong constraint?

Parity change and Lie algebroids

Mackenzie, Xu, Weinstein, Liu, Roytenberg, Voronov

- $V = \mathrm{span}(e_1, \ldots, e_n)$ finite dimensional vector space with dual $V^* = \mathrm{span}(e^1, \ldots, e^n)$.
- n variables x^i for $v = x^i e_i$, corresponding to e^i by $e^i(v) = x^i$.
- Consider polynomials $p = a_{i_1 \ldots i_k} x^{i_1} \cdots x^{i_k} \in \mathrm{Pol}^\bullet(V)$.
- Taking x^i to be Grassmannian, p corresponds to wedge products of e^i, i.e.

$$\mathrm{Pol}^\bullet(\Pi V) \simeq \wedge^\bullet(V^*) \ , \quad \Pi : \ \text{parity reversion} \, .$$

- Derivations on $\mathrm{Pol}^\bullet(\Pi V)$ correspond to derivations on $\wedge^\bullet V^*$, i.e. differentials if they square to zero.

Definition

A vector bundle $A \to M$ is called Lie algebroid, if there exists a homological vector field d_A (also expressed as a derivation of square 0) on the supermanifold ΠA, i.e. $[d_A, d_A] = 0$.

An example: (M, π) Poisson manifold

Mackenzie, Xu, Weinstein, Liu, Roytenberg, Voronov

- $\underline{A = TM}$, basis of sections e_i, $[e_i, e_j]_A = f_{ij}^k e_k$
  label coordinates on ΠA by (x^i, ξ^i) and the anchor $a : TM \to TM$ by a_j^i, then the de Rham differential is

$$d_A = a_j^i(x)\xi^j \partial_i - \frac{1}{2} f_{ij}^k(x)\xi^i \xi^j \frac{\partial}{\partial \xi^k} \ . \tag{5}$$

- $\underline{A^* = T^*M}$, basis e^i, $[e^i, e^j]_{A^*} = Q_k^{ij} e^k$,
  label coordinates on ΠA^* by (x^i, θ_i) and the anchor $a : T^*M \to TM$ by a^{ij}, then the Poisson-Lichnerowicz differential is

$$d_{A^*} = a^{ij}(x)\theta_i \partial_j - \frac{1}{2} Q_k^{ij}(x)\theta_i \theta_j \frac{\partial}{\partial \theta_k} \ . \tag{6}$$

The pair (A, A^*) is an example of a *Lie bialgebroid*.

Taking the cotangent bundle

Roytenberg, arXiv:math/9910078

Why? Idea: d_A gives ∂_i on functions. Relation of d_{A^*} to $\tilde{\partial}^i$? - If so, we need to get expressions like $\partial_i f + \tilde{\partial}^i f$, i.e. *define a meaningful sum of d_A and d_{A^*}!*

- $T^*\Pi A$: Coordinates $(x^i, \xi^i, x_i^*, \xi_i^*)$. Poisson structure:

$$\{x^j, x_i^*\} = \delta_i^j \ , \quad \{\xi^j, \xi_i^*\} = \{\xi_i^*, \xi^j\} = \delta_i^j \ .$$

  De Rham differential is now given by taking the Poisson bracket with the function

$$h_{d_A} = a_j^i(x) x_j^* \xi^i - \frac{1}{2} f_{ij}^k \xi^i \xi^j \xi_k^* \ , \tag{7}$$

- $T^*\Pi A^*$: Coordinates $(x^i, \theta_i, x_i^*, \theta_*^i)$. Poisson structure:

$$\{x^j, x_i^*\} = \delta_i^j \ , \quad \{\theta_i, \theta_*^j\} = \{\theta_*^j, \theta_i\} = \delta_i^j \ .$$

  Poisson-Lichnerowicz differential "lifted" to

$$h_{d_{A^*}} = a^{ij}(x) \theta_i x_j^* - \frac{1}{2} Q_k^{ij} \theta_i \theta_j \theta_*^k \ . \tag{8}$$

Legendre transform and Drinfel'd double

Roytenberg, arXiv:math/9910078

Relation between the two bundles: *Legendre transform*:

$$L : T^*\Pi A \to T^*\Pi A^* \ , L(x^i, \xi^j, x_i^*, \xi_j^*) = (x^i, \xi_j^*, x_i^*, \xi^j) \ .$$

Gives the possibility to pull-back $h_{d_{A^*}}$ and add it to d_A:

$$\mu := \ h_{d_A} + L^* \, h_{d_{A^*}} \ . \tag{9}$$

Theorem

A pair of Lie algebroids (A, A^*) is a Lie bialgebroid iff $\{\mu, \mu\} = 0$.

Thus the following definition is justified:

Definition

The Drinfel'd double of a Lie bialgebroid (A, A^*) is given by $T^*\Pi A$ together with the homological vector field $\{\mu, \cdot\}$.

Application to double field theory

Deser, Stasheff, arXiv:1406.3601

Two sets of momenta:

$$h_{d_A} = \xi^i \left(a_i^j x_j^* - \frac{1}{2} f_{ij}^k \xi^j \xi_k^* \right) =: \xi^i p_i \,,$$
$$L^* \, h_{d_{A^*}} = \xi_i^* \left(a^{ij} x_j^* + Q_k^{ij} \xi^k \xi_j^* \right) =: \xi_i^* \tilde{p}^i \,.$$

Thus, we get two derivative operators for $f \in \mathcal{C}^\infty(M)$, seen as $f \in \mathcal{C}^\infty(T^*\Pi A)$:

$$\partial_i f := \{p_i, f\} \,, \quad \tilde{\partial}^i f := \{\tilde{p}^i, f\} \,, \tag{10}$$

i.e. think of f as $f(x, \tilde{x})$. More general: Lift of a generalized vector field:

$$V^m \partial_m + V_m dx^m \to V^m(x, \tilde{x}) \xi_m^* + V_m(x, \tilde{x}) \xi^m \ \in \mathcal{C}^\infty(T^*\Pi A) \,.$$

Now, what is the C-bracket and the strong constraint?

Main results

Deser, Stasheff, arXiv:1406.3601

Theorem

Let $V^m e_m + V_m e^m$ and $W^m e_m + W_m e^m$ be generalized vectors with corresponding lifts to $T^*\Pi A$ given by $V = V^m \xi^*_m + V_m \xi^m$ and $W = W^m \xi^*_m + W_m \xi^m$. In addition let the operation $\circ$ be defined by:

$$V \circ W = \left\{ \{ \xi^i p_i + \xi^*_i \tilde{p}^i, V \}, W \right\},$$

Then the C-bracket of V and W is given by

$$[V, W]_C = \frac{1}{2} \Big(V \circ W - W \circ V \Big) . \quad (11)$$

Thus, the C-bracket can be seen as a Courant bracket, written in a form appropriate to DFT.

Main results

Deser, Stasheff, arXiv:1406.3601

Theorem

Let $\phi(x, \tilde{x}), \psi(x, \tilde{x})$ be two double scalar fields and $D = \{\mu, \cdot\}$ the homological vector field on $T^*\Pi A$. Then we have

$$0 = \{D^2\phi, \psi\} = \partial_i\phi\tilde{\partial}^i\psi + \tilde{\partial}^i\phi\partial_i\psi \ . \qquad (12)$$

Thus, the strong constraint is a consequence of the condition on $T^*\Pi A$ being the Drinfel'd double of a Lie bialgebroid.

Conclusions and outlook

Interpreting double scalars and double vector fields as functions on the Drinfel'd double of a Lie bialgebroid enabled us to

- Give a mathematical interpretation of "double fields".
- Express the C-bracket of DFT by Poisson brackets on $T^*\Pi A$ and thus identify it as a Courant bracket.
- Get the strong constraint as a consequence of the structure of $T^*\Pi A$.

Lots of work ahead:

- Interpretation of $O(d,d)$ transformations on the Drinfel'd double
- What are the non-geometric fluxes in this framework?
- Relation to BFV and BV theories?
- α'-corrections to double field theory?