Transverse momentum dependent PDFs at NNLO

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Motivation

Hadron colliders

- $\Rightarrow$ extract σ_{process}, $\frac{d\sigma_{\text{process}}}{dv}$, ...

  e.g. for $pp \to Z + X$ differential in q_T

- $\Rightarrow$ results depend on many mass scales

  s, M_Z, q_T, ...

  can be of very different size.

Motivation

- Excellent performance of LHC: high experimental precision.
- Need high theory precision to
  - identify and measure signal (SM, BSM) on top of large (QCD) background,
  - extract SM parameters to high accuracy.

$\Rightarrow$ Need multi-loop calculations. Especially in QCD.

- Preferable: Analytic calculations, since:
  - offer many checks,
  - exact result,
  - elegant and explicit representation,
  - can learn more about theory, structure.

Perturbation theory

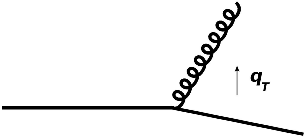
- perturbative expansion:

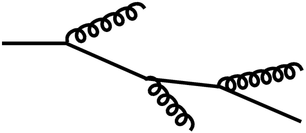
$$\frac{d\sigma}{dv} = \sum_{n=0}^{\infty} \alpha_s^n \frac{d\sigma}{dv}^{(n)}$$

  fixed order: stop at e.g. $n = 2$ (NNLO).
- For each power in α_s IR divergences, cancel between real and virtual corrections by KLN theorem.
- However, large logarithms of scale ratios can remain, which
  - spoil convergence,
  - need to be resummed to all orders in α_s.

Large logarithms

Momenta of emitted particle soft, collinear
$\Rightarrow$ up to $\log^2 \frac{q_T^2}{q^2}$ for each power α_s.





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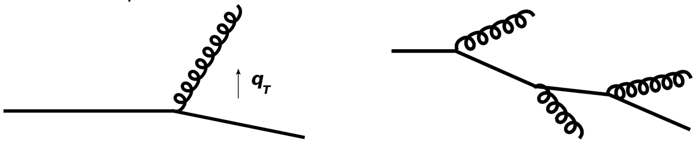


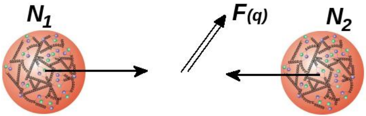
Due to specific structure, factorize, can account for them to all orders in α_s (resum).

In expansion $$\frac{d\sigma}{dv} = \sum_{n=0}^{\infty}\sum_{k=0}^{2n} \alpha_s^n L^k \frac{d\sigma}{dv}^{(n,k)} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \alpha_s^n (\alpha_s L^2)^k C^{(k)}_{N^n LL_e}.$$

$\Rightarrow$ Stabilizes spectrum at low q_T.

Example

Consider:
$N_1 + N_2 \to F(q) + X$



- E.g. Drell-Yan, vector boson, Higgs, ... production.
- Test SM to high precision.

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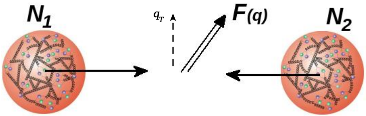
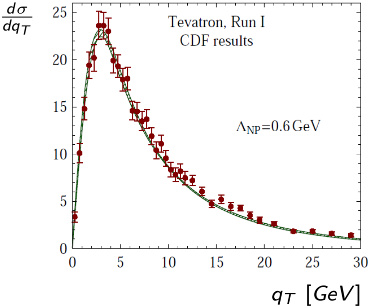


- E.g. Drell-Yan, vector boson, Higgs, ... production.
- Test SM to high precision.
- $d\sigma/dq_T$ in region $q_T^2 \ll q^2$.

$\Rightarrow$ Need to resum large logarithms.

$\Rightarrow$ TPDFs (Beam functions).



Outline

0. Motivation

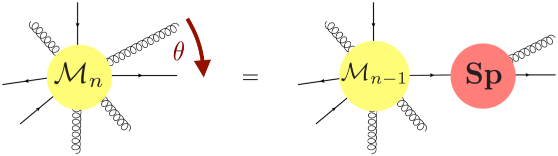
1. k_T factorization
   - SCET
   - Factorization theorems
2. Perturbative calculation
   - Rapidity divergences
   - NNLO
3. Checks and resummation
   - RGEs
   - Counting Logs
   - Relation to other frameworks

(k_T) factorization

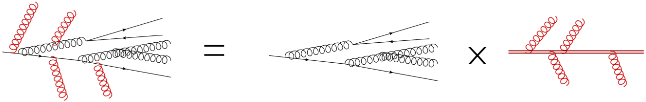
QCD simplified

Soft and collinear emissions: Important contributions to high-energy processes; have characteristic structure.
In both limits interactions simplify:

- Collinear limit, multiple particles move in similar direction



- Soft limit, particles with small energy and momentum are emitted $\Rightarrow$ Eikonal interactions.



Soft-collinear effective theory

EFT of QCD.

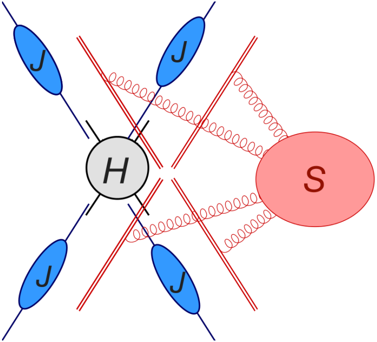
- Structure of soft and collinear interactions implemented on Lagrangian level.

⇒ Soft and collinear fields with definite interactions and power counting.

- Efficient formalism to (re-)derive factorization theorems for multi scale problems.

+: Gauge invariant operator definitions for the soft and collinear contributions.

- Resummation via renormalization group.



$$d\sigma \sim H(s_{ij}^2)S(\Lambda_{ij}^2) \otimes \prod_i J_i(M_i^2)$$

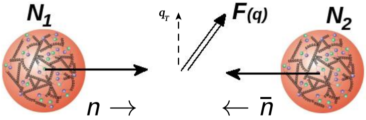
[Bauer, Fleming, Pirjol, Stewart, . . .]

Notation

- Consider:

$$N_1 + N_2 \to F(q) + X$$

- LC vectors, $n^2, \bar{n}^2 = 0$, $n \cdot \bar{n} = 2$



  - LC basis: $v^\mu = (\bar{n}v)\frac{n^\mu}{2} + (nv)\frac{\bar{n}^\mu}{2} + v_\perp^\mu = (v_-, v_+, v_\perp)^\mu$

$$v \cdot x = \frac{v_- x_+}{2} + \frac{v_+ x_-}{2} + v_\perp \cdot x_\perp$$

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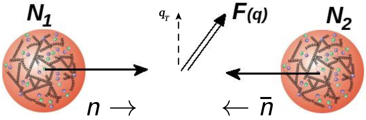

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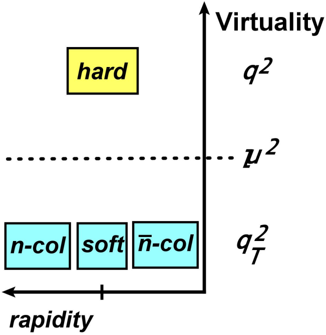
  - Consider process for $\lambda^2 = \frac{q_T^2}{q^2} \ll 1$.
  - Distinguish fields with different momentum scaling

hard (h):	p_h	$\sim \sqrt{q^2}(1, 1, 1)$	$p_h^2 \sim q^2$
$n-$collinear (n):	p_n	$\sim \sqrt{q^2}(1, \lambda^2, \lambda)$	$p^2 \sim q_T^2$
$\bar{n}-$collinear ($\bar{n}$):	$p_{\bar{n}}$	$\sim \sqrt{q^2}(\lambda^2, 1, \lambda)$	
soft (s):	p_s	$\sim \sqrt{q^2}(\lambda, \lambda, \lambda)$	

  - Last three are relevant modes for SCET.

SCET: Soft collinear EFT of QCD

- Start from $\mathcal{L}_{\text{QCD}}$.
- Split fields in individual modes.
- Integrate out hard modes.
- Expand in λ.



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$\Rightarrow$ EFT of $n-$, $\bar{n}-$col. and soft fields.

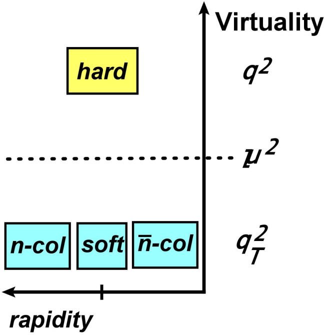
- Perform soft decoupling transform..

$\Rightarrow$ $\mathcal{L}_{\text{SCET}} = \mathcal{L}_n + \mathcal{L}_{\bar{n}} + \mathcal{L}_s + \mathcal{O}(\lambda^2)$

! Different regions do not interact.

$\Rightarrow$ Factorization

- $\mathcal{L}_{\text{SCET}} \supset$ non-local operators associated with large light like momenta.



Match operators

- Include $\mathcal{O} \in \big(\mathcal{L}_{\mathrm{SM}} - \mathcal{L}_{\mathrm{QCD}}\big)$.
- E.g. $\mathcal{O} = ig_e A^{\gamma}_{\mu}(x) J^{\mu}(x)$
  with e.m. current $J^{\mu} = \sum_f e_f \bar{q}_f \gamma^{\mu} q_f$.

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- Match to SCET ($v = n, \bar{n}$):
  $q_f \stackrel{v}{\to} \chi_{f,v} = S_v W_v^\dagger \xi_{f,v}$; $\xi_v = \frac{\not{v}\,\not{\bar{v}}}{2} q_v$;
  Wilson lines: $W_v(x) = P \exp\left[ig \int_{-\infty}^0 ds\, \bar{v}\cdot A_v(x + s\bar{v})\right]$,
  $S_v(x) = P \exp\left[ig \int_{-\infty}^0 ds\, v\cdot A_s(x + sv)\right]$,

$\Rightarrow$ $J^\mu \to \sum_f e_f C_\gamma(-q^2) \left(\bar{\xi}_{f,\bar{n}} W_{\bar{n}}\right) \gamma^\mu_\perp \left(S^\dagger_{\bar{n}} S_n\right) \left(W^\dagger_n \xi_{f,n}\right)$,

with hard Wilson coefficient $C_\gamma(-q^2) \hat{=}$ form factor.

- Fully analogous for $V = Z, W^\pm, Z'$, multiple V bosons ...

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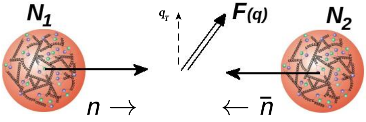
  with hard Wilson coefficient $C_{\gamma}(-q^2) \hat{=}$ form factor.
- Fully analogous for $V = Z, W^{\pm}, Z'$, multiple V bosons ...
- Similarly for operators containing gluons
  $A^{\mu} \xrightarrow{v} \mathcal{A}^{\mu}_v = S^{\text{adj}}_v W^{\text{adj}\dagger}_v A^{\mu}_v\ ;$
  with Wilson lines in adjoint repr.: $T^C_{ab} \to -if^{CAB}$.

Derive factorization

Process:

$N_1(p{\sim}n) + N_2(\bar{p}{\sim}\bar{n}) \to \gamma(q) + X$



Cross section:

$$d\sigma = \frac{4\pi\alpha_{em}^2}{3q^2 s} \frac{d^4q}{(2\pi)^4} \int d^4x \, e^{-iq\cdot x} (-g_{\mu\nu}) \times \sum_X \langle N_1 N_2 | \, J^{\mu\dagger}(x) \, | X \rangle \langle X | \, J^{\nu}(0) \, | N_1 N_2 \rangle$$

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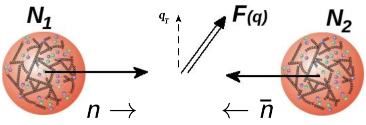

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- $J^\mu \to \sum_f e_f\, C(-q^2)\big(\bar{\xi}_{f,\bar{n}} W_{\bar{n}}\big)\gamma_\perp^\mu\big(S_{\bar{n}}^\dagger S_n\big)\big(W_n^\dagger \xi_{f,n}\big)$
- $|N_1 N_2\rangle \to |N_1\rangle_n \otimes |N_2\rangle_{\bar{n}}$
- $|X\rangle \to |X_n\rangle \otimes |X_{\bar{n}}\rangle \otimes |X_s\rangle$

$\Rightarrow$ Factorizes into 4 factors (color average).

Generalized PDFs

Factorized cross section

$$d\sigma = \frac{4\pi\alpha_{em}^2}{3q^2 s}\frac{d^4q}{(2\pi)^4}\left|C(-q^2)\right|^2\int d^4x\, e^{-iq\cdot x}\sum_f e_f^2\ \tilde{B}^n_{q_f/N_1}(x)\tilde{B}^{\bar{n}}_{\bar{q}_f/N_2}(x)\tilde{\mathcal{S}}(x)$$

with

$$\tilde{B}^n_{q_f/N_1}(x) = \sum_{X_n}\frac{\bar{\not{n}}_{\alpha\beta}}{2}\left\langle N_1(p)\right|(\bar{\xi}_{f,n}W_n)_\alpha(x)\left|X_n\right\rangle\left\langle X_n\right|(W_n^\dagger\xi_{f,n})_\beta(0)\left|N_1(p)\right\rangle,$$

$$\tilde{B}^{\bar{n}}_{\bar{q}_f/N_2}(x) = \sum_{X_{\bar{n}}}\frac{\not{n}_{\alpha\beta}}{2}\left\langle N_2(\bar{p})\right|(\xi_{f,\bar{n}}W_{\bar{n}}^\dagger)_\beta(x)\left|X_{\bar{n}}\right\rangle\left\langle X_{\bar{n}}\right|(\bar{\xi}_{f,\bar{n}}W_{\bar{n}})_\alpha(0)\left|N_2(\bar{p})\right\rangle,$$

$$\tilde{\mathcal{S}}(x) = \frac{1}{N_c}\sum_{X_s} Tr\left\langle 0\right|(S_n^\dagger S_{\bar{n}})(x)\left|X_s\right\rangle\left\langle X_s\right|(S_{\bar{n}}^\dagger S_n)(0)\left|0\right\rangle.$$

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Multipole expand $x^\mu = (x_-, x_+, x_\perp)$ according to power counting:
For TPDFs, drop subleading x_- in $\tilde{B}^{\bar{n}}$; x_+ in $\tilde{B}^n$; x_-, x_+ in $\tilde{\mathcal{S}}$.
Expand $\int d^4x\, e^{-iq\cdot x}$ in LC basis, absorb x_+ (x_-) part to $\tilde{B}^n$ ($\tilde{B}^{\bar{n}}$).

Differential cross section

$$\frac{d^3\sigma}{dq_T^2\,dq^2\,dy} = \frac{\alpha_{ew}^2}{3N_c s q^2}\left|C_V(-q^2)\right|^2 \int d^2x_\perp\, e^{-iq_\perp\cdot x_\perp} \sum_f e_f^2$$

$$\times \left[\mathcal{S}(x_T^2)\mathcal{B}_{q_f/N_1}(z_1,x_T^2)\bar{\mathcal{B}}_{\bar{q}_f/N_2}(z_2,x_T^2) + (q \leftrightarrow \bar{q})\right]_* ,$$

with ([Becher, Neubert], ...)

TPDF (quark, n collinear)

$$\mathcal{B}_{q_f/N_1}(z,x_T^2) = \frac{1}{2\pi}\int \mathrm{d}t\; e^{-izt\bar{n}\cdot p} \sum_{X_n} \frac{\bar{\not{n}}_{\alpha\beta}}{2}$$

$$\times \langle N_1(p)|\,(\bar{\xi}_{f,n}W_n)_\alpha(t\bar{n}+x_\perp)\,|X_n\rangle\,\langle X_n|\,(W_n^\dagger \xi_{f,n})_\beta(0)\,|N_1(p)\rangle_* .$$

- Anti-collinear $\bar{\mathcal{B}}_{\bar{q}/N_2}$ correspondingly with $p \sim n \leftrightarrow \bar{n} \sim \bar{p}$ & $q \leftrightarrow \bar{q}$.
- Valid up to corrections $(q_T^2/q^2)^m$.
- $z_{1,2} = \sqrt{\tau}e^{\pm y}$, $\tau = (q^2+q_T^2)/s$.

*: To care for rapidity divergences.

Differential cross section

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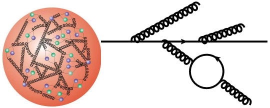
$$\times \langle N_1(p)| (\bar{\xi}_{f,n} W_n)_\alpha (t\bar{n} + x_\perp) |X_n\rangle \langle X_n| (W_n^\dagger \xi_{f,n})_\beta(0) |N_1(p)\rangle_* .$$

- k = momentum of X_n: k_- and Fourier partner of $k_\perp$ fixed:
  $\int dt e^{-iztp_-} \bar{\chi}(t\bar{n} + x_\perp) \Rightarrow \delta(k_- - (1-z)p_-)$,
  $\bar{\chi}(t\bar{n} + x_\perp) \Rightarrow e^{-ik_\perp x_\perp} \overset{\text{F.T.}}{\to} \delta^{(2)}(k_\perp + q_\perp)$.

*: To care for rapidity divergences.

Factorization, pictorially

$\mathcal{B}_{q_f/N_1}(z_1, x_T^2)$



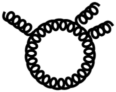
'transverse PDF'

$n^\mu \longrightarrow$

$C_V(-q^2)$

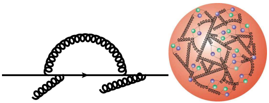
'hard function'

$\mathcal{S}(x_T^2)$



'soft function'

$\bar{\mathcal{B}}_{\bar{q}_f/N_2}(z_2, x_T^2)$



'transverse PDF'

$\longleftarrow \bar{n}^\mu$

$$\frac{d^2\sigma}{dq_T^2\, dy\, dq^2} = C_V \otimes S \otimes \mathcal{B}_{q/N_1} \otimes \bar{\mathcal{B}}_{\bar{q}/N_2}$$

Other GPDFs/beam functions

Back to intermediate step

$$d\sigma = \frac{4\pi\alpha_{em}^2}{3q^2 s}\frac{d^4 q}{(2\pi)^4}\left|C(-q^2)\right|^2\int d^4x\, e^{-iq\cdot x}\sum_f e_f^2\left[\tilde{B}^n_{q_f/N_1}(x)\tilde{B}^{\bar{n}}_{\bar{q}_f/N_2}(x)\tilde{\mathcal{S}}(x) + (q_f \leftrightarrow \bar{q}_f)\right].$$

Analogous, adjust multipole expansion, if measure q^2, y and

- $q_T^2 \;\Rightarrow$ TPDFs $B(z, x_T^2)$: in $\frac{d^3\sigma}{dq^2 dq_T^2 dy}$
- Nothing $\Rightarrow$ PDFs $\phi(z)$: in $\frac{d^2\sigma}{dq^2 dy}$
  $\Rightarrow$ Operator defintion as for TPDFs w/o $x_\perp$ dependence.
- virtuality $t \Rightarrow$ Virtuality dep. PDFs $B(z,t)$: in $\frac{d^3\sigma}{dq^2 dt\, dy}$
- q_T^2 and $t \Rightarrow$ Fully unintegrated PDFs $B(z, x_T^2, t)$: in $\frac{d^4\sigma}{dq^2 dq_T^2 dt\, dy}$
- ...

[Gaunt, Stahlhofen, Tackmann], [Diehl], ...

Gluons

- Similar fact. theorems for other $q\bar{q}$ and gg initiated processes.
- If gg initiated: C, $\mathcal{B}$ & $\bar{\mathcal{B}}$ become Lorentz tensors.

gluon TPDF (n collinear)

$$\mathcal{B}^{\mu\nu}_{g/N}(z, x_\perp) = \frac{-z\bar{n}\cdot p}{2\pi} \int dt\, e^{-izt\bar{n}\cdot p} \sum_X \langle N| \mathcal{A}^{\mu a}_{n,\perp}(t\bar{n} + x_\perp) |X\rangle \langle X| \mathcal{A}^{\nu a}_{n,\perp}(0) |N\rangle$$

- with gauge invariant gluon field $\mathcal{A}^\mu_{n\perp}(x) = \left(W_n^{adj\dagger} A^\mu_{n\perp}\right)(x)$.
- Decompose tensor as

$$\mathcal{B}^{\mu\nu}_{g/N}(z, x_\perp) = \frac{g_\perp^{\mu\nu}}{d-2} \mathcal{B}_{g/N}(z, x_T^2) + \left[\frac{g_\perp^{\mu\nu}}{d-2} + \frac{x_\perp^\mu x_\perp^\nu}{x_T^2}\right] \mathcal{B}'_{g/N}(z, x_T^2)\,.$$

- Discussion below for $\mathcal{B}$ only. $\mathcal{B}'$ analogous.

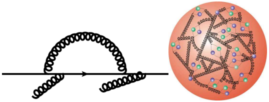
Matching kernel

If $x_T^{-2} \gg \Lambda^2_{\mathrm{QCD}}$, refactorize these scales:

Matching kernel $\mathcal{I}_{i/k}$

$$\mathcal{B}_{i/N}(z, x_T^2) = \sum_k \int_z^1 \frac{d\rho}{\rho} \mathcal{I}_{i/k}(\rho, x_T^2) \phi_{k/N}(z/\rho) + \mathcal{O}(\Lambda^2 x_T^2)$$

$$= \sum_k \mathcal{I}_{i/k}(z, x_T^2) \otimes \phi_{k/N}(z) \, .$$



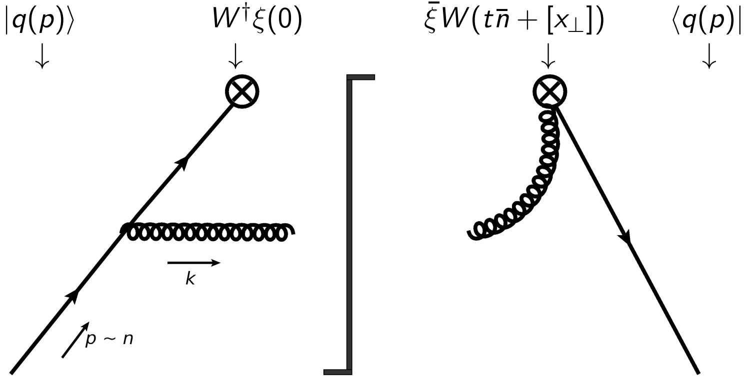
- $\mathcal{I}_{i/k}$ perturbative: Extract from perturbative $\mathcal{B}_{i/j}$ and $\phi_{k/j}$.
- Determine to NNLO: [Gehrmann,TL,Yang].

Perturbative calculation

Perturbative calculation

- Explicit operator definitions.
  $\mathcal{B}_{q_f/j} = \ldots \langle j(p)| \, (\bar{\xi}_{f,n} W_n)_\alpha (t\bar{n} + x_\perp) \, |X_n\rangle \, \langle X_n| \, (W_n^\dagger \xi_{f,n})_\beta(0) \, |j(p)\rangle_*$
  $W_n(x) = P \exp \left[ig \int_{-\infty}^0 ds \, \bar{n} \cdot A_n(x + s\bar{n}) \right] .$
- Here discuss n-collinear case. Other case via $n \leftrightarrow \bar{n}$.
- Gauge independent due to W_n.
- Calculate perturbatively $\mathcal{B} = \sum_m (\frac{\alpha_s}{4\pi})^m \mathcal{B}^{(m)}$.
- Use dim. reg. $d = 4 - 2\epsilon$. *
- n-collinear fields only: Can use QCD Feynman rules.
- General gauge: $W \Rightarrow \quad \bar{n} \cdot (\ldots)$ denominators.
- Special gauge: LC gauge using $\bar{n}$
  $\Rightarrow W = 1$, but LC propagators.
- Do both: LC and Feynman gauge.

Example diagram at NLO



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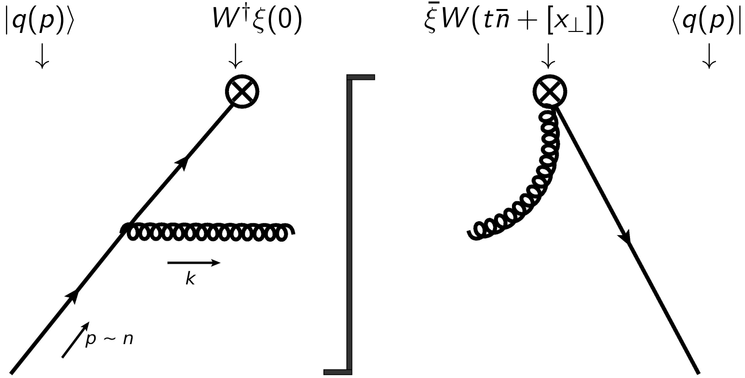
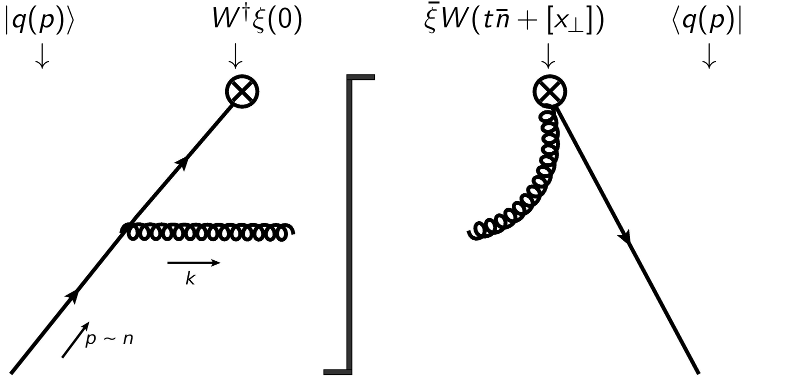


$$\frac{\alpha_s}{4\pi}\left(\mathcal{B}_{qq}^{(1)},\phi_{qq}^{(1)}\right) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}(2\pi)\delta^+(k^2)\delta(\bar{n}\cdot[k-(1-z)p])\left(e^{-ik_\perp\cdot x_\perp},1\right)\mathcal{M}$$

Example diagram at NLO



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Integrals for ϕ and purely virtual contributions vanish (scaleless).

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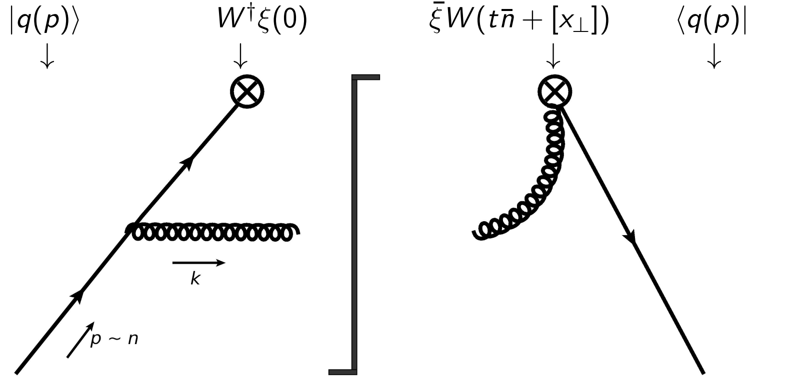

$$\frac{\alpha_s}{4\pi}\left(\mathcal{B}_{qq}^{(1)},\phi_{qq}^{(1)}\right) = \int\frac{\mathrm{d}^d k}{(2\pi)^d}(2\pi)\delta^+(k^2)\delta(\bar{n}\cdot[k-(1-z)p])\left(e^{-ik_\perp\cdot x_\perp},1\right)\mathcal{M}$$

Integrals for ϕ and purely virtual contributions vanish (scaleless).
$\mathcal{B}$ contains integrals unregulated in dimensional regularization!

Rapidity divergences

- Arise in integrals along LC-direction: $\int d^d k = \frac{1}{2} \int dk_+ dk_- d^{d-2} k_\perp$.
- Recall: $e^{-ik_\perp \cdot x_\perp} \overset{\text{F.T.}}{\to} \delta^{(d-2)}(k_\perp + q_\perp) \;\Rightarrow$ kills $\int d^{d-2} k_\perp$ without putting ϵ to k_+ and k_- denominators.

$\Rightarrow$ Unregulated divergences (by ϵ). Originating from extreme rapidities $y(k) = \frac{1}{2} \log \frac{k_+}{k_-}$.

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$\Rightarrow$ Unregulated divergences (by ϵ). Originating from extreme rapidities $y(k) = \frac{1}{2} \log \frac{k_+}{k_-}$.

$\Rightarrow$ Need additional regulator. Various possibilities:

  - Avoid divergent $k_\pm$ denominators: Use W away from n, $\bar{n}$; Introduce 'LC-mass', ...
  - Regulate $k_\pm$ integrals: Analytic regulator in phase space, ...

- For each various ways.
- Combination $\mathcal{S}_i \mathcal{B}_{i/j} \bar{\mathcal{B}}_{\bar{i}/k}$ must not dependent on this regularization.
- Similarly for other observables in SCET-II.

Analytic regulator

- We use analytic regulator α [Becher, Bell]: $\times\left(\frac{\nu}{n \cdot l_i}\right)^{\alpha}$ for each external parton.
- Same LC vector n for all functions. Breaks symmetry $(n \leftrightarrow \bar{n})$.
- Simple soft function $\mathcal{S} = 1$.
- α poles cancel in product

Refactorization

$$\left[\mathcal{S}(x_T^2)\mathcal{B}_{i/j}(z_1, x_T^2)\bar{\mathcal{B}}_{\bar{i}/k}(z_2, x_T^2)\right]_{q^2} \overset{\alpha=0}{=} \left(\frac{x_T^2 q^2}{4e^{-2\gamma_e}}\right)^{-F_{i\bar{i}}^b(x_T^2)} B_{i/j}^b(z_1, x_T^2) B_{\bar{i}/k}^b(z_2, x_T^2)\,,$$

- On refactorized RHS no α and ν dependence left.
- Hard scale q^2 generated.
- 'Collinear anomaly' [Becher, Neubert].

Analytic regulator

- Phase space integral $\left(\hat{k}_{z,n} = \bar{n}\cdot[k-(1-z)p],\ \hat{k}_{z,\bar{n}} = n\cdot[k-(1-z)\bar{p}]\right)$:

$$\int d\Pi_{n_r}^{\mathrm{TD}} = \left[\prod_i \int \frac{d^d l_i}{(2\pi)^{d-1}}\,\delta^+(l_i^2)\left(\frac{\nu}{n\cdot l_i}\right)^{\alpha}\right]\int d^d k\,\delta^d\Big(k-\sum_i l_i\Big)\,e^{-ik_\perp\cdot x_\perp}\,\delta(\hat{k}_z)\,.$$

- Generates $\left(\frac{\nu}{n\cdot\bar{p}}\right)^{n_r\alpha} f_{\bar{n}}^{(n_r)}(\alpha,\ldots)$ for $\bar{n}$-col. function; $\left(\frac{\nu x_T^2\,\bar{n}\cdot p}{4e^{-2\gamma_e}}\right)^{n_r\alpha} f_n^{(n_r)}(\alpha,\ldots)$ for $\bar{n}$-col. function.
- Combine and expand in $\alpha \Rightarrow \alpha$ poles cancel; α^0 term contains difference of corresponding logarithms: ν cancels, $\log\frac{x_T^2 q^2}{4e^{-2\gamma_e}}$ remains.
- Identify F, B, B.

Contributions at NNLO

$$\mathcal{B}_{q/q}(z,x_T^2)=\frac{1}{2\pi}\int dt\ e^{-izt\bar{n}\cdot p}\sum_X \frac{\bar{\not{n}}_{\alpha\beta}}{2}\ \langle q|\,(\bar{\xi}_n W_n)_\alpha(t\bar{n}+x_\perp)\,|X\rangle\,\langle X|\,(W_n^\dagger \xi_n)_\beta(0)\,|q\rangle$$

Virtual-Real:

Real-Real:

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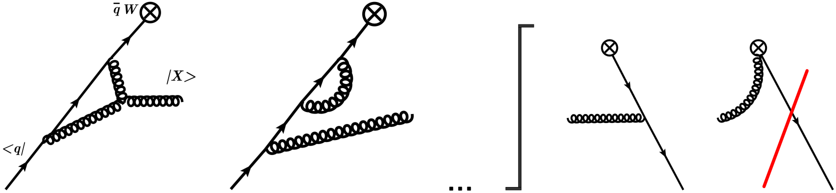

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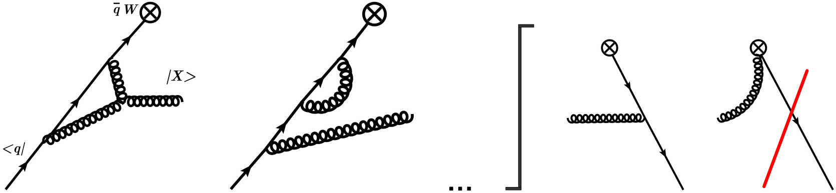

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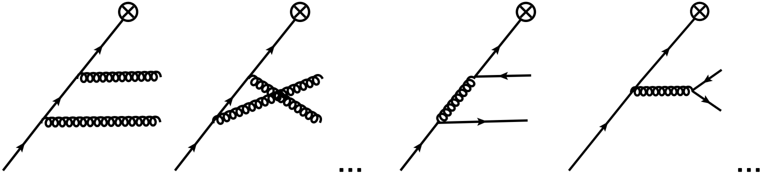

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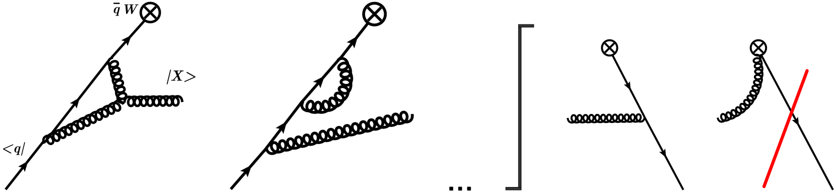

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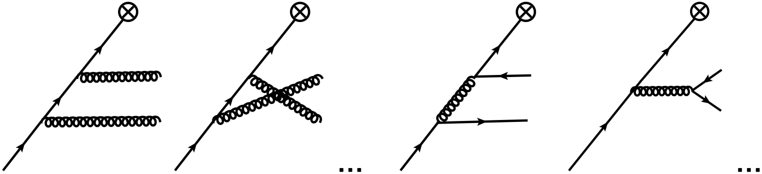

- α not enter loop integral $\Rightarrow$ VR integrals standard techniques.
- Main difficulty RR integrals.

RR integration

- Phase space integral ($\hat{k}_z = \bar{n}\cdot[k-(1-z)p]$):

$$\int d\Pi_{n_r}^{\mathrm{TD}} = \left[\prod_i \int \frac{d^d l_i}{(2\pi)^{d-1}}\,\delta^+(l_i^2)\left(\frac{\nu}{n\cdot l_i}\right)^{\alpha}\right]\int d^d k\,\delta^d\Big(k-\sum_i l_i\Big)e^{-ik_\perp\cdot x_\perp}\delta(\hat{k}_z)\,.$$

- LC and full denominators build from p, k, $\{l_i\}$.
- Unusual integral. Many standard reduction and integration techniques do not work well.

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- LC and full denominators build from p, k, $\{l_i\}$.
- Unusual integral. Many standard reduction and integration techniques do not work well.
- Partly performing & reformulating $\Rightarrow$ typical class:

$$\sim\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}t\, y^{1-a_2-a_3-a_5}(1-y)^{1-a_1-a_2-a_6-2\epsilon}\left[1-(1-z)(1-y)\right]^{-a_7}$$
$$\times(1-t)^{1-2a_2-2\epsilon}{}_2F_1(1-a_2-\epsilon,1-a_2-2\epsilon;1-\epsilon;t)\left[t^{-a_3-\epsilon}\{1-y(1-t)\}^b\right.$$
$$\left.+t^{-a_1-\epsilon}\{1-(1-y)(1-t)\}^b\right]\ ,\ b=-2+a_1+a_2+a_3+2\epsilon\,.$$

exponent	a_1	a_3	a_5
integer part	n_1	n_3	n_5
regulator part	α	α	(-2α)

[Gehrmann,TL,Yang]

Refactorize and renormalize

- After much work: Obtain $\mathcal{B}$, $\bar{\mathcal{B}}$, ϕ up to α_s^2 for all cases: $i/j \in \{g/g,\ g/q,\ q/g,\ q/q,\ \bar{q}/q,\ q'/q,\ \bar{q}'/q\}$.
- Extract F and B

$$\left[\mathcal{B}_{i/j}(z_1, x_T^2)\bar{\mathcal{B}}_{\bar{\imath}/k}(z_2, x_T^2)\right] \stackrel{\alpha=0}{=} \left(\frac{x_T^2 q^2}{4e^{-2\gamma_e}}\right)^{-F^b_{i\bar{\imath}}(x_T^2)} B^b_{i/j}(z_1, x_T^2) B^b_{\bar{\imath}/k}(z_2, x_T^2)\,.$$

- UV poles removed by renormalization ($\overline{\text{MS}}$)

$$F^{\text{b}}_{i\bar{\imath}}(x_T^2) = F_{i\bar{\imath}}(x_T^2, \mu) + Z_i^F(\mu)\,,$$
$$B^{\text{b}}_{i/j}(z, x_T^2) = Z_i^B(x_T^2, \mu)\, B_{i/j}(z, x_T^2, \mu)\,,$$
$$\phi^{\text{b}}_{i/j}(z) = \sum_k Z^{\phi}_{i/k}(z, \mu) \otimes \phi_{k/j}(z, \mu),,$$

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$$\phi^{\mathrm{b}}_{i/j}(z) = \sum_k Z^\phi_{i/k}(z, \mu) \otimes \phi_{k/j}(z, \mu),,$$

- Note $|Z^C|^2 Z_i^B Z_{\bar{\imath}}^B e^{-L_Q Z_i^F} = 1$.
- Eqs. for $I_{i/k}$ in $B_{i/j}(z) = \sum_k I_{i/k}(z, x_T^2) \otimes \phi_{k/j}(z)$ implied.
- Extract Z factors and renormalized functions.

Example result

Scale independent part of $I^{(2)}_{g/g}$ at $\mu_x = \frac{2e^{-\gamma_e}}{x_T}$. Expressed via HPLs $H_{\{m\}} \equiv H(\{m\}, z)$ and $\tilde{p}_{gg}(z) = \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)$.

$$
\begin{aligned}
I^{(2)}_{g/g}(z, x_T^2, \mu_x) = {} & C_A^2 \Big\{ \delta(1-z)\Big[\frac{25}{4}\zeta_4 - \frac{77}{9}\zeta_3 - \frac{67}{6}\zeta_2 + \frac{1214}{81}\Big] + \tilde{p}_{gg}(z)\Big[-4H_{0,0,0} + 8H_{0,1,0} + 8H_{0,1,1} \\
& - 8H_{1,0,0} + 8H_{1,0,1} + 8H_{1,1,0} + 52\zeta_3 - \frac{808}{27}\Big] + \tilde{p}_{gg}(-z)\Big[-16H_{-1,-1,0} + 8H_{-1,0,0} + 16H_{0,-1,0} \\
& - 4H_{0,0,0} - 8H_{0,1,0} - 8H_{-1}\zeta_2 + 4\zeta_3\Big] + \Big[-16(1+z)H_{0,0,0} + \frac{8(1-z)(11-z+11z^2)}{3z}(H_{1,0} + \zeta_2) \\
& + \frac{2(25-11z+44z^2)}{3}H_{0,0} - \frac{2z}{3}H_1 - \frac{(701+149z+536z^2)}{9}H_0 + \frac{4(-196+174z-186z^2+211z^3)}{9z}\Big]\Big\} \\
& + C_A T_F N_f \Big\{\delta(1-z)\Big[\frac{28}{9}\zeta_3 + \frac{10}{3}\zeta_2 - \frac{328}{81}\Big] + \frac{224}{27}\tilde{p}_{gg}(z) + \Big[\frac{8(1+z)}{3}H_{0,0} + \frac{4z}{3}H_1 + \frac{4(13+10z)}{9}H_0 \\
& - \frac{4(-65+54z-54z^2+83z^3)}{27z}\Big]\Big\} \\
& + C_F T_F N_f \Big\{8(1+z)H_{0,0,0} + 4(3+z)H_{0,0} + 24(1+z)H_0 - \frac{8(1-z)(1-23z+z^2)}{3z}\Big\} .
\end{aligned}
$$

Results of all $I_{i/j}$ and $F_{i\bar{i}}$ up to NNLO ⇒ [Gehrmann,TL,Yang]

With 1403.6451, full results in electronic form.

Checks and resummation

RGEs

- Renormalization $\Rightarrow$ μ dependence, described by RGEs:

$$\frac{d}{d\log\mu} C_i(-q^2,\mu) = \left[\Gamma^i(\alpha_s)\log\frac{-q^2}{\mu^2} + 2\gamma^i(\alpha_s)\right] C_i(-q^2,\mu)\,,$$

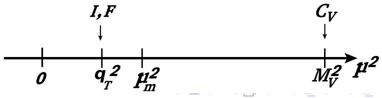
$$\frac{d}{d\log\mu} F_{i\bar{i}}(x_T^2,\mu) = \;2\,\Gamma^i(\alpha_s)\,,$$

$$\frac{d}{d\log\mu} B_{i/j}(z,x_T^2,\mu) = \left[\Gamma^i(\alpha_s)\log\frac{x_T^2\mu^2}{4e^{-2\gamma_e}} - 2\gamma^i(\alpha_s)\right] B_{i/j}(z,x_T^2,\mu)\,,$$

$$\frac{d}{d\log\mu}\phi_{i/j}(z,\mu) = \;2\sum_k P_{ik}(z,\mu)\otimes\phi_{k/j}(z,\mu)\,.$$

§ Each function depends only on M_{typical} and μ.

$\rightarrow$ Consistently determine each to fixed order at $\mu \sim M_{\text{typical}}$.



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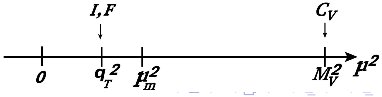
$$\frac{d}{d\log\mu} \phi_{i/j}(z,\mu) = \;2\sum_k P_{ik}(z,\mu)\otimes\phi_{k/j}(z,\mu)\,.$$

§ Each function depends only on M_{typical} and μ.

$\rightarrow$ Consistently determine each to fixed order at $\mu \sim M_{\text{typical}}$.

$\rightarrow$ Subsequently evolve to common scale μ_m with RGEs
e.g. $C_i(-q^2,\mu_m) = C_i(-q^2,\mu_h)\exp\left[\int_{\mu_h}^{\mu_m}\frac{d\mu}{\mu}\left(\Gamma^i\log\frac{-q^2}{\mu^2} + 2\gamma^i\right)\right]$.

$\Rightarrow$ Resums logarithms.



RGEs continued

Using RGEs:

✓ Check result functions obey corresponding RGE.

- Express all $L = \log \frac{M^2_{\text{typical}}}{\mu^2}$ dependent terms by anomalous dimensions and lower order L independent terms,
  e.g.: $F_i^{(n,l+1)} = \frac{1}{l+1}\left[\delta_{l,0}\Gamma^i_{n-1} + n\epsilon F_i^{(n,l)} + \sum_{s=0}^{n-1} s\,\beta_{n-s-1} F_i^{(s,l)}\right].$
- From their corresponding RGEs can obtain Z factors,
  e.g.: $Z_{F_i}^{(n,0)} = \frac{1}{n\epsilon}\left[\Gamma^i_{n-1} - \sum_{s=0}^{n-1} s\,\beta_{n-s-1} Z_{F_i}^{(s,0)}\right].$

✓ All as checks for results.

$\Rightarrow$ RGEs very useful tool.

Rephrased factorization formula

- At $\Lambda \ll q_T \ll M$:

$$\frac{d^3\sigma}{dq_T^2\,dy\,dq^2} = \frac{\alpha_{ew}^2}{3N_c s q^2}\sum_{i,j}\sum_q e_q^2 \int d^2x_\perp\, e^{-iq_\perp\cdot x_\perp} \times \left[\widetilde{C}_{q\bar{q}\leftarrow ij}(z_1,z_2,x_T^2,q^2,\mu) + (q\leftrightarrow\bar{q})\right] \otimes \phi_{i/N_1}(z_1,\mu) \otimes \phi_{j/N_2}(z_2,\mu)\,,$$

with perturbative

$$\widetilde{C}_{q\bar{q}\leftarrow ij}(z_1,z_2,x_T^2,q^2,\mu) = \left|C_V(-q^2,\mu)\right|^2 \left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F_{q\bar{q}}(x_T^2,\mu)} I_{q\leftarrow i}(z_1,x_T^2,\mu) I_{\bar{q}\leftarrow j}(z_2,x_T^2,\mu)\,.$$

Rephrased factorization formula

- At $\Lambda \ll q_T \ll M$:

$$\frac{d^3\sigma}{dq_T^2\,dy\,dq^2} = \frac{\alpha_{ew}^2}{3N_c s q^2}\sum_{i,j}\sum_q e_q^2 \int d^2x_\perp\, e^{-iq_\perp\cdot x_\perp}$$
$$\times\left[\widetilde{C}_{q\bar{q}\leftarrow ij}(z_1,z_2,x_T^2,q^2,\mu) + (q\leftrightarrow\bar{q})\right] \otimes \phi_{i/N_1}(z_1,\mu) \otimes \phi_{j/N_2}(z_2,\mu)\,,$$

with perturbative

$$\widetilde{C}_{q\bar{q}\leftarrow ij}(z_1,z_2,x_T^2,q^2,\mu) = \left|C_V(-q^2,\mu)\right|^2 \left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F_{q\bar{q}}(x_T^2,\mu)} I_{q\leftarrow i}(z_1,x_T^2,\mu)\, I_{\bar{q}\leftarrow j}(z_2,x_T^2,\mu)\,.$$

- Determine each factor at appropriate scale μ.
- Solving RGEs $\Rightarrow$ resummation of $L = \log(x_T^2 q^2)$.

$$\widetilde{C}_{q\bar{q}\leftarrow ij} = \tilde{c}(\alpha_s)\exp\left[L g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \alpha_s^2 g_4(\alpha_s L) + \dots\right]$$

- N^3LL requires determination of $g_1, \dots, g_4$.

Towards N^{3}LL, required elements

Numbers refer to power n in expansion $X = \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n X^{(n)}$.

expression	needed to	known to	
Γ^i	4	3	for RGEs
γ_i	3	3	
$P_{i/j}(z)$	3	3	
β	4	4	
$C(q^2)$	2	2 ± 1	at appr. μ
$F_{i\bar{i}}(x_T^2)$	3	2	
$I_{i/j}(z, x_T^2)$	2	2	

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$F_{i\bar{i}}(x_T^2)$	3	2	
$I_{i/j}(z, x_T^2)$	2	2	
$I'_{g/j}(z, x_T^2)$	1 (2)*	1 (+)	

*: $B'_{g/j} = \sum_k I'_{g/k} \otimes \phi_{k/j}$. I' starts at α_s^1.
$\Rightarrow$ $n = 1$ sufficient if C does not mix I' & I. (E.g. for Higgs.)

Relation to formula by Collins, Soper, Sterman

$$\widetilde{C}_{q\bar{q}\leftarrow ij}(z_1,z_2,x_T^2,q^2,\mu)$$

$$= |C_V(-q^2,\mu)|^2 \left(\frac{x_T^2 q^2}{b_0^2}\right)^{-F_{q\bar{q}}(x_T^2,\mu)} I_{q\leftarrow i}(z_1,x_T^2,\mu) I_{\bar{q}\leftarrow j}(z_2,x_T^2,\mu) \,,$$

compare this to [Collins, Soper, Sterman, 1985]:

$$= \exp\left\{ -\int_{\mu_b}^{q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\log \frac{q^2}{\bar{\mu}^2} A(\alpha_s(\bar{\mu})) + B(\alpha_s(\bar{\mu}))\right]\right\}$$

$$\times\, C_{qi}(z_1, \alpha_s(\mu_b)) C_{\bar{q}j}(z_2, \alpha_s(\mu_b)) \,,$$

x_T dependence via $\mu_b = b_0 x_T^{-1}$ $(b_0 = 2e^{-\gamma_e})$.

Relations:

- $C_{qi}(z, \alpha_s(\mu_b)) = \left| C_V(-\mu_b^2, \mu_b) \right| I_{q/i}(z, x_T^2, \mu_b) \,,$
- A & B related to F and RGE of F & C_V.

Dictionary CSS vs BN

Using $b_0 = 2e^{-\gamma_e}$, $\mu_b = b_0 x_T^{-1}$ and $\bar{x}_T = b_0 \bar{\mu}^{-1}$:

$$C_{qi}(z, \alpha_{s(\mu_b)}) = \left| C_V(-\mu_b^2, \mu_b) \right| I_{q/i}(z, \bar{x}_T^2, \mu_b)\,,$$

$$A(\alpha_{s(\bar{\mu})}) = \Gamma^q_{\rm cusp}(\alpha_s) - \bar{\mu}^2 \frac{dF_{q\bar{q}}(\bar{x}_T, \bar{\mu})}{d\bar{\mu}^2}\,,$$

$$B(\alpha_{s(\bar{\mu})}) = 2\gamma_q(\alpha_s) + F_{q\bar{q}}(\bar{x}_T, \bar{\mu}) - \bar{\mu}^2 \frac{d \log |C_V(-\bar{\mu}^2, \bar{\mu})|^2}{d\bar{\mu}^2}\,.$$

$\Rightarrow$ Apply results in preferred resummation framework.

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$\Rightarrow$ Apply results in preferred resummation framework.

Moreover, can reconstruct $\mathcal{H}^{DY(2)}_{q\bar{q}\leftarrow i\bar{j}}$ and $\mathcal{H}^{H(2)}_{gg\leftarrow i\bar{j}}$ in [Catani, Cieri, de Florian, Ferrera, Grazzini] (✓).
$\Rightarrow$ Apply for q_T subtraction.

Conclusions

- Reviewed basic ideas of factorization and SCET.
- q_T resummation requires TPDFs.
- Calculate matching kernels perturbatively using
  - gauge invariant operator definitions,
  - analytic regulator (rapidity singularities).
- NNLO results for all $I_{i/j}$.
- Generic ingredients to obtain N^{2}LO+N^{3}LL precision of $d\sigma/dq_T$ for large class of processes at hadron colliders.

Thank you.

Appendix

Overview: Perturbative determination

- Aim: NNLO+N^{3}LL for $d\sigma/dq_T$.
- Calculate $\mathcal{B}_{i/j}$ and $\phi_{i/j}$ to NNLO regulated by α & ϵ.
- $\mathcal{B}_{i/j}\bar{\mathcal{B}}_{\bar{\imath}/k} = (\ldots q^2)^{-F_{i\bar{\imath}}} B_{i/j} B_{\bar{\imath}/k}$. $\alpha \to 0$.
- Obtain $I_{i/j}$ from $B_{i/k} = \sum_j I_{i/j} \otimes \phi_{j/k}$.
- Renormalize I and F. $\epsilon \to 0$.
- With $I^{(2)}$ obtained generic ingredients for N^{3}LL resummation.

NLO results

$$\phi^{(1)}(z,\mu) = 0 \text{ in dim. reg.} \qquad L_\perp = \log\left(\frac{x_T^2\mu^2}{4e^{-2\gamma_E}}\right)$$

$$\mathcal{B}^{(1)}_{i/j}(z,\perp,\mu) = e^{(\epsilon+\alpha)L_\perp-(\epsilon+2\alpha)\gamma_E}\frac{\Gamma(-\epsilon-\alpha)}{\Gamma(1+\alpha)}\left(\frac{\nu x_T^2\,\bar{n}\cdot p}{4e^{-2\gamma_e}}\right)^{\alpha}(1-z)^{\alpha} f_{i/j}(z,\epsilon)\ ,$$

$$\bar{\mathcal{B}}^{(1)}_{\bar{i}/\bar{j}}(z,\perp,\mu) = e^{\epsilon L_\perp-\epsilon\gamma_E}\Gamma(-\epsilon)\left(\frac{\nu}{n\cdot\bar{p}}\right)^{\alpha}(1-z)^{-\alpha} f_{i/j}(z,\epsilon)\ ,$$

$$f_{q/q}(z,\epsilon) = C_F(1-z)^{-1}\left[4z+2(1-\epsilon)(1-z)^2\right],$$

$$f_{q/g}(z,\epsilon) = 2T_F\left[1-\frac{2}{1-\epsilon}z(1-z)\right],$$

$$f_{g/q}(z,\epsilon) = C_F\left[2\frac{1+(1-z)^2}{z}-2\epsilon z\right],$$

$$f_{g/g}(z,\epsilon) = C_A(1-z)^{-1}\left[4\frac{\left(1-z+z^2\right)^2}{z}\right],$$

Another example: Scale independent part of $I^{(2)}_{q/q}$

$$
\begin{aligned}
&I^{(2)}_{q/q}(z,0) = \\
&\delta(1-z)\left[C_F^2\frac{5\zeta_4}{4} + C_F C_A\left(\frac{3032}{81} - \frac{67\zeta_2}{6} - \frac{266\zeta_3}{9} + 5\zeta_4\right) + C_F T_F n_f\left(-\frac{832}{81} + \frac{10\zeta_2}{3} + \frac{28\zeta_3}{9}\right)\right] \\
&+ P^{(0)}_{qq}(z)\left[C_F\left(12\zeta_3 + 4H_0 + \frac{3}{2}H_{0,0} + 4H_{0,1,0} + 2H_{0,1,1} - 2H_{1,0,0} + 4H_{1,0,1} + 4H_{1,1,0}\right)\right. \\
&+ C_A\left(\zeta_3 - \frac{202}{27} - \frac{38}{9}H_0 - \frac{11}{6}H_{0,0} - H_{0,0,0} - 2H_{0,1,0} - 2H_{1,0,1} - 2H_{1,1,0}\right) \\
&\left. + T_F n_f\left(\frac{56}{27} + \frac{10}{9}H_0 + \frac{2}{3}H_{0,0}\right)\right] + P^{(0)}_{qg}(z)C_F\left[-\frac{68}{27} + \frac{4\zeta_2}{3} + \frac{32}{9}H_0 - \frac{4}{3}H_{0,0} + \frac{4}{3}H_{1,0}\right] \\
&+ P^{(0)}_{gq}(z)T_F\left[\frac{86}{27} - \frac{4\zeta_2}{3} - \frac{4}{3}H_{1,0}\right] + C_F^2\Big[(2-24z)H_0 + (3+7z)H_{0,0} + 2(1+z)H_{0,0,0} \\
&+ 2zH_1 + (1-z)\left(6\zeta_2 - 22 + 4H_{0,1} + 12H_{1,0}\right)\Big] + C_F C_A\Big[(2+10z)H_0 - 4zH_{0,0} - 2zH_1 \\
&+ (1-z)\left(\frac{44}{3} - 6\zeta_2 - 4H_{1,0}\right)\Big] + C_F T_F\Big[\frac{-50+38z}{9} + \frac{20+8z}{9}H_0 + \frac{2-22z}{3}H_{0,0} \\
&+ 4(1+z)H_{0,0,0}\Big] - \frac{4}{3}C_F T_F n_f(1-z)\,,
\end{aligned}
$$

where $P^{(0)}_{qq}(z) = 2C_F\left[(1+z^2)/(1-z)\right]_+$, $P^{(0)}_{qg}(z) = 2T_F\left[z^2 + (1-z)^2\right]$, $P^{(0)}_{gq}(z) = 2C_F\left[1+(1-z)^2\right]/z$,

and $H_{\{m\}} \equiv H(\{m\}, z)$.

Reconstruct scale logarithms

For

$$F_{i\bar{i}}(L_\perp, \alpha_s) = \sum_{n\geq 1} \sum_{l\geq 0} F_{i\bar{i}}^{(n,l)} \left(\frac{\alpha_s}{4\pi}\right)^n L_\perp^l \,,$$

$$I_{i/j}(z, L_\perp, \alpha_s) = \sum_{n\geq 0} \sum_{l\geq 0} I_{i/j}^{(n,l)}(z) \, \left(\frac{\alpha_s}{4\pi}\right)^n L_\perp^l \,,$$

solving RGEs imply recursion relations

$$F_i^{(n,l+1)} = \frac{1}{l+1} \left[\delta_{l,0} \Gamma^i_{n-1} + n\epsilon F_i^{(n,l)} + \sum_{s=0}^{n-1} s\, \beta_{n-s-1} F_i^{(s,l)}\right],$$

$$I_{i/j}^{(n,l+1)} = \frac{1}{l+1} \sum_{s=0}^{n-1} \left[\frac{1}{2} \Gamma^i_{n-s-1} I_{i/j}^{(s,l-1)} + \left(s\, \beta_{n-s-1} - \gamma^i_{n-s-1}\right) I_{i/j}^{(s,l)} - \sum_k I_{i/k}^{(s,l)} \otimes P_{k/j}^{(n-s-1)}\right] + \frac{\epsilon n}{l+1} I_{i/j}^{(n,l)} \,.$$