LOOPS AND LEGS 2016

LONG-DISTANCE SINGULARITIES IN MULTI-LEG SCATTERING AMPLITUDES

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FINAL RESULTS FOR THE 3-LOOP SOFT ANOMALOUS DIMENSION - WORK WITH ØYVIND ALMELID AND CLAUDE DUHR

28 APRIL, 2016

LONG-DISTANCE SINGULARITIES IN MULTI-LEG SCATTERING AMPLITUDES

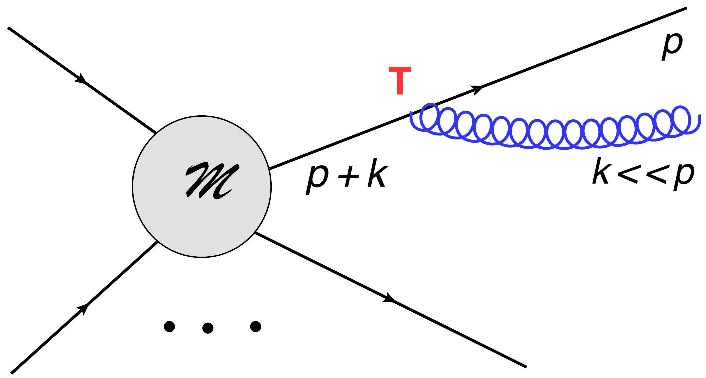
Plan of the talk

- Soft singularities from Wilson lines: fixed-angle factorization and rescaling symmetry.
- The soft anomalous dimension for massless partons: the dipole formula.
- Calculation of connected webs in near light-like kinematics.
- The complete 3-loop soft anomalous dimension.
- Special kinematics: collinear limit, Regge limit.

The soft (eikonal) approximation and rescaling symmetry

Eikonal Feynman rules:

Assuming $k \ll p$ such that all components of k are small:



$$\bar{u}(p)\left(-\mathrm{i}g_s T^{(a)}\gamma^\mu\right)\frac{\mathrm{i}(\not{p}+\not{k}+m)}{(p+k)^2-m^2+\mathrm{i}\varepsilon} \quad \longrightarrow \quad \bar{u}(p) g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon}$$

Rescaling invariance: only the direction and the colour charge of the emitter matter.

$$g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon} = g_s T^{(a)}\frac{\beta^\mu}{\beta\cdot k+\mathrm{i}\varepsilon}$$

equivalent to emission from a Wilson line:

$$\Phi_{\beta_i}(\infty,0) \equiv P\exp\left\{\mathrm{i}g_s\int_0^\infty d\lambda\beta\cdot A(\lambda\beta)\right\}$$

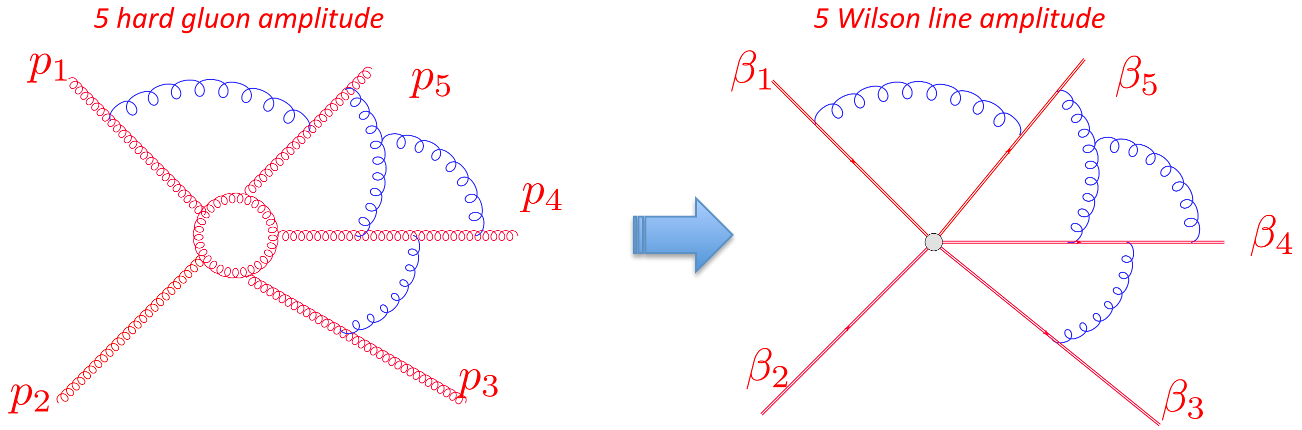
This symmetry is realised differently for lightlike and massive Wilson lines.

IR singularities from Wilson lines

Factorization at fixed angles:

all kinematic invariants are simultaneously taken large $p_i \cdot p_j = Q^2 \beta_i \cdot \beta_j \gg \Lambda^2$

Soft singularities factorise to all orders & computed from a product of Wilson lines:



$$\mathcal{M}_J(p_i, \epsilon_{\rm IR}) = \sum_K \mathcal{S}_{JK}\left(\gamma_{ij}, \epsilon_{\rm IR}\right) H_K(p_i)$$

$\mathcal{S}$ is a product of Wilson lines: $\mathcal{S} = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$

The *soft anomalous dimension* Γ is the logarithmic derivative of $\mathcal{S}$

Due to rescaling symmetry it only depends on angles: $\gamma_{ij} = \dfrac{2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \, \beta_j^2}}$

IR SINGULARITIES FOR AMPLITUDES WITH MASSLESS LEGS

Solving a renormaliaztion-group equation → **Exponentiation:**

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s,\epsilon\right) = \exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma\left(\lambda^2/s_{ij},\alpha_s(\lambda^2,\epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu},\alpha_s\right)$$

The Dipole Formula:

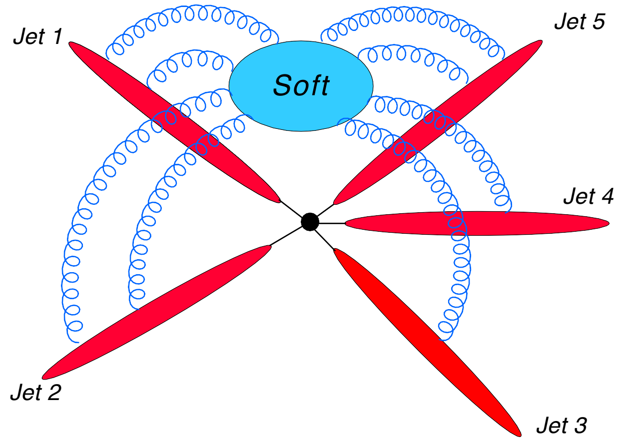
simple ansatz for the singularity structure of multi-leg massless amplitudes

$$\Gamma_{\rm Dip.}(\lambda,\alpha_s) = \frac{1}{4}\widehat{\gamma}_K(\alpha_s)\sum_{(i,j)}\ln\left(\frac{\lambda^2}{-s_{ij}}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}(\alpha_s)$$

Becher & Neubert,
EG & Magnea (2009)

Complete two-loop calculation by
Dixon, Mert-Aybat and Sterman in 2006
(confirming Catani's predictions from 1998).

Generalization to all orders motivated
by constraints based on **soft/jet factorisation**
and **rescaling symmetry.**



Corrections to the Dipole Formula

First possible corrections to the Dipole Formula:
Functions of **conformally-invariant cross ratios** at 3-loops, 4 legs:

$$\Gamma = \Gamma_{\rm Dip.} + \Delta(\rho_{ijkl})$$

$$\rho_{ijkl} = \frac{(p_i \cdot p_j)(p_k \cdot p_l)}{(p_i \cdot p_k)(p_j \cdot p_l)}$$

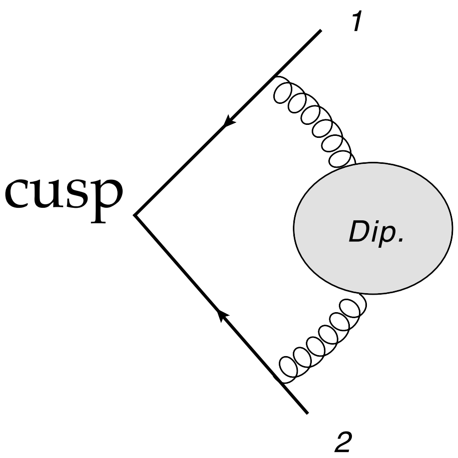
Other constraints on $\Delta(\rho_{ijkl})$:

Non-Abelian exponentiation theorem [EG, Smillie, White (2013)] **implies that** $\Delta(\rho_{ijkl})$ **has *fully connected* colour factors, such as** $f^{abe} f^{cde}\, \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d$

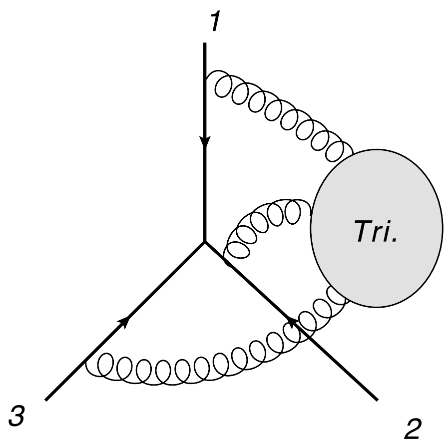
Bose symmetry
Transcendental weight
Collinear limits
Regge limit

EG & Magnea, Becher & Neubert (2009)
Dixon, EG & Magnea (2010)
Del Duca, Duhr, EG, Magnea & White (2011)
Ahrens & Neubert & Vernazza (2012)
Caron-Huot (2013)

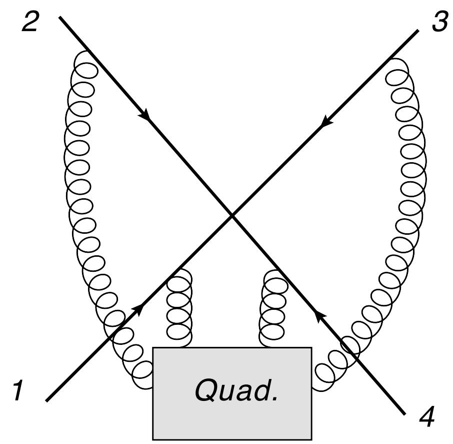
The structure of the soft anomalous dimension: massless vs. massive partons



$\mathbf{T}_i \cdot \mathbf{T}_j$ / planar



$f^{abc}\,\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c$



$f^{abe}f^{cde}\,\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d$

massless	known @ 3-loop	forbidden to all loops	**3-loop done!** **
massive	known @ 3-loop*	known @ 2-loops progress @ 3-loop	starts @ 3-loop - in progress

* Grozin, Henn, Korchemsky & Marquard, Phys. Rev. Lett. 114, 062006 (2015)

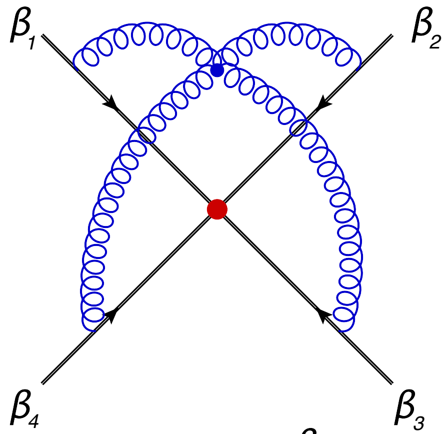
****Almelid, Duhr, EG - 1507.00047 v2 to appear**

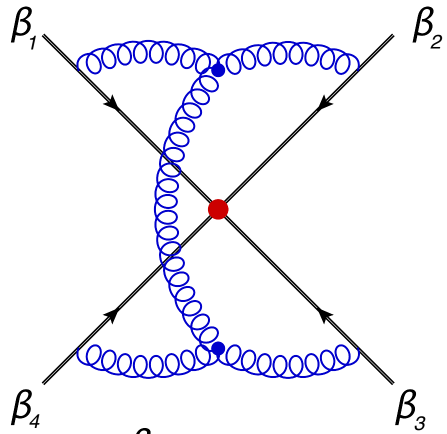
Computing IR singularities at 3-loops

Classes of three-loop webs connecting four Wilson lines

Single connected subgraph

Each web depends on all six angles - **can form conformally-invariant cross ratios** (cicrs)

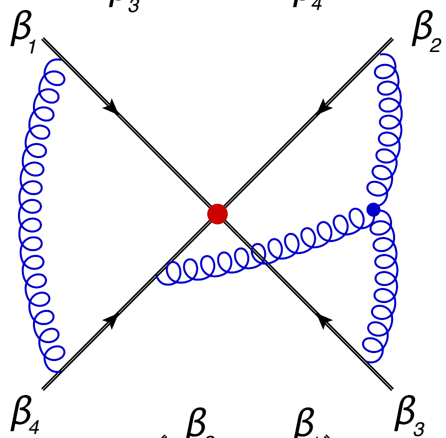




Two connected subgraphs

Depends on $\gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34}$ only.

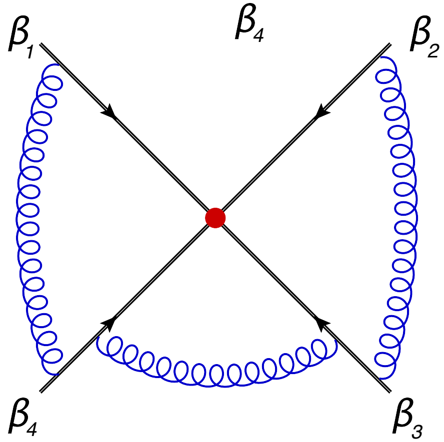
Cannot form cicrs - yields products of logs for near lightlike kinematics

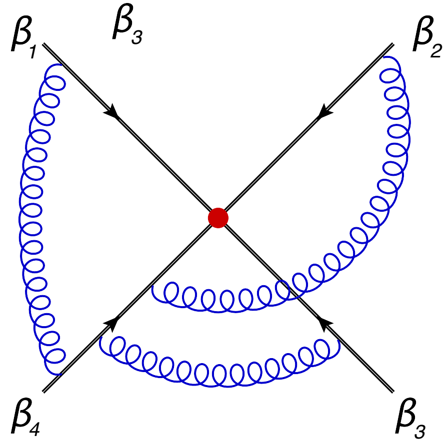


Three connected subgraphs (multiple gluon exchanges)

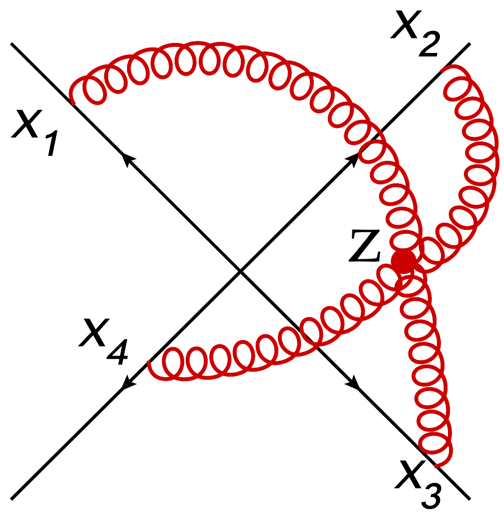
Depends on 3 angles only!

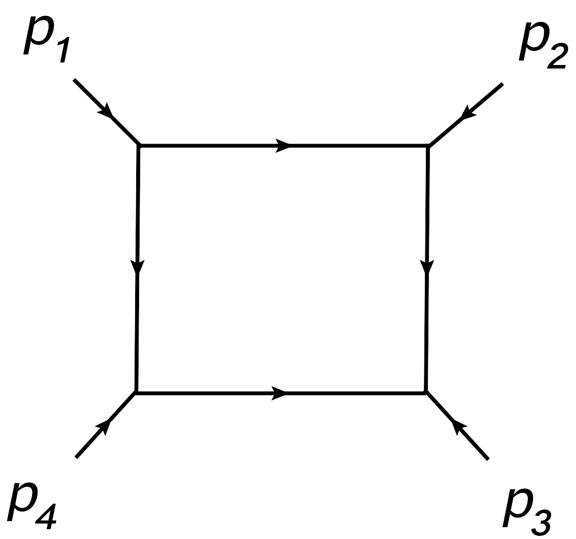
Cannot form cicrs - yields products of logs for near lightlike kinematics





DUAL MOMENTUM BOX INTEGRAL

$W_{4g} =$ 



Parametrise the positions along the Wilson lines by $x_i^\mu = \beta_i^\mu s_i$

Define auxiliary momenta $p_i = x_i - x_{i-1}$

The z integral is a 4-mass $\mathrm{Box}(p_1, p_2, p_3, p_4)$

$$C_{4g} = T_1^a T_2^b T_3^c T_4^d \Big[f^{abe} f^{cde} (\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23}) + f^{ade} f^{bce} (\gamma_{12}\gamma_{34} - \gamma_{13}\gamma_{24}) + f^{ace} f^{bde} (\gamma_{12}\gamma_{34} - \gamma_{14}\gamma_{23}) \Big]$$

$$W_{4g} = g_s^6 \mathcal{N}^4 C_{4g} \int_0^{\text{``}\infty\text{''}} ds_1 ds_2 ds_3 ds_4 \ \mathrm{Box}(x_1 - x_4, x_2 - x_1, x_3 - x_2, x_4 - x_3)$$

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \lambda \begin{pmatrix} ca \\ c(1-a) \\ (1-c)b \\ (1-c)(1-b) \end{pmatrix}$$

Integration over λ yields an overall $1/\epsilon$ UV pole.
Remaining integrations can be done in 4 dimensions.

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CONNECTED THREE-LOOP WEBS WITH TWO 3-GLUON VERTICES

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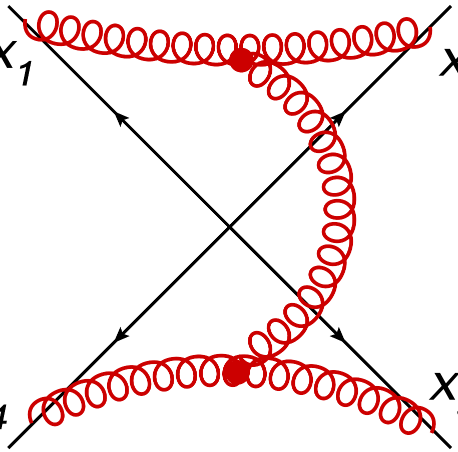
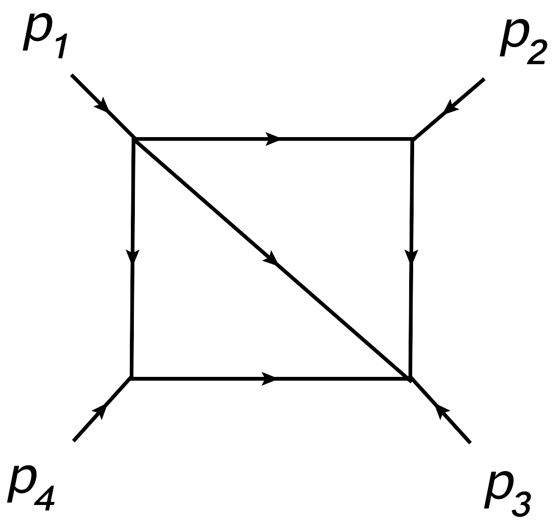
A similar mapping - but with a diagonal box

$W_{(3g)^2} =$ 



W_{4g} and $W_{(3g)^2}$ may have non-trivial kinematic dependence in the limit $\beta_i^2 \to 0$

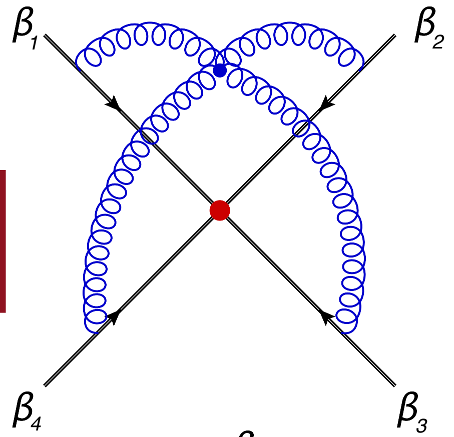
$$\rho_{ijkl} = \frac{\gamma_{ij}\,\gamma_{kl}}{\gamma_{ik}\,\gamma_{jl}} = \frac{(\beta_i \cdot \beta_j)\,(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$$

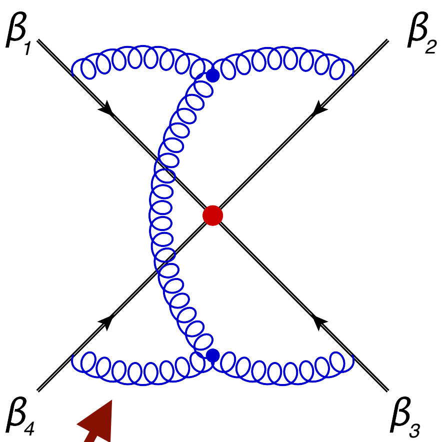
$$\rho_{1234} = z\bar{z}$$
$$\rho_{1432} = (1-z)(1-\bar{z})$$

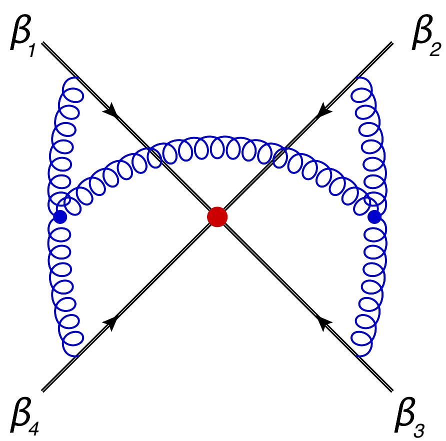
We extract the asymptotic near-lightlike behaviour using the Mellin-Barnes technique. The remaining MB integral is three-fold, and can be converted into an iterated parameter integral and be expressed in terms of polylogarithms.

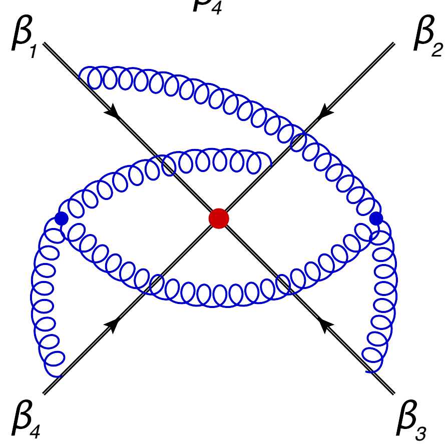
Summing the connected webs results

$$w_{4g}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \Big[f^{abe} f^{cde} \left(z\bar{z} - z - \bar{z}\right) + f^{ade} f^{bce} \left(1 - z\bar{z}\right) + f^{ace} f^{bde} \left(1 - z - \bar{z}\right) \Big] \boxed{\frac{1}{z - \bar{z}}\, g_1(z, \bar{z}, \{\gamma_{ij}\})}$$









$$w_{(12)(34)}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \; f^{abe} f^{cde} \Big[g_0(z, \bar{z}, \{\gamma_{ij}\}) - \boxed{\frac{z\bar{z} - z - \bar{z}}{z - \bar{z}}\, g_1(z, \bar{z}, \{\gamma_{ij}\})} \Big]$$

cancels in the sum!

Pure function of uniform weight 5 ($\mathcal{N}$=4 SYM property)

Symbol alphabet $\{z, \bar{z}, 1 - z, 1 - \bar{z}\}$ relating to collinear / Regge limits

From the connected webs to the full quadrupole term in the soft anon. dim.

After applying Jacobi Identity one finds

$$w_{\text{con.}}^{(3,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \Big[f^{ade} f^{bce}\ \mathcal{F}_1{}^{\text{con.}}(z,\bar{z},\{\gamma_{ij}\}) + f^{abe} f^{cde}\ \mathcal{F}_2{}^{\text{con.}}(z,\bar{z},\{\gamma_{ij}\}) \Big]$$

and the functions <u>separate</u> into a polylogarithmic function of depending only on conformally invariant cross ratios via $\{z,\bar{z}\}$, and a function involving purely logarithmic dependence on individual cusp angles:

$$\mathcal{F}_n{}^{\text{con.}}(z,\bar{z},\{\gamma_{ij}\}) = \mathcal{F}_n{}^{\text{con.}}(z,\bar{z}) + Q_n^{\text{con.}}(\{\log(\gamma_{ij})\})$$

<u>Rescaling symmetry</u> implies that the quadrupole contribution to the light-like soft anomalous dimension would depend <u>**exclusively**</u> on $\{z,\bar{z}\}$!

So far put aside non-connected webs, and webs connecting fewer than 4 lines. All these, in the light-like asymptotics, <u>**involve only logarithms**</u>, $\ln(\gamma_{ij})$.

Any kinematic dependence which isn't rescaling invariant must cancel out!

Colour conservation

Colour conservation for n Wilson lines: $\left(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3+\ldots\mathbf{T}_n\right)|\mathcal{H}\rangle=0$

Considering the diagrams that connect 4 lines

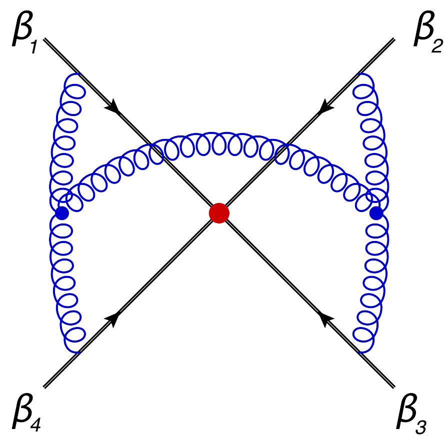
$$\Gamma_4(1,2,3,4)=\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\left(f^{abe}f^{cde}H_4[(1,2),(3,4)]+f^{ace}f^{bde}H_4[(1,3),(2,4)]+f^{ade}f^{bce}H_4[(1,4),(2,3)]\right)$$

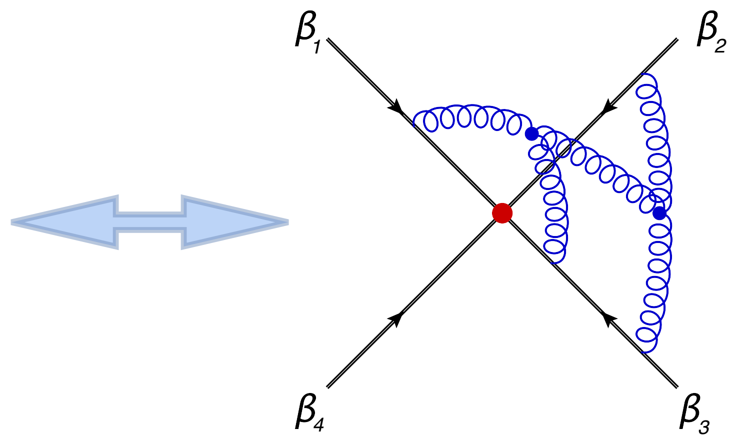
with permutation symmetry $H_4[(i,j),(k,l)]=-H_4[(j,i),(k,l)]=H_4[(k,l),(i,j)]$

Applying colour conservation to eliminate $\mathbf{T}_4$ — the 4-line result may be expressed as

$$\Gamma_4(1,2,3,4)=-\frac{1}{2}f^{abe}f^{cde}\sum_{\substack{(i,j,k)\in(1,2,3)\\ j<k}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\left(H_4[(i,j),(k,4)]+H_4[(i,k),(j,4)]\right)$$

Colour conservation relates 4- and 3-line colour factors:

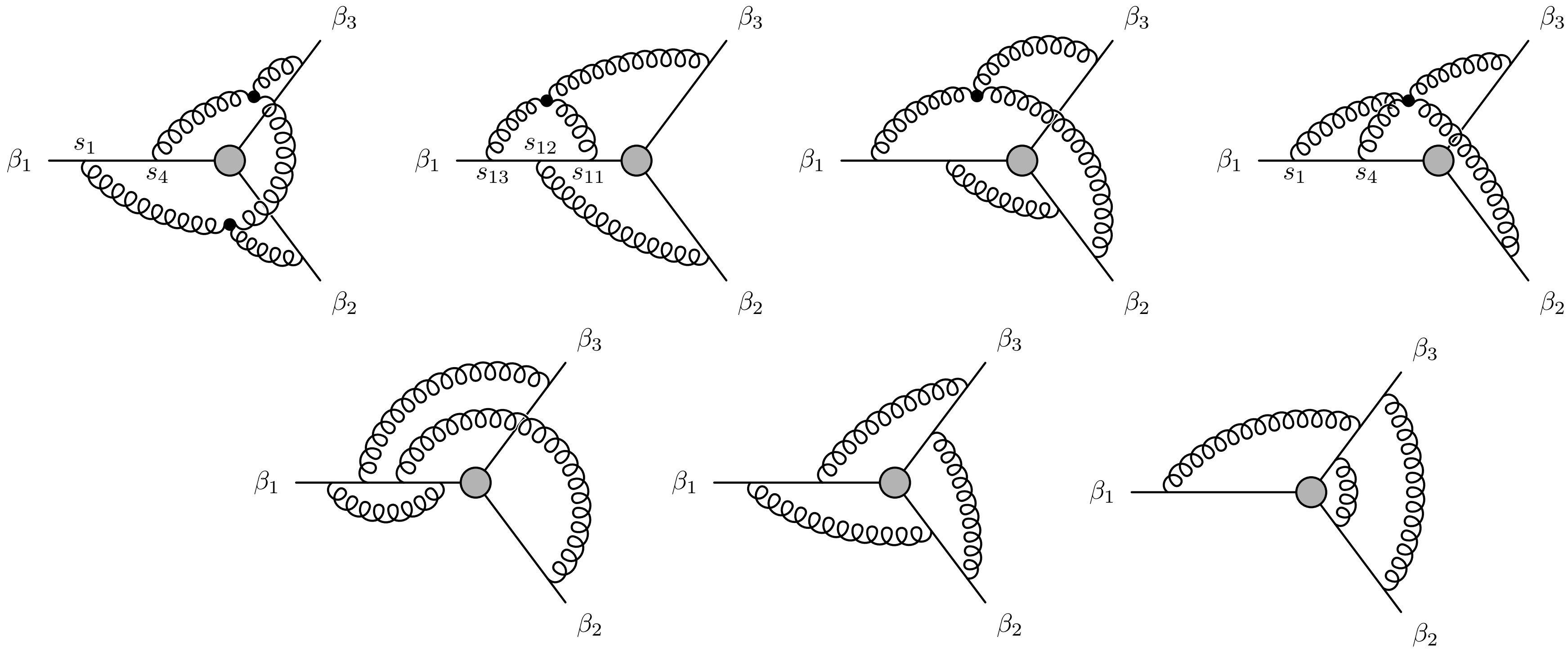




Diagrams connecting fewer Wilson lines are also relevant for Δ_n

Webs with three lines

So we also computed three-line diagrams:



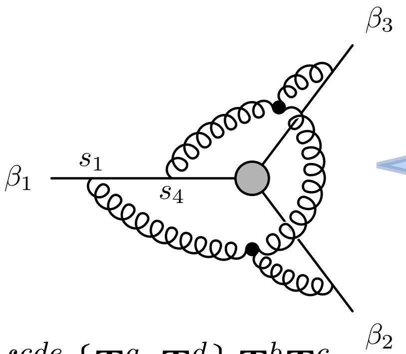
<u>Colour basis:</u>

$f^{abe} f^{cde} \left\{ \mathbf{T}_i^a , \mathbf{T}_i^d \right\} \mathbf{T}_j^b \mathbf{T}_k^c$ → contributes to Δ_n

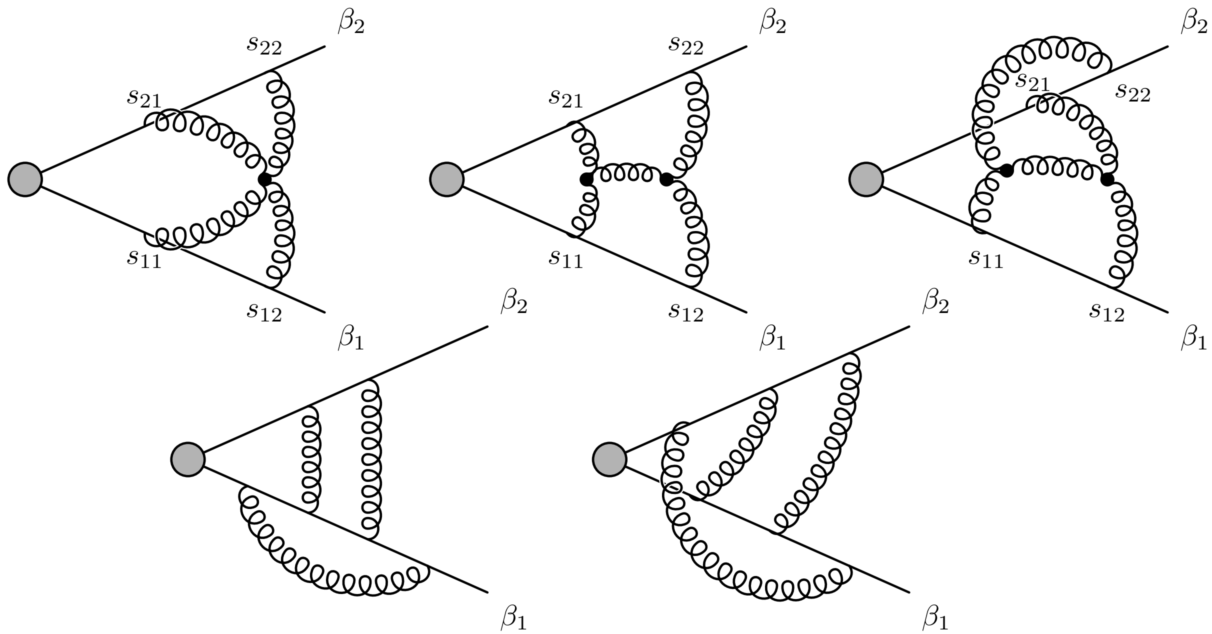
$N_c \, f^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c$ → does not contribute

Webs with two lines

Similarly colour conservation on 3 lines relates to 2 lines:



$$f^{abe} f^{cde} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \mathbf{T}_j^b \mathbf{T}_k^c$$



Colour basis: $f^{abe} f^{cde} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \left\{\mathbf{T}_j^c, \mathbf{T}_j^d\right\}$ → contributes to Δ_n

$\mathbf{T}_i \cdot \mathbf{T}_j$ → does not contribute

Surprise with three lines

Consider now the soft anomalous dimension for three coloured lines.
Given that **no conformal cross ratios can be formed**, the expectation was:
no corrections beyond the dipole formula, i.e. $\Delta_3 = 0$.

Summing all 2- and 3-line webs we get, instead, a constant:

$$\Delta_3 = -16 \left(\frac{\alpha_s}{4\pi}\right)^3 (\zeta_5 + 2\zeta_2\zeta_3)\, f^{abe} f^{cde} \sum_{\substack{(i,j,k)\in(1,2,3)\\ j<k}} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \mathbf{T}_j^b \mathbf{T}_k^c$$

THE COMPLETE 3-LOOP CORRECTION TO THE DIPOLE FORMULA

$$\Delta(z,\bar{z}) = 16\left(\frac{\alpha_s}{4\pi}\right)^3 f_{abe}f_{cde}\Bigg\{\sum_{1\le i<j<k<l\le n}\Big[\ \mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\left(F\left(1-1/z\right)-F\left(1/z\right)\right)$$

$$+\mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\left(F(1-z)-F(z)\right)$$

$$+\mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\left(F\left(1/(1-z)\right)-F\left(1-1/(1-z)\right)\right)\Big]$$

$$-\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\left(\zeta_5+2\zeta_2\zeta_3\right)\Bigg\}$$

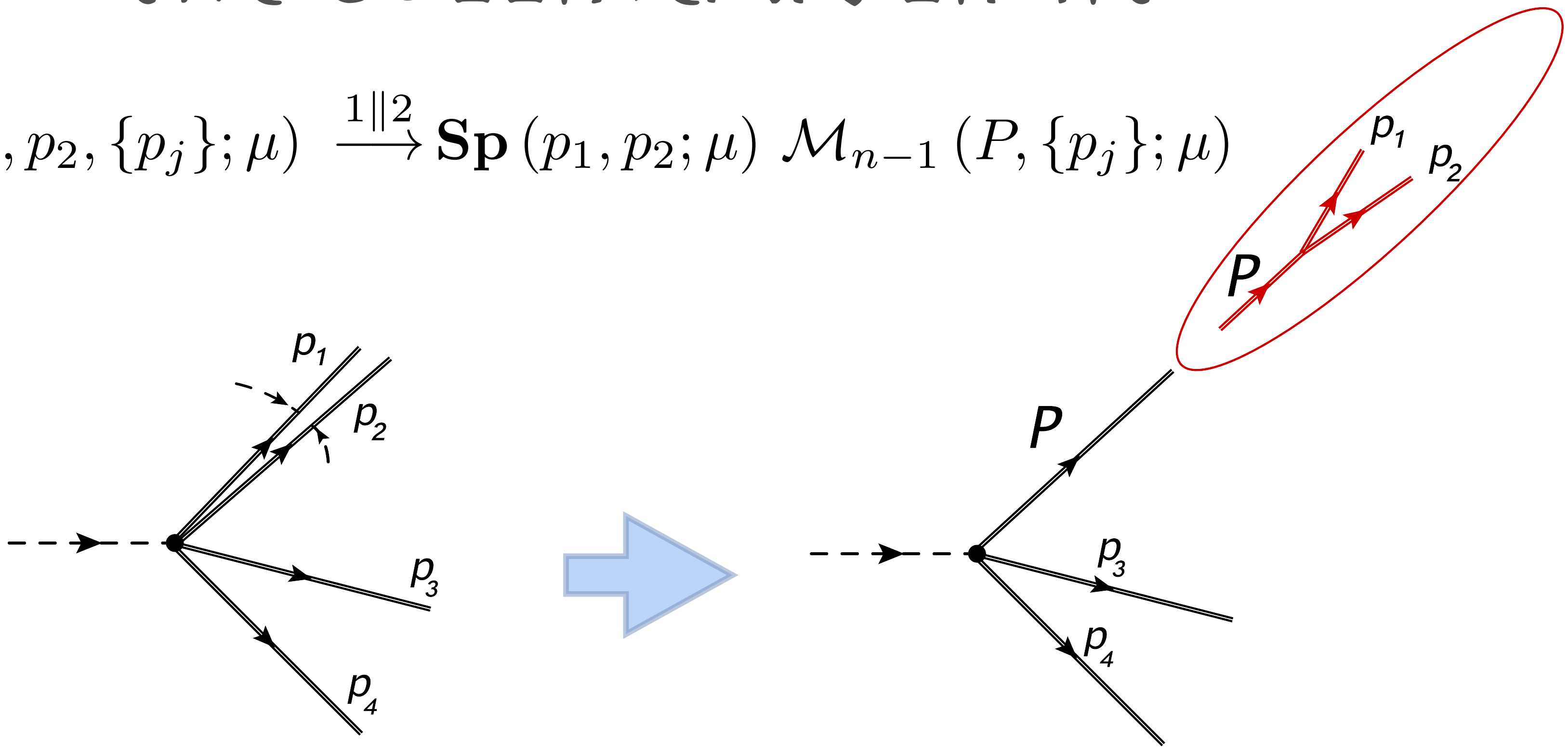
$$F(z,\bar{z}) = \mathcal{L}_{1,0,1,0,1}(z,\bar{z}) + 2\zeta_2\left(\mathcal{L}_{0,1,1}(z,\bar{z})+\mathcal{L}_{0,0,1}(z,\bar{z})\right)$$

Ø. Almelid, C. Duhr, EG

$\mathcal{L}_{10\ldots}(z)$ are the single-valued harmonic polylogarithms introduced by Francis Brown in 2009. They are defined in the region where $\bar{z}=z^*$

THE COLLINEAR LIMIT

$$\mathcal{M}_n\left(p_1, p_2, \{p_j\}; \mu\right) \xrightarrow{1\|2} \mathbf{Sp}\left(p_1, p_2; \mu\right) \mathcal{M}_{n-1}\left(P, \{p_j\}; \mu\right)$$



In particular, IR singularities of the splitting amplitude are those present in n parton scattering (with 1| |2) and not in n-1 parton scattering:

$$\Gamma_{\mathbf{Sp}} = \Gamma_n - \Gamma_{n-1}$$

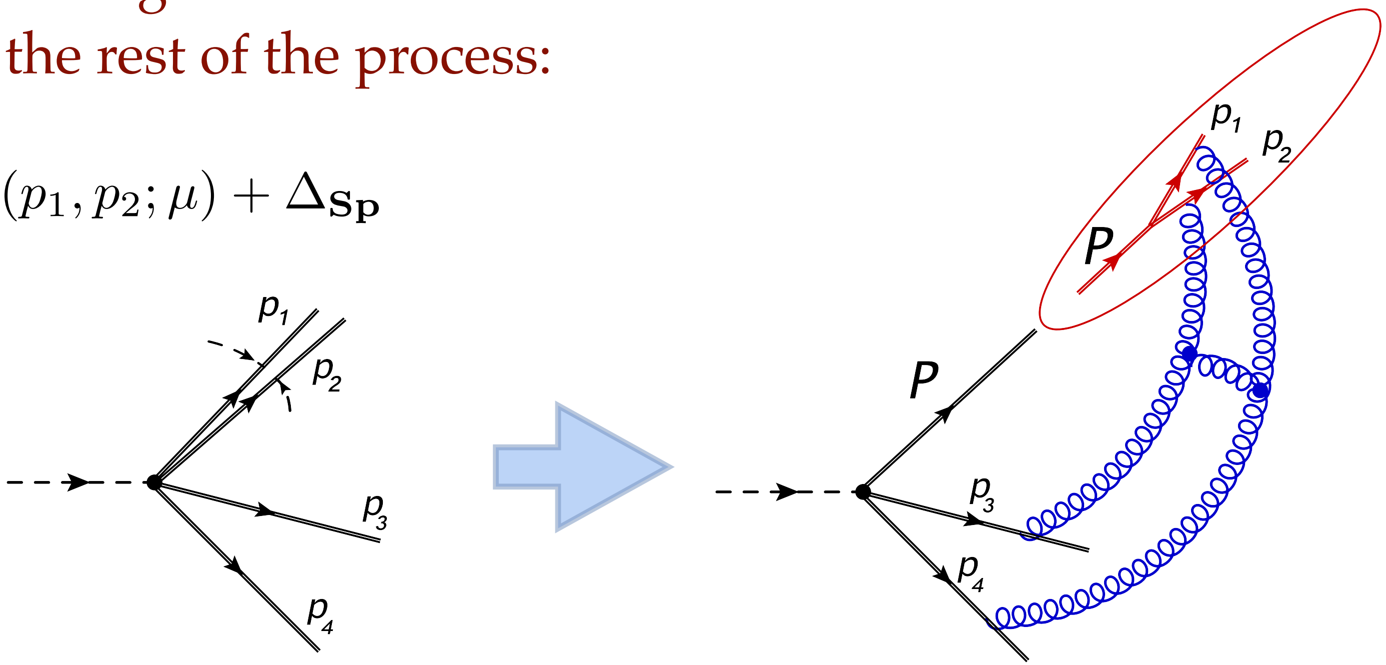
Becher & Neubert (2009)
Dixon, EG & Magnea (2010)

The general expectation (& recent proof by [Feige & Schwartz 1403.6472]) is that the splitting amplitude depends exclusively on the variables of the collinear pair. This is *automatically realised* by the dipole formula for the singularities.

The collinear limit at 3 loops

At three loops there are diagrams that could introduce correlation between collinear partons and the rest of the process:

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \mu) = \Gamma^{\text{dip.}}_{\mathbf{Sp}}(p_1, p_2; \mu) + \Delta_{\mathbf{Sp}}$$



But through intricate cancellations the correction is a *constant* depending **only** on the colour degrees of freedom of the collinear pair:

$$\Delta_{\mathbf{Sp}} = (\Delta_n - \Delta_{n-1})|_{1\|2} = -24\left(\frac{\alpha_s}{4\pi}\right)^3 (\zeta_5 + 2\zeta_2\zeta_3)\left[f^{abe}f^{cde}\left\{\mathbf{T}_1^a, \mathbf{T}_1^c\right\}\left\{\mathbf{T}_2^b, \mathbf{T}_2^d\right\} + \frac{1}{2}C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2\right]$$

Ø. Almelid, C. Duhr, EG

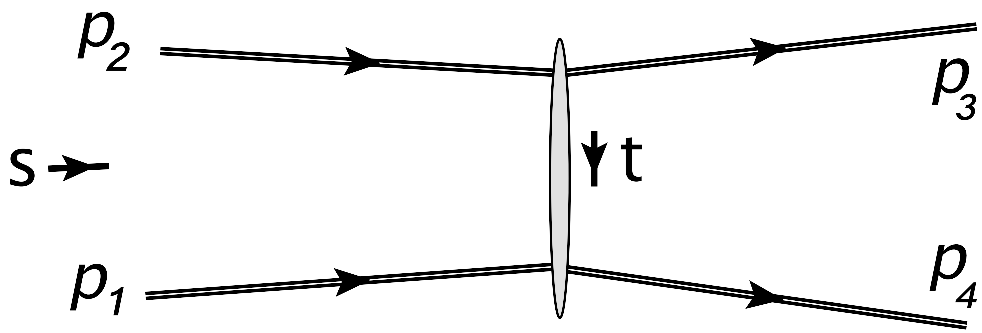
Conclusion: IR singularities of the splitting amplitudes are independent of the rest of the process.
Consistent with expectations!

HIGH-ENERGY (REGGE) LIMIT

Expanding Δ_4 at large s/(-t) we get ***no log-enhanced terms***, just a constant. This can be contrasted with dedicated calculations of the high-energy limit.

The Regge limit is dominated by t-channel gluon exchange. Leading logs of (-t/s) are summed through Reggeization:

$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$



The gluon Regge pole is

$$\alpha(t) = \frac{1}{4}(\mathbf{T}_2+\mathbf{T}_3)^2 \int_0^{-t} \frac{d\lambda^2}{\lambda^2}\widehat{\gamma}_K(\alpha_s(\lambda^2,\epsilon))$$

Korchemskaya and Korchemsky (1996)
Del Duca, Duhr, EG, Magnea & White (2011)

which is **fully consistent with the dipole formula**. This consideration excludes any quadrupole contribution $\alpha_s^3 \log^n(-t/s)$ with $n \geq 2$ for the Re part.

$\mathrm{i}\alpha_s^3 \log^2(-t/s)$ is excluded by an explicit **two Reggeon** calculation

Caron-Huot 1309.6521

$\alpha_s^3 \ln(-t/s)$ is excluded by a dedicated **three Reggeon** calculation.

Caron-Huot, EG, Vernazza - to appear

CONCLUSIONS

- IR singularities of massless scattering amplitudes are now known to **3-loops**.
- As expected, the first correction to the dipole formula occurs at three loops. For three partons it is a **constant**, while for four or more, a quadrupole interaction correlating simultaneously colour and kinematics of 4 patrons.
- The quadrupole term is expressed in terms of single-valued harmonic polylogarithms of weight 5, depending on $\{z, \bar{z}\}$. These variables are simple algebraic functions of conformally-invariant cross ratios, and they manifest the symmetries and analytic structure of the quadruple interaction.
- Splitting amplitudes receive a kinematic-independent correction beyond the dipole formula at 3-loop, but remains independent of the rest of the process!
- Regge limit: consistency with known results at LL and NLL and new predictions at NNLL and beyond.