Bonn 7 April. 2017

Bethe Forum: Discrete Symmetries

F-theory GUTs: *Some ways of being discrete*

$\Gamma\epsilon\omega\rho\gamma\iota o\varsigma\, \mathcal{K}.\ \Lambda\epsilon o\nu\tau\alpha\rho\eta\varsigma$

($\mathcal{G}eorge\ \mathcal{K}.\ \mathcal{L}eontaris$)

The University of Ioannina

$I\omega\alpha\nu\nu\iota\nu\alpha$

GREECE

Outline of the Talk

- $\mathcal{F}-$Theory: A few basic notions...
- Model building with F-theory
- $SU(5) \times PSL_2(p)$ and Neutrinos ...
- Concluding Remarks

$\mathcal{PART} - \mathcal{I}$

F-Theory

why ?

★ **Advantages**

Consistent framework for unification

Calculability

testable predictions

Basic features of F-theory:

⋆ Geometrization of Type II-B String Theory

⋆ Elliptically fibred 8-dimensional compact space

⋆ Fibration described by a simple well known model
(*Weierstraß model*)

$\mathcal{A}$

... a short geometric description of the fibration ...

Any cubic equation with a rational point can be written in:

★ Weierstraß form:

$$y^2 = x^3 + fx + g$$

▲ Two important quantities characterising elliptic curves:

1. The Discriminant:

$$\Delta = 4f^3 + 27g^2$$

... classifies the curves with respect to its singularities

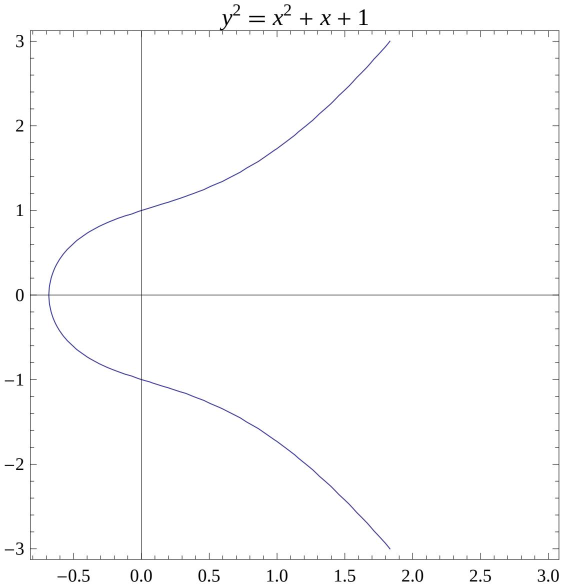
2. The j-invariant function:

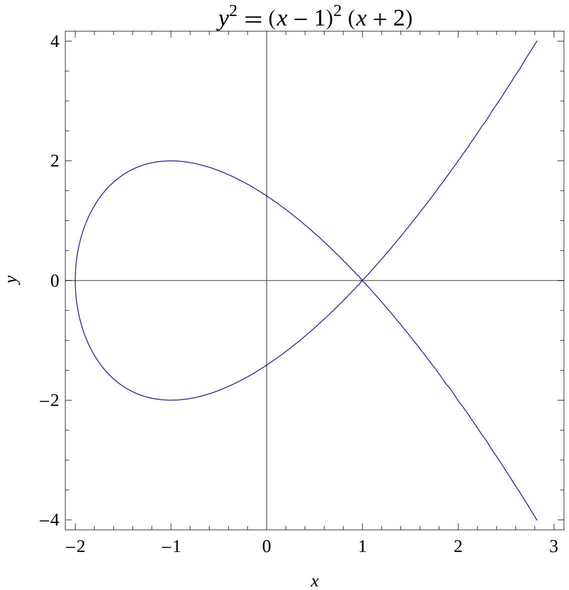
$$j = 4\,\frac{(24f)^3}{4f^3 + 27g^2}$$

... takes the same value for equivalent elliptic curves

basic ingredients: the elliptic curve equ and its discriminant:

$$y^2 = x^3 + fx + g, \quad \Delta = 4f^3 + 27g^2$$

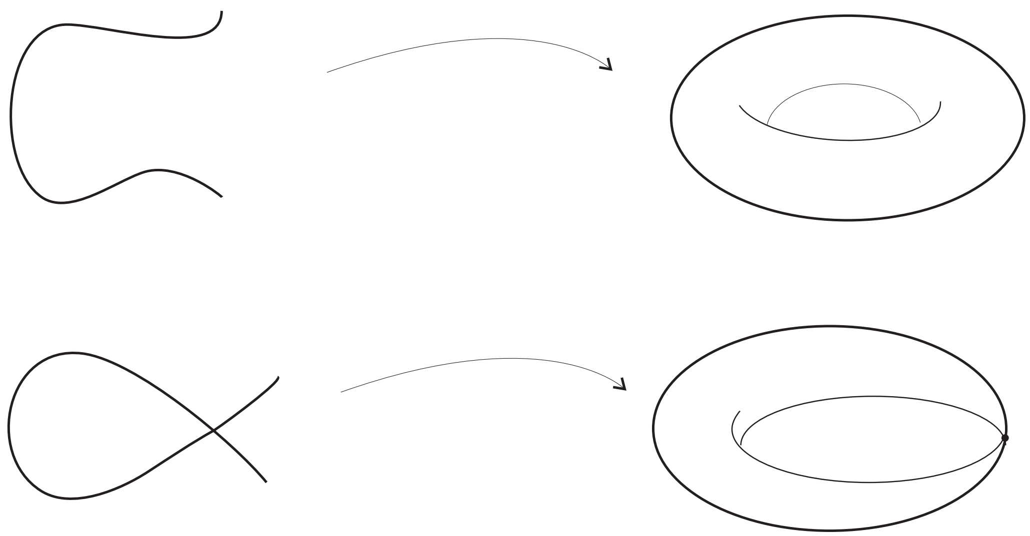


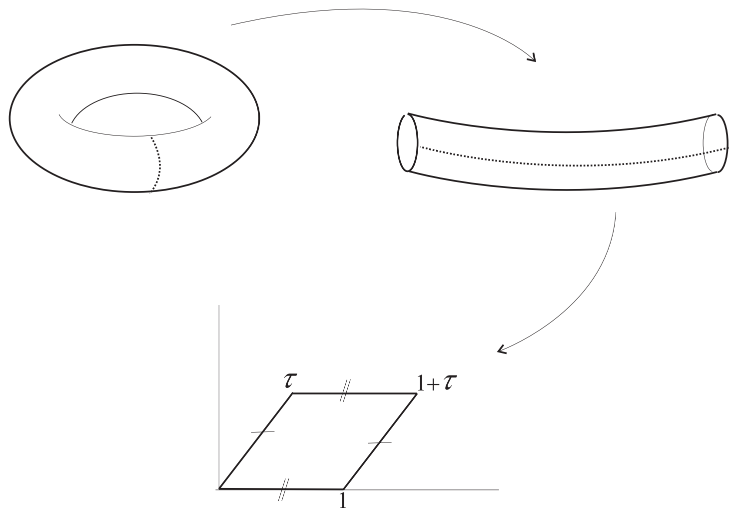


non-singular $\Delta \neq 0 \longleftarrow$ Elliptic Curves $\longrightarrow$ *singular* $\Delta = 0$

Geometric Objects described by Elliptic Curves:

Real ***Complex***





Weierstraß model associated with Torus

Torus described by Complex Modulus: $\tau = \alpha + \beta\, i$.

j-function $\rightarrow j(\tau)$ and $\Delta \rightarrow \Delta(\tau)$

★ F-theory ★

(*Vafa hep-th/9602022*)

⇓

Geometrisation of Type II-B superstring

II-B: *closed string spectrum obtained by combining left and right moving open strings with NS and R-boundary conditions:*

$$(NS_+, NS_+),\ (R_-, R_-),\ (NS_+, R_-),\ (R_-, NS_+)$$

Bosonic spectrum:

(NS_+, NS_+): graviton, dilaton and 2-form Kalb-Ramond-field:

$$g_{\mu\nu},\ \phi,\ B_{\mu\nu} \to B_2$$

(R_-, R_-): scalar, 2- and 4-index fields (*p-form potentials*)

$$\mathbf{C_0}, C_{\mu\nu}, C_{\kappa\lambda\mu\nu} \to C_p,\ p = 0, 2, 4$$

Definitions (*F-theory bosonic part*)

1. *String coupling:* $g_s = e^{-\phi}$

2. *Combining the two scalars* C_0, ϕ *to one modulus:*

$$\tau = C_0 + i\, e^{\phi} \to C_0 + \frac{i}{g_s}$$

(*recall that* τ *can describe a torus*)

$$\Downarrow$$

1. *Theory can be described by consistent properly invariant action* (*see for example arXiv:0803.1194*)

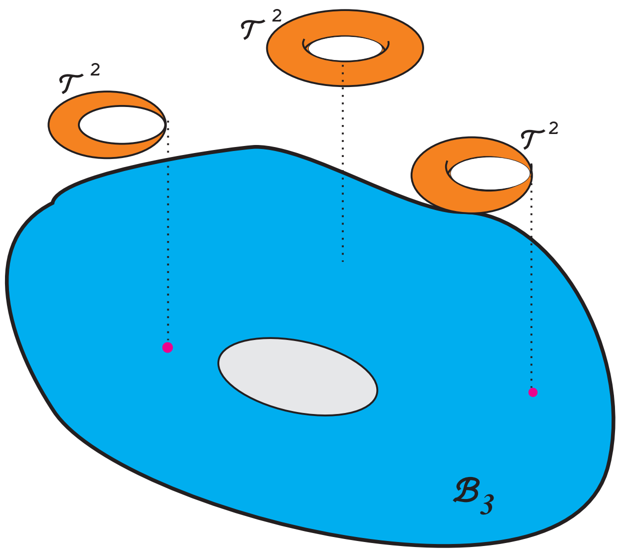
2. ... gives the correct EoM

3. *Consistent with* $N = 1$ Supersymmetry

$\mathcal{FIBRATION}$

- 6-d compact space described by 3-complex dim. manifold $\mathcal{B}_3$
- At each point on $\mathcal{B}_3$ assign a torus with modulus:

$$\tau = C_0 + \imath/g_s$$



$\Rightarrow$ F-theory defined on $\mathcal{R}^{3,1} \times \mathcal{X}$

$\mathcal{X}$, *is called elliptically* fibered **CY** 4-fold over B_3

Elliptic Fibration

described by $\mathcal{W}$eierstraß $\mathcal{E}$quation

$$y^2 \quad = \quad x^3 + f(z)\,x + g(z)$$

For each point of B_3, the above equation describes a torus

1. Discriminant

$$\Delta(z) = 4\,f^3 + 27\,g^2$$

Fiber **singularities** at zeros of *Discriminant*:

$$\Delta(z) = 0 \rightarrow 24 \text{ roots } z_i$$

$$\Downarrow$$

The fiber degenerates at the zeros of the discriminant

$$\Delta(z) = 0 \rightarrow 24 \text{ roots } z_i$$

j-invariant function can be written in terms of modulus τ

$$j(\tau) = 4\,\frac{(24f)^3}{\Delta} \tag{1}$$

$$\propto e^{-2\pi i\tau} + 744 + \mathcal{O}(e^{2\pi i\tau}) \tag{2}$$

$$\Delta = \prod_{i=1}^{24}(z - z_i) \tag{3}$$

Solving

$$\tau \approx \frac{1}{2\pi i}\log(z - z_i)$$

Circling around z_i: (*recall* $\tau = C_0 + i/g_s$)

$$\tau \to \tau + 1 \;\Rightarrow C_0 \to C_0 + 1$$

$\to$ τ and C_0 (*potential*) undergo Monodromy.

At $z = z_i$ $\exists$ source of RR-flux which is interpreted as a:

$D7$-brane at $z = z_i$

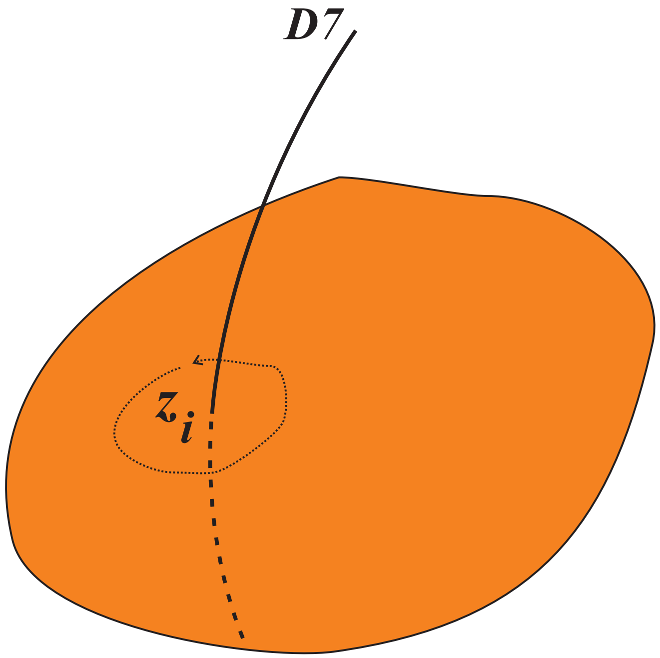


Figure 1: Moving around z_i, $\log(z) \to \log|z| + i(2\pi + \theta)$ and $\tau \to \tau + 1$

Kodaira classification:

- Type of Manifold **singularity** is specified by the **vanishing order** of $f(z)$, $g(z)$ and $\Delta(z)$
- **Geometric Singularities** classified in terms of $\mathcal{A}\,\mathcal{D}\,\mathcal{E}$ Lie groups (**Kodaira**$\sim$ 1960...).

Interpretation of geometric singularities

$$\Downarrow$$

$$\boxed{CY_4\text{-}\textbf{Singularities} \rightleftarrows \textbf{gauge symmetries}}$$

$$\textbf{Groups} \quad \rightarrow \quad \begin{cases} SU(n) \\ SO(m) \\ \mathcal{E}_n \end{cases}$$

Singularities are classified in terms of the vanishing order of

$$f(z),\ g(z),\ \Delta(z)$$

Example:

$$f = z^3(b_3 + b_4 z + \cdots)$$

$$g = z^4(c_4 + c_5 z + \cdots),$$

$$\Delta = 4\,f^3 + 27\,g^2 = z^8(d_8 + d_9 z + \cdots)$$

$$\rightarrow \mathcal{E}_6$$

ord(f)	ord(g)	ord(Δ)	fiber type	Singularity
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Table 1: Vanishing order of the polynomials f, g and the discriminant Δ. (Kodaira classification)

Tate's Algorithm

$$y^2 + \alpha_1 x\, y\, z + \alpha_3 y\, z^3 = x^3 + \alpha_2\, x^2 z^2 + \alpha_4 x\, z^4 + \alpha_6 z^6$$

Table: *Classification of Elliptic Singularities w.r.t. vanishing order of Tate's form coefficients* α_i*:*

Group	α_1	α_2	α_3	α_4	α_6	Δ
$SU(2n)$	0	1	n	n	$2n$	$2n$
$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$
$SU(5)$	0	1	2	3	5	5
$SO(10)$	1	1	2	3	5	7
$\mathcal{E}_6$	1	2	3	3	5	8
$\mathcal{E}_7$	1	2	3	3	5	9
$\mathcal{E}_8$	1	2	3	4	5	10

Basic ingredient in F-theory:

$D7$ - brane

GUTs are associated with 7-branes wrapping certain classes of *'internal'* **2-complex dim.** surface:

$$\mathbf{S} \subset B_3$$

▲ Gauge symmetry embedded in maximal exceptional group:

$$\mathcal{E}_8 \to \mathbf{G_{GUT}} \times \mathcal{C}$$

▲ $G_{GUT} = SU(5),\ SO(10),\ \ldots$

★ $\mathcal{C}$ *Symmetry can be reduced by* $\Rightarrow$ *monodromies or some symmetry breaking mechanism to:*

$$U(1)^n, \text{or some discrete symmetry}\, A_4, S_4, \ldots$$

... these act as family or discrete symmetries :

Karozas, King, GKL, Meadowcroft
JHEP **1409** (2014) 107

Crispim Romao, Karozas, King, GKL, Meadowcroft
Phys. Rev. D **93** (2016) no.12, 126007

$\exists$ additional $U(1)$ and discrete symmetries form elliptic curves:

$$E_8 \times U(1)^n \times Z_n \times Z_m$$

(Mordell-Weil group) (*see refs in : arXiv:1501.06499*)

$\mathcal{B}$

Models

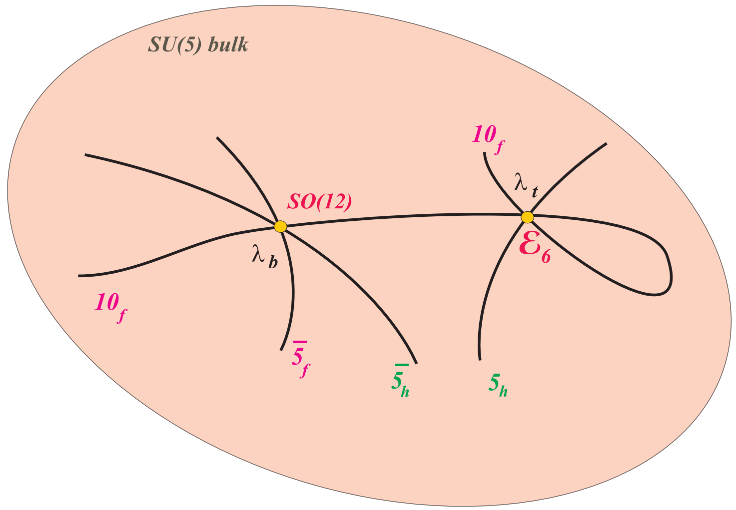
An $SU(5)$ Model

$$\mathcal{E}_8 \to SU(5) \times SU(5)_\perp \to \mathcal{C} = SU(5)_\perp.$$

Spectral Cover description: $SU(5)_\perp \to$ described by Cartan roots:

$$t_i \;=\; SU(5) - \text{roots} \to \sum_i t_i = 0$$

Matter resides in 10 and $\bar{5}$ along intersections with other 7-branes



$\lambda_{t,b}$-Yukawas at intersections and gauge symmetry enhancements

Fluxes:

$SU(5)$ Chirality

$SU(5)$ Symmetry Breaking

Splitting of $SU(5)$-reps

Two types of fluxes:

M_{10}, M_5:

associated with flux-restrictions on $U(1)$'s$\in SU(5)_\perp$:

determine the chirality of complete $10, 5 \in SU(5)$.

N_Y:

related to Cartan generators of $SU(5)_{GUT}$.

They are taken along $U(1)_Y \in SU(5)_{GUT}$ and **split** $SU(5)$-reps.

$SU(5)$ chirality from $U(1)_\perp$ Flux

$U(1)_\perp$–Flux on SM reps $\in$ **10**'s:

$$\#10-\#\overline{10}=\begin{cases} n_{(3,2)_{\frac{1}{6}}}-n_{(\bar{3},2)_{-\frac{1}{6}}} & = & M_{10}\\ n_{(\bar{3},1)_{-\frac{2}{3}}}-n_{(3,1)_{\frac{2}{3}}} & = & M_{10}\\ n_{(1,1)_1}-n_{(1,1)_{-1}} & = & M_{10}\end{cases}$$

$U(1)_\perp$– Flux on SM reps $\in$ **5**'s:

$$\#5-\#\bar{5}=\begin{cases} n_{(3,1)_{-\frac{1}{3}}}-n_{(\bar{3},1)_{\frac{1}{3}}} & = & M_5\\ n_{(1,2)_{\frac{1}{2}}}-n_{(1,2)_{-\frac{1}{2}}} & = & M_5\end{cases}$$

(*...subject to:* $\sum_i M_{10}^i+\sum_j M_5^j=0$)

SM chirality form Hypercharge Flux

$U(1)_Y-$**Flux**-splitting of **10**'s:

$$\begin{aligned} n_{(3,2)_{\frac{1}{6}}} - n_{(\bar{3},2)_{-\frac{1}{6}}} &= M_{10} \\ n_{(\bar{3},1)_{-\frac{2}{3}}} - n_{(3,1)_{\frac{2}{3}}} &= M_{10} - N_{Y_{10}} \\ n_{(1,1)_1} - n_{(1,1)_{-1}} &= M_{10} + N_{Y_{10}} \end{aligned}$$

$U(1)_Y-$ **Flux**-splitting of **5**'s:

$$\begin{aligned} n_{(3,1)_{-\frac{1}{3}}} - n_{(\bar{3},1)_{\frac{1}{3}}} &= M_5 \\ n_{(1,2)_{\frac{1}{2}}} - n_{(1,2)_{-\frac{1}{2}}} &= M_5 + N_{Y_5} \end{aligned}$$

(... *for the Higgs* $M_{10} = 0, N_{Y_5} = \pm 1 \rightarrow$ doublet-triplet splitting...)

▲ **Spectrum** (... *in brief*)▼

- MSSM spectrum + natural doublet-triplet splitting
- vector-like fields $f+\bar{f}$ (always present for $G_S \geq SO(10)$)
- singlets + KK-modes ...(*good for RH neutrinos*)

Two ways to obtain Fermion Mass Hierarchy in F-theory

▲▼ All families on the same curve(s) ($\Sigma_{10}, \Sigma_{\bar{5}}$)

non-commutative geometry, ...**Flux** corrections ⇒ Hierarchy...

▲▼ Families assigned on different matter curves ($\Sigma_{10}^{1,2,3}, \Sigma_{\bar{5}}^{1,2,3}$)

Monodromy → Rank one mass matrices at tree level.
Hierarchy organised by $U(1)$**'s** *(Froggatt Nielsen mechanism)*
from underlying E_8 via Singlet vevs $\langle\theta_{ij}\rangle$

Choice: $\langle\theta_{14}\rangle \cdot \langle\theta_{43}\rangle \neq 0$

- **Rank one** Quark mass matrices (*GKL and GG Ross*) JHEP02(2011)108

$$M_d = \begin{pmatrix} \lambda^d_{11}\theta^2_{14}\theta^2_{43} & \lambda^d_{12}\theta_{14}\theta^2_{43} & \lambda^d_{13}\theta_{14}\theta_{43} \\ \lambda^d_{21}\theta^2_{14}\theta_{43} & \lambda^d_{22}\theta_{14}\theta_{43} & \lambda^d_{23}\theta_{14} \\ \lambda^d_{31}\theta_{14}\theta_{43} & \lambda^d_{32}\theta_{43} & 1\times\lambda^d_{33} \end{pmatrix} v_b, \tag{4}$$

$$M^u = \begin{pmatrix} \lambda^u_{11}\theta^2_{14}\theta^2_{43} & \lambda^u_{12}\theta^2_{14}\theta_{43} & \lambda^u_{13}\theta_{14}\theta_{43} \\ \lambda^u_{21}\theta^2_{14}\theta_{43} & \lambda^u_{22}\theta^2_{14} & \lambda^u_{23}\theta_{14} \\ \lambda^u_{31}\theta_{14}\theta_{43} & \lambda^u_{32}\theta_{14} & 1\times\lambda^u_{33} \end{pmatrix} v_u \tag{5}$$

- *Yukawa strengths* λ_{ij} computed from overlapping Ψ_fs-integrals $\mathcal{O}(1)$.
- Singlet vevs θ_{ij} fixed by F- and D-flatness.

Particles' Wavefunctions: solving EoM → Gaussian profile:
$\psi \sim f(z_i) e^{-M|z_i|^2}$

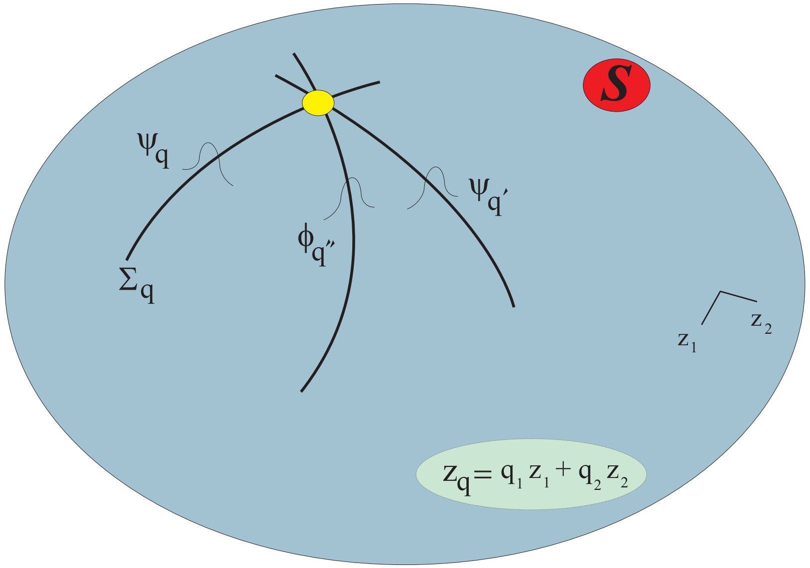


Figure 2: Overlapping of three wavefunctions at triple intersection (Yukawa coupling)

Strength of Yukawa coupling ∝ integral of overlapping ψ's at

3-intersection:

$$\lambda_{ij} \propto \int \psi_i(z_1, z_2)\psi_j(z_1, z_2)\psi_H(z_1, z_2) dz_1 \wedge dz_2 \approx 0.3 - 0.5$$

F-$SU(5)$ interesting low energy implications ($\exists$ vector-like pairs, RPV suppressed...)

$\mathcal{PART-II}$

F-models Discrete Symmetries and Neutrinos

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E.G.Floratos, GKL *arXiv:1511.01875*

Phys.Lett. B755 (2016) 155-161

PSL(2,7) Representations and their relevance to Neutrino Physics

Aliferis, GKL, Vlachos arXiv:1612.06161

▲ neutrino oscillations tightly connected to non-zero neutrino masses and the mixing

▲ Old data ($\sim$ 15 yrs ago) consistent with simple Tri-Bimaximal mixing

$$V_{TB} = V_l^\dagger V_\nu = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

▲ ... theoretical interpretation → invariance under some discrete group:

$$S_4,\ A_4,\ Z_2 \times Z_2,\ A_5, \ldots$$

▲ Recent data show that the actual case is far more complicated...

Neutrino data: parametrization of mixing angles

$$U_\nu = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{23}s_{12}s_{13} - c_{12}s_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \tag{6}$$

Experimental data (3σ range) of the angles ($c_{ij} \equiv \cos\theta_{ij}$)

$$\begin{aligned} \sin^2\theta_{12} &= [0.259 - 0.359] \\ \sin^2\theta_{23} &= [0.331 - 0.637] \\ \sin^2\theta_{13} &= [0.0169 - 0.0313] \\ \delta &= 0.77\pi - 1.36\pi \end{aligned} \tag{7}$$

Working with $M = m_\nu m_\nu^\dagger$ (*assuming Hermitian matrix*):

Hermitian matrix $\rightarrow f(U)$ ($\cdots +$ *Cayley Hamilton theorem:*)

$$M = i\log(U) = c_0 I + c_1 U + c_2 U^2 \qquad (\mathcal{R}_1)$$

Assuming invariance under group generator(s) A_i

$$[M, A_i] = 0 \rightarrow [U, A_i] = 0 \qquad (\mathcal{R}_2)$$

▲ $(\mathcal{R}_1) \rightarrow$ disentangles mixing from eigenvalues ...
$m_{\nu_i} = m_{\nu_i}(c_{0,1,2})$ *(see hep-ph 1103.6178)*

▲ $(\mathcal{R}_2) \rightarrow M, U, A_i$ **common** system of eigenvectors:
i) search for groups with $3-d$ irreps A_i and the right eigenvectors
$\rightarrow$ ν-mixing *or* ...
ii)...try to express $M = \sum_i \alpha_i A_i$.

... a unified method to construct discrete group representations...
required.

... this is **feasible** for a wide class of **Discrete Groups**

$PSL_2(p)$, p **prime**

▲ **Requirements:** ▲

▲ ... of physical interest only those with $3-dim.$ representations

▲ *GUT and "perpendicular"-group embedded in maximal symmetry E_8:*

$$E_8 \supset \begin{cases} E_6 \times SU(3)_\perp \\ SO(10) \times SU(4)_\perp \\ SU(5) \times SU(5)_\perp \end{cases} \tag{8}$$

→ In the context of F-theory, $PSL_2(p)$ must be subgroups of

$$\mathbf{SU(5)_\perp,\ SU(4)_\perp,\ SU(3)_\perp}$$

$$\cdots \to p \leq 11$$

Definition of $SL_2(p)$ $p \in \mathbb{Z}/p\mathbb{Z}$

$SL_2(p)$: group of 2×2 matrices with integer entries

$$\mathfrak{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; ad - bc = 1 \mathrm{mod}(p), p \in \mathbb{Z}/p\mathbb{Z}$$

Group generated by two 2×2 **generators** (**Artin's** rep.):

$$\mathfrak{a} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; \mathfrak{b} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathfrak{a}^2 = \mathfrak{b}^3 = -\mathcal{I} \equiv - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

... *additional conditions depend on* p.

Observing that $Z_2 = \{I, -I\}$ is normal subgroup $\in SL_2(p)$...
...Quotient defines the projective linear group

$$SL_2(p)/\{I, -I\} = PSL_2(p)$$

AIM: construction of 3-dim. representations of $PSL_2(p)$.
Method: use of Weil's Metaplectic Representation
(based on work of *Balian & Itzykson* Acad. Dc. Paris 303 (1986).)

...this method provides the p-dimensional *reducible* representation of $SL_2(p)$...
... p-dim. splits to two lower dimensional **irreducible** representations:

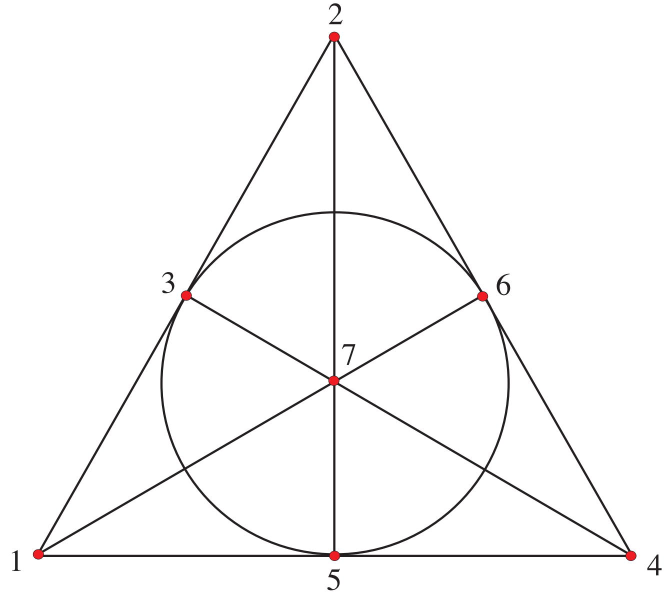
$$p = \frac{p+1}{2} + \frac{p-1}{2}$$

of discrete groups $\in SU(\frac{p+1}{2})$ and $SU(\frac{p-1}{2})$

Cases of Physical Interest: $p = 3, 5, 7$

- $PSL_2(3) \sim A_4$, (and $SL_2(3)$ its double covering)
- $PSL_2(5) \sim A_5$ (*smallest non-abelian simple group*)
- $SL_2(7)$ and its projective $PSL_2(7) \subset SU(3)$ with 168 elements...

...isomorphic to the group preserving the discrete projective geometry of Fano plane.



Fano plane

BACKGROUND

Consider the *GF=Galois field* of discrete circle $GF[p]$ and position eigenfunctions $|q\rangle$

$$GF[p] = \{0, 1, 2, \ldots, p-1\}, \; |q\rangle_i = \delta_{ij}, \; (i, j) = 1, 2, \ldots, p-1$$

Define **Translation** and **Momentum** operators :

$$P|q\rangle = |q+1\rangle; \quad Q|q\rangle = \omega|q\rangle \tag{9}$$

with

$$\omega = e^{2\pi i/p}, \; P_{kl} = \delta_{k-1,l}, \; Q_{kl} = \omega^k \delta_{kl}$$

Properties : Commutation Relation

$$QP = \omega PQ$$

Associated with each-other through the Discrete Fourier Transform (DFT) $F_{kl} = \frac{1}{\sqrt{p}}\omega^{kl}$: $P = F^{-1}QF$

Heisenberg Group $\mathcal{H}$

P and Q generate $\mathcal{H}$ with elements of the form:

$$J_{n_1,n_2,t} = \omega^t P^{n_1} Q^{n_2},$$

with $t \in \mathbb{Z}/p\mathbb{Z}$, $n_1, n_2 \in (\mathbb{Z}/p\mathbb{Z})^2$. Isomorphic to the group of matrices:

$$J_{n_1,n_2,t} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ n_1 & 1 & 0 \\ t & n_2 & 1 \end{pmatrix}$$

Working with a subset of it ($t \to \frac{n_1 n_2}{2}$):

$$J_{\vec{n}} \equiv J_{n_1,n_2} = \omega^{\frac{n_1 n_2}{2}} P^{n_1} Q^{n_2}, \quad \vec{n} = (n_1, n_2)$$

which obeys the 'multiplication' law

$$J_{\vec{m}} J_{\vec{n}} = \omega^{\frac{\vec{n} \times \vec{m}}{2}} J_{\vec{m}+\vec{n}}$$

... magnetic translation operators...

Metaplectic Representation

... the action of an $SL_2(p)$ element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on coordinates (r, s) of periodic lattice $\mathbb{Z}_p \times \mathbb{Z}_p$ induces unitary automorphism $U(A)$:

$$U(A) J_{r,s} U^{\dagger}(A) = J_{r',s'}, \quad \text{where } (r',s') = (r,s) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Formula of $U(A)$ has been given by *Balian and Itzykson (1986)*:

$$U(A) = \frac{\sigma(1)\sigma(\delta)}{p} \sum_{r,s} \omega^{[br^2 + (d-a)rs - cs^2]/(2\delta)} J_{r,s}$$

for $\delta = 2 - a - d \neq 0$, and:

$$\begin{aligned} \delta = 0,\ b \neq 0: &\qquad U(A) = \frac{\sigma(-2b)}{\sqrt{p}} \sum_s \omega^{s^2/(2b)} J_{s(a-1)/b,s} \\ \delta = b = 0,\ c \neq 0: &\qquad U(A) = \frac{\sigma(2c)}{\sqrt{p}} \sum_r \omega^{-r^2/(2c)} P^r \\ \delta = b = 0 = c = 0: &\qquad U(1) = I \end{aligned} \tag{10}$$

A few clarifications on notation

$\sigma(a)$ is the Quadratic Gauss Sum,

$$\sigma(a) \quad = \quad \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \omega^{ak^2} = (a|p) \times \begin{cases} 1 & \text{for } p = 4k+1 \\ i & \text{for } p = 4k-1 \end{cases}$$

and $(a|p)$ the Legendre symbol

$$\left(\frac{a}{p}\right) \quad = \quad \begin{cases} 0 & \text{if} a \text{ devides } p \\ +1 & \text{if } a = \mathcal{QR}\ p \\ -1 & \text{if } a \neq \mathcal{QR}\ p \end{cases} \tag{11}$$

(*integer* a *is* $\mathcal{QR} \rightarrow$ *Quadratic Residue iff* $\exists\, x : x^2 = a \bmod p$.)

for $x = 0, 1, 2, 3, \ldots, \quad x^2 \bmod 5 = 0, 1, 4, 4, 1, 0, 1, 4, 4, 1, \ldots$

The construction of the $SL_2(p)$ representations

... it suffices to construct only the two generators
$\mathfrak{a} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{b} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Observe that

$$U(\mathfrak{a}) = (-1)^{k+1}\, i^n F, \quad \begin{cases} n = 0 & \text{for } p = 2k+1 \\ n = 1 & \text{for } p = 2k-1 \end{cases}$$

Observe also that DFT generates an *Abelian group* with four elements

$$F,\ S = F^2,\ F^3 = F^*,\ S^2 \equiv F^4 = I$$

and... since $S^2 = I \Rightarrow S :$ can be used to define projection operators

$$P_\pm = \frac{1 \pm S}{2} \rightarrow U(A)_\pm = U(A)P_\pm$$

... *split* $SL_2(p)$ *reducible* representations to $\frac{p+1}{2}$ & $\frac{p-1}{2}$ dim. **irreps**

... $U(A)_{\pm}$ *block diagonal form achieved by orthogonal matrix* $\mathcal{O}$ *of* S *eigenvectors*:

$$
\begin{aligned}
(e_0)_k &= \delta_{k0}, \\
(e_j^+)_k &= \frac{1}{\sqrt{2}}(\delta_{k,j} + \delta_{k,-j}),\ j = 1, \ldots, \frac{p-1}{2} \\
(e_j^-)_k &= \frac{1}{\sqrt{2}}(\delta_{k,j} - \delta_{k,-j}),\ j = \frac{p+1}{2}, \ldots, p
\end{aligned}
$$

Example. $SL_2(7)$ *case*:

$$
\mathcal{O} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

Final block-diagonal form:

$$
V_{\pm}(A) = \mathcal{O} U(A)_{\pm} \mathcal{O}
$$

▲ $SL_2(7)$ has $(p^2-1)p = 336$ elements

▲ $PSL_2(7)$ has 168 elements ($= 7 \times 24$ *hours* $= 1$ *week!*)

Construction of 3-d. irreducible representation of $PSL_2(7)$ *satisfying:*

$$\mathfrak{a}^2 = \mathfrak{b}^3 = (\mathfrak{a}\mathfrak{b})^7 = ([\mathfrak{a},\mathfrak{b}])^4 = I$$

from $7-d.$ reducible rep. of $SL_2(7)$.

Defining $\eta = e^{2\pi i/7}$, (7^{th} *root of unity*)

$$\mathfrak{a} \to A^{[3]} \quad = \quad \frac{i}{\sqrt{7}} \begin{pmatrix} \eta^2-\eta^5 & \eta^6-\eta & \eta^3-\eta^4 \\ \eta^6-\eta & \eta^4-\eta^3 & \eta^2-\eta^5 \\ \eta^3-\eta^4 & \eta^2-\eta^5 & \eta-\eta^6 \end{pmatrix}$$

and

$$\mathfrak{b} \to B^{[3]} \quad = \quad \frac{i}{\sqrt{7}} \begin{pmatrix} \eta-\eta^4 & \eta^4-\eta^6 & \eta^6-1 \\ \eta^5-1 & \eta^2-\eta & \eta^5-\eta \\ \eta^2-\eta^3 & 1-\eta^3 & \eta^4-\eta^2 \end{pmatrix}$$

Observation. generators have Latin square structure:

$$U \quad \propto \quad \begin{pmatrix} r_1 & r_2 & r_3 \\ r_2 & r_3 & r_1 \\ r_3 & r_1 & r_2 \end{pmatrix}$$

Imposing conditions: orthogonality, unitarity , . . .

$$r_1^2 + r_2^2 + r_3^2 = 1$$

$$r_1 r_2 + r_1 r_3 + r_2 r_3 = 0$$

$$r_1 + r_2 + r_3 = -1$$

$$x^3 + x^2 - r_1 r_2 r_3 = 0$$

for $PSL_2(7)$, $r_1 r_2 r_3 = \frac{1}{7}$

A toy example:

The following elements give the correct mixing

$$U_1 = \begin{pmatrix} r_3 & -r_1 & -r_2 \\ -r_1 & r_2 & r_3 \\ -r_2 & r_3 & r_1 \end{pmatrix}, \; U_2 = \begin{pmatrix} 0 & 0 & -e^{\frac{6\pi i}{7}} \\ e^{-\frac{2\pi i}{7}} & 0 & 0 \\ 0 & e^{-\frac{4\pi i}{7}} & 0 \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} 0.80217e^{0.5667i} & 0.57735e^{2.3948i} & 0.152283e^{-1.27039i} \\ 0.36647e^{0.106487i} & 0.57735e^{-0.8735i} & 0.729634e^{-0.3499i} \\ 0.471405e^{-1.6582i} & 0.57735e^{3.05416i} & 0.666667e^{0.635302i} \end{pmatrix} \tag{13}$$

Comparison with experimental data:

- ▲ $\theta_{12}, \theta_{23}, \theta_{13}$ in agreement with experimental values.
- ▲ θ_{13} *automatically* non-zero (*see arXiv:1612.06161*)

F-theory models :

$\Downarrow\Downarrow\Downarrow$

Geometric interpretation of GUTs

Calculability, form handful of topological properties, natural Doublet-Triplet splitting...

Prediction of Vector-like pairs and singlets ...

hints for New Physics

such as ... resonances, diphoton events at a few TeV...

Discrete Symmetries interpreting the Neutrino data naturally incorporated in E_8 singularity